Superpositions, transition probabilities and primitive observables in infinite quantum systems

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Motivation

Simple systems ("particles") in quantum mechanics are

- described by *pure* states (maximal information)
- satify superposition principle (interference, entanglement)
- admit statistical interpretation (transition probabilities)

Meaningful for finite systems; but problems in QFT:

- localized (partial) states are *never* pure (Reeh-Schlieder)
- inevitable loss of information due to radiation (Huygens)
- appearance of horizons (Unruh . . .)

Simple systems are often to be described by non–pure states.

Question: Status of superpostions, transition probabilities *etc* ?

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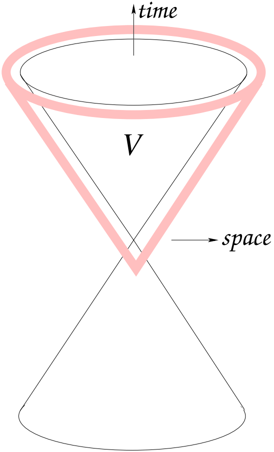

Fig. Restricted information in V due to outgoing radiation

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Basics: (finite quantum systems)

- observables: $\mathcal{N} \simeq \mathcal{B}(\mathcal{H})$ type I factors
- states: $\omega : \mathcal{N} \to \mathbb{C}$ positive, linear, normalized functionals
- pure states: $\omega \neq p_1\,\omega_1 + p_2\,\omega_2$ (not mixed)

Pure states admit:

- bijective lifts: $\omega \to \mathbb{T}\,\Omega \subset \mathcal{H}$ s.t. $\omega(A) = \langle \Omega, A\,\Omega \rangle$, $A \in \mathcal{N}$
- superpositions: $\omega_1, \omega_2 \to \mathbb{T}\Omega_1, \mathbb{T}\Omega_2 \to \mathbb{T}(c_1\Omega_1 + c_2\Omega_2) \to \omega_{12}$
  resulting states are again pure (relative phases matter)
- transition probabilites: $\omega_1, \omega_2 \to \omega_1 \cdot \omega_2 \doteq |\langle \Omega_1, \Omega_2 \rangle|^2$
  expectation values of observables $\omega_1 \cdot \omega_2 = \omega_1(P_2) = \omega_2(P_1)$

Fact: Algebras of observables $\mathcal{M}$ are not always factors of type I

Task: Characterization of "simple systems" on non–type I factors $\mathcal{M}$ and determination of their properties

Observation: Algebras $\mathcal{M}$ of interest are hyperfinite factors

Ingredients for solution:

- funnels of type I algebras (replace $\mathcal{B}(\mathcal{H})$ for finite systems)
- generic states on funnels (replaces concept of pure states)
- primitive observables (replace minimal projections)

Message: Generic states on funnels describe "simple systems"

Funnels

Hyperfinite factors $\mathcal{M}$ generated by

- $\mathcal{N}_1 \subset \mathcal{N}_2 \cdots \subset \mathcal{N}_n \cdots$ type I_∞ factors with common identity
- s.t. $\mathcal{N}_n' \bigcap \mathcal{N}_{n+1}$ infinite dimensional (hence type I_∞), $n \in \mathbb{N}$,
- $\boldsymbol{\mathcal{N}} = \bigcup_n \mathcal{N}_n$ dense in $\mathcal{M}$ in strong operator topology

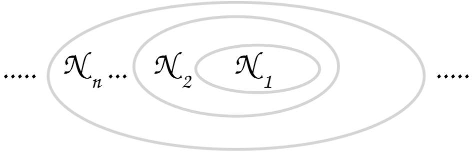
Remarks:

- $\boldsymbol{\mathcal{N}}$ called proper sequential type I_∞ **funnel** [Takesaki]
- C*–algebras generated by funnels $\boldsymbol{\mathcal{N}}$ are isomorphic
- different funnels generating $\mathcal{M}$ are related elements of In $\mathcal{M}$

Present context:

$\boldsymbol{\mathcal{N}}$ **not** closed, allowing unified analysis of states of any infinite type

Physical interpretation:



Observables localized in increasing regions

- relativistic QFT's having split property (semilocal nets)
- non–relativistic QFT's
- infinite lattice systems . . .

Generic states

States $\omega : \mathcal{N} \to \mathbb{C}$, GNS–representation $(\pi, \mathcal{H}, \Omega)$

- locally normal, *i.e.* weakly continuous on unit balls of $\mathcal{N}_n$, $n \in \mathbb{N}$,
- faithful, *i.e.* $\omega(A^* A) = 0$ implies $A = 0$
- **generic**, *viz.* representing vector Ω cyclic for $\mathcal{N}_n' \bigcap \mathcal{N}_{n+1}$, $n \in \mathbb{N}$

Remark: Generic vector states form dense G_δ set [Dixmier, Marechal]

Definition

Let ω be generic. Its **orbit** under non–mixing operations is given by

$$\boldsymbol{\omega_{\mathcal{N}}} \doteq \{\omega_A = \omega \circ \mathrm{Ad}\, A : A \in \mathcal{N} \, , \ \omega_A(1) = 1\} \, ,$$

where $\mathrm{Ad}\, A\,(B) \doteq A^* B\, A$, $B \in \mathcal{N}$.

Physical interpretation:

Generic state ω describes “global background” in which physical operations are performed (“state of the world”). These operations produce the corresponding orbit $\omega_{\mathcal{N}}$.

Examples:

- vacuum state in QFT
- thermal equilibrium states in QFT
- Hadamard states in QFT on curved spacetime

Superpositions

Fix generic state ω (any type), orbit $\boldsymbol{\omega_{\mathcal{N}}}$. Norm distance

$$\|\omega_A - \omega_B\| \doteq \sup_{C \in \mathcal{N}_1} |\omega_A(C) - \omega_B(C)| \,, \quad \omega_A, \omega_B \in \boldsymbol{\omega_{\mathcal{N}}} \,.$$

Proposition

There exists a lift from $\boldsymbol{\omega_{\mathcal{N}}}$ *to rays in* $\mathcal{N}$ *which is*

1. *bijective:* $\omega_A = \omega_B$ *iff* $B = t\,A$ *for* $t \in \mathbb{T}$
2. *locally continuous: if* $\|\omega_{A_m} - \omega_A\| \to 0$ *for (bounded)* $A_m, A \in \mathcal{N}_{\boldsymbol{n}}$*,* $\boldsymbol{n}$ *fixed, then* $t_m\,A_m \to A$ *in the strong operator topology*
3. *locally complete: if* $\|\omega_{A_l} - \omega_{A_m}\| \to 0$ *for (bounded)* $A_l, A_m \in \mathcal{N}_{\boldsymbol{n}}$*, there is* $A \in \mathcal{N}_{\boldsymbol{n}}$ *such that* $t_m\,A_m \to A$ *and* $\|\omega_{A_m} - \omega_A\| \to 0$.

Note that $\mathcal{N}$ is a pre–Hilbert space, *i.e.* result analogous to lifting pure states to rays in Hilbert space

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Physical interpretation:

3. $\boldsymbol{\omega}_{\mathcal{N}}$ maximal set of states arising from local non-mixing operations
1. superpositions of states in $\boldsymbol{\omega}_{\mathcal{N}}$ are meaningful,

$$\omega_A, \omega_B \to \mathbb{T}\,A,\, \mathbb{T}\,B \to \mathbb{T}\,(c_A\,A + c_B\,B) \to \omega_{(c_A\,A + c_B\,B)},$$

and relative phase between $c_A, c_B \in \mathbb{C}$ matters

Mixtures:

$$\text{Conv}\,\boldsymbol{\omega}_{\mathcal{N}} \doteq \{\,\textstyle\sum_m p_m\,\omega_{A_m} : \; \omega_{A_m} \in \boldsymbol{\omega}_{\mathcal{N}}\,,\; p_m > 0\,,\; \textstyle\sum_m p_m = 1\}$$

Proposition

Let $\omega_A = \sum_{m=1}^M p_m\,\omega_{A_m} \in \boldsymbol{\omega}_{\mathcal{N}}$; *then* $\omega_{A_1} = \cdots = \omega_{A_M} = \omega_A$.

$\boldsymbol{\omega}_{\mathcal{N}}$ are the extreme points of $\text{Conv}\,\boldsymbol{\omega}_{\mathcal{N}}$ in complete analogy to case of pure states

Transition probabilities

Definition

Transition probability for $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$: $\quad \omega_A \cdot \omega_B \doteq |\omega(A^* B)|^2$

Remark: comparison with Uhlmann transition probability

$$\omega_A \cdot \omega_B \leq \omega_A \overset{U}{\cdot} \omega_B = \sup_{\Omega_A, \Omega_B} |\langle \Omega_A, \Omega_B \rangle|^2.$$

Proposition

Let $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$.

1. $0 \leq \omega_A \cdot \omega_B \leq 1$ *(notion of* ***orthogonality****),* $\omega_A \cdot \omega_B = \omega_B \cdot \omega_A$
2. $\omega_A \cdot \omega_B \leq 1 - \frac{1}{4} \|\omega_A - \omega_B\|^2$ *where equality holds iff ω is pure*
3. $\omega_A, \omega_B \mapsto \omega_A \cdot \omega_B$ ***locally continuous***
4. *there exist* ***complete families*** *of orthogonal states $\omega_{A_m} \in \boldsymbol{\omega}_{\mathcal{N}}$, $m \in \mathbb{N}$, i.e. $\sum_m \omega_B \cdot \omega_{A_m} = 1$ for any $\omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$.*

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Algebra of states

Input: Span $\boldsymbol{\omega}_{\mathcal{N}}$. By polarization formula

$$\omega(B^* \cdot A) = \frac{1}{4} \sum_{k=0}^{3} i^k \omega\big((B + i^k A)^* \cdot (B + i^k A)\big) \in \text{Span}\, \boldsymbol{\omega}_{\mathcal{N}} \,.$$

Note: Expression not invariant under substitutions $A \mapsto t_A\, A$, $B \mapsto t_B\, B$.

Definition

Let $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$. Corresponding product $\omega_A \times \omega_B \in \text{Span}\, \boldsymbol{\omega}_{\mathcal{N}}$ given by

$$\omega_A \times \omega_B(C) \doteq \omega(A^* B)\omega(B^* CA)\,, \quad C \in \mathcal{N}\,,$$

is well defined (phases t_A, t_B cancel).

Proposition

1. *The map* $\omega_A, \omega_B \mapsto \omega_A \times \omega_B$ *extends linearily in both entries to an associative product, i.e. Span* $\omega_{\mathcal{N}}$ *is an algebra.*
2. *The antilinear involution* $\dagger$: *Span* $\omega_{\mathcal{N}} \to$ *Span* $\omega_{\mathcal{N}}$ *given by*
$$\left(\sum_m c_m \omega_{A_m}\right)^\dagger \doteq \left(\sum_m \overline{c}_m \omega_{A_m}\right)$$
is consistent with the product, i.e. $\mathcal{C} \doteq$ *Span* $\omega_{\mathcal{N}}$ *is a *–algebra*
3. $\mathcal{C}$ *is an* $\mathcal{N}$*–bimodule*

Further structure:

- spectral theorem exists in $\mathcal{C}$ (in particular: unique decomposition of elements of Conv $\omega_{\mathcal{N}}$ into orthogonal states)
- there exists spatial isomorphism $\mathcal{C} \leftrightarrow \mathcal{C}_{\mathcal{H}}$ where $\mathcal{C}_{\mathcal{H}} \subset \mathcal{B}(\mathcal{H})$ is $\mathcal{N}$–bimodule of finite rank operators.

Question: How is the type of ω, respectively of $\mathcal{M}$, encoded in the structure of the “skeleton” $\mathcal{C}$ of $\mathcal{C}^{-\omega} = \mathcal{M}_*$?

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Primitive observables

Question: (When) are the transition probabilities observable?

Recall: Non–mixing operations, $V \in \mathcal{N}$,

$$\omega \mapsto (1/\omega(V^* V))\, \omega \circ \mathrm{Ad}\, V\,.$$

Restrict to unitaries $U^* U = 1 = UU^*$ (observable) inducing transitions

$$\omega_A \mapsto \omega_A \circ \mathrm{Ad}\, U = \omega_{UA}\,, \quad \omega_A \in \boldsymbol{\omega}_{\mathcal{N}}\,.$$

Standard examples: Effects of temporary perturbations of dynamics

Transition probability between initial and final states:

$$\omega_A \cdot (\omega_A \circ \mathrm{Ad}\, U) = \omega_A \cdot \omega_{UA} = |\omega_A(U)|^2$$

Alternative interpretation: “Fidelity” of operation U in given state ω_A

$\omega_A \cdot \omega_{UA}$ can be determined by measurements of U in state ω_A.

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A primitive observable is fixed by a unitary $U \in \mathcal{N}$. For any $\omega_A \in \boldsymbol{\omega}_{\mathcal{N}}$,

- $\omega_A \mapsto \omega_{UA}$ (result of operation)
- $\omega_A \cdot \omega_{UA} = |\omega_A(U)|^2$ (transition probability/fidelity of operation)

Primitive observables replace (generalize) minimal projections

Standard expectation values of observables can be recovered:

Proposition

Given projection $E \in \mathcal{N}$, (finite number of) states $\omega_A \in \boldsymbol{\omega}_{\mathcal{N}}$, and $\varepsilon > 0$. There are unitaries $U \in \mathcal{N}$

1. $|\omega_A \cdot \omega_{UA} - \omega_A(E)^2| < \varepsilon$, *i.e. "standard probatilities $\approx \sqrt{\text{fidelities}}$"*
2. $\omega_{UA}(1 - E) < \varepsilon$ *(compare von Neumann projection postulate)*

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Question: Is $\omega_A \cdot \omega_B$ operationally defined for each pair $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$?
(This would require that there are unitaries $U \in \mathcal{N}$ such that $\|\omega_B - \omega_{UA}\| < \varepsilon$.)

Theorem (Connes, Haagerup, Størmer)

Let ω *be of type* III_λ *and let*

1. $0 \leq \lambda < 1$. *There are* $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$ *s.t.* $\inf_U \|\omega_B - \omega_{UA}\| > \varepsilon(\lambda)$.
2. $\lambda = 1$. *Then* $\inf_U \|\omega_B - \omega_{UA}\| = 0$ *for any* $\omega_A, \omega_B \in \boldsymbol{\omega}_{\mathcal{N}}$.

Concept of transition probabilities (operationally) meaningful for

- pure states ω on $\mathcal{N}$ [Kadison]
- generic states ω on $\mathcal{N}$ of type III_1 [Connes, Størmer]

These are exactly the two cases of interest in infinite quantum systems.

Summary

Generic states ω on funnels: generalization of concept of pure states

- excitations – non–mixing operations $\omega \mapsto \omega_A$
- superpositions – bijective lifts $\omega_A \mapsto \mathbb{T}\, A$
- transition probabilities – product $\omega_A \cdot \omega_B = |\omega(A^* B)|^2$
- primitive observables – replace projections $\omega_A \cdot \omega_{UA} = |\omega_A(U)|^2$

Meaningful framework for simple (elementary) quantum systems