

Integrable systems on (multiplicative) quiver varieties

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Calogero-Moser space (1)

$\mathcal{M} := \{X, Y \in \mathfrak{gl}_n(\mathbb{C}), V \in \text{Mat}_{1 \times n}(\mathbb{C}), W \in \text{Mat}_{n \times 1}(\mathbb{C})\}$

Action of $\text{GL}_n(\mathbb{C})$, $g \cdot (X, Y, V, W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$

Calogero-Moser space $\mathcal{C}_n := \{[X, Y] - WV = \text{Id}_n\} // \text{GL}_n$ [Wilson,'98]

Calogero-Moser space (1)

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Calogero-Moser space $\mathcal{C}_n := \{[X, Y] - WV = \text{Id}_n\} // \text{GL}_n$ [Wilson, '98]

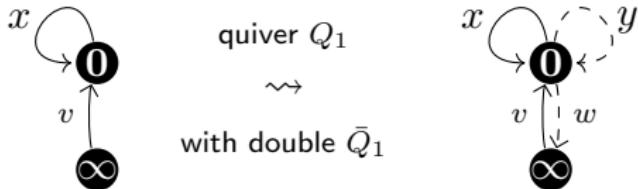
On open dense subset : $X = \text{diag}(q_1, \dots, q_n)$, $V = (1, \dots, 1)$

from which $W = -(1, \dots, 1)^T$ $Y_{ij} = \delta_{ij} p_j - \delta_{(i \neq j)} \frac{1}{q_i - q_j}$

Calogero-Moser Hamiltonian

$$\frac{1}{2} \text{tr } Y^2 = \frac{1}{2} \sum_{1 \leq j \leq n} p_j^2 - \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}$$

Calogero-Moser Space (2)



$$\text{Rep}(\mathbb{C}\bar{Q}_1, (1, n)) = \{X, Y \in \mathfrak{gl}_n, V \in \text{Mat}_{1 \times n}, W \in \text{Mat}_{n \times 1}\} = \mathcal{M}$$

↪ The Calogero-Moser space \mathcal{C}_n is a *quiver variety* obtained by reduction from the representation space of a quiver

The Poisson bracket can also be understood on \bar{Q}_1 !

Same for the moment map $[X, Y] - WV$ (cf. part 2)

Plan for the talk

- ① **Integrable systems from quiver varieties**
- ② Method : Noncommutative Poisson geometry
- ③ Integrable systems from multiplicative quiver varieties

How to read a quiver ? (1)

Quiver Q : directed graph (vertices $s \in I$, arrows $a \in Q$)

Double quiver \bar{Q} : for each $s \xrightarrow{a} t \in Q$, add $t \xrightarrow{a^*} s$

Fix a dimension vector $\mathbf{n} := (n_s) \in \mathbb{N}^I$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \text{Mat}(n_s \times n_t, \mathbb{C})$$

Such matrices obtained from the arrows parametrise $\text{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

How to read a quiver ? (2)

Given Q , $\mathbf{n} = (n_s) \in \mathbb{N}^I$, we got $\text{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \text{Mat}(n_s \times n_t, \mathbb{C})$$

Embed matrices as elements of $\text{End}(\bigoplus_{s \in I} \mathbb{C}^{n_s})$

\rightsquigarrow natural $\text{GL}(\mathbf{n}) := \prod_{s \in I} \text{GL}(n_s)$ action by conjugation

There is a simple* Poisson structure with moment map $\sum_{a \in Q} [A, A^*]$

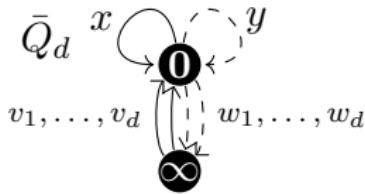
Fix parameters $(\lambda_s) \in \mathbb{C}^I$.

Quiver variety : $\{\sum_{a \in Q} [A, A^*] = \prod_{s \in I} \lambda_s \text{Id}_{\mathbb{C}^{n_s}}\} // \text{GL}(\mathbf{n})$

* Nonzero Poisson bracket only between the matrices A, A^* of an arrow a and its double a^*

Calogero-Moser space with spin (1)

[Wilson,~'98 ; Bielawski-Pidstrygach,'10 ; Tacchella,'15 ; Chalykh-Silantyev,'17]



$$d \geq 2. \quad \mathcal{M} = \text{Rep} (\mathbb{C}\bar{Q}_d, (1, n))$$

space parametrised by :

$$X, Y \in \mathfrak{gl}_n$$

$$V_\alpha \in \text{Mat}_{1 \times n}, W_\alpha \in \text{Mat}_{n \times 1}$$

$$\mathcal{C}_{n,d} := \left\{ [X, Y] - \sum_{1 \leq \alpha \leq d} W_\alpha V_\alpha = \lambda_0 \text{Id}_n \right\} // \text{GL}_n \quad (\lambda_0 \neq 0)$$

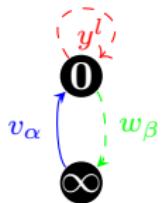
Calogero-Moser space with d spins/degrees of freedom [Gibbons-Hermsen,'84]

On open dense subset where $X = \text{diag}(q_1, \dots, q_n)$:

$$Y_{ij} = \delta_{ij} p_j - \delta_{(i \neq j)} \frac{f_{ij}}{q_i - q_j}, \quad f_{ij} := \sum_{\alpha=1}^d (W_\alpha V_\alpha)_{ij}, \quad \text{Lax matrix for spin CM}$$

Calogero-Moser space with spin (2)

We can compute on $\mathcal{C}_{n,d}$ the Poisson brackets of $\text{tr } Y^k$ and $t_{\alpha\beta}^l = V_\alpha Y^l W_\beta$.



$$\{\text{tr } Y^k, \text{tr } Y^l\}_P = 0 = \{\text{tr } Y^k, t_{\alpha\beta}^l\}_P$$

$$\{t_{\alpha\beta}^k, t_{\gamma\epsilon}^l\}_P = \delta_{\beta\gamma} t_{\alpha\epsilon}^{k+l} - \delta_{\alpha\epsilon} t_{\gamma\beta}^{k+l}$$

Proposition

1. The commutative algebra generated by $(\text{tr } Y^k, t_{\alpha\alpha}^k)$, $1 \leq \alpha \leq d$, is an abelian Poisson algebra of dimension nd .
2. The commutative algebra generated by $(\text{tr } Y^k, t_{\alpha\beta}^k)$ (any indices), is a Poisson algebra of dimension $2nd - n$, with centre of dimension n containing the $(\text{tr } Y^k)$.

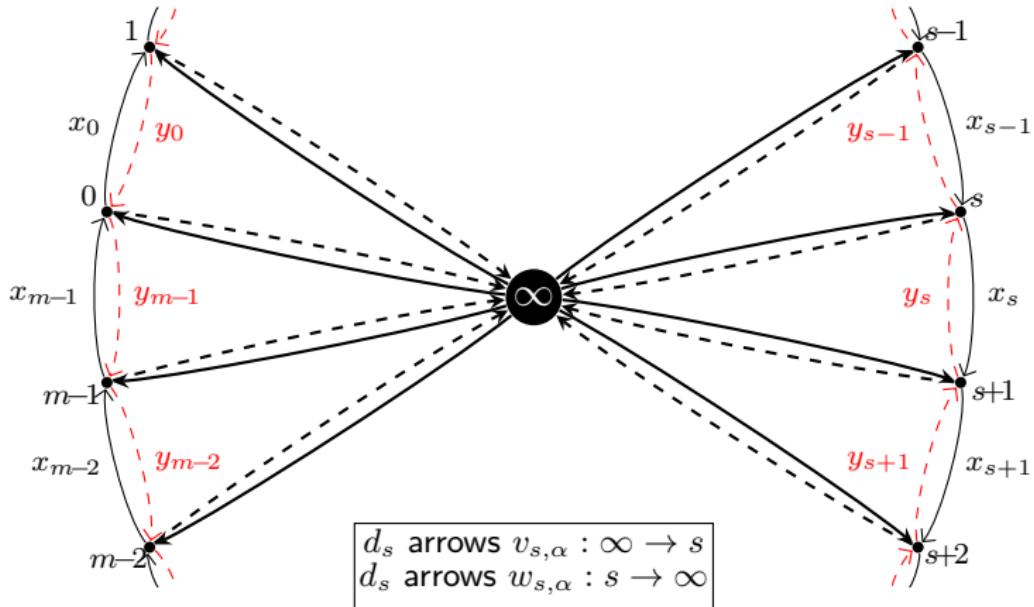
1. \Rightarrow Liouville integrability of the $(\text{tr } Y^k)$.
2. \Rightarrow Degenerate integrability of the $(\text{tr } Y^k)$.

In fact, can get maximal degenerate integrability of each $\text{tr } Y^k$ by [Wojciechowski, '83]

CM spaces from cyclic quivers (1)

Extended cyclic quiver \Rightarrow system of CM type [Chalykh-Silantyev,'17]

arbitrary extension in [F.-Görbe,'21] for spin vector $\mathbf{d} \in \mathbb{N}^{\mathbb{Z}_m}$



We are interested in $\text{Rep}(\mathbb{C}\bar{Q}_{\mathbf{d}}, (1, n, \dots, n))$ and $y_{\bullet} = \sum_s y_s$

CM spaces from cyclic quivers (2)

$\mathcal{M} = \text{Rep}(\mathbb{C}\bar{Q}_{\mathbf{d}}, (1, n, \dots, n))$ parametrised by

$$X_s \in \mathfrak{gl}_n, \quad Y_s \in \mathfrak{gl}_n, \quad V_{s,\alpha} \in \text{Mat}_{1 \times n}, \quad W_{s,\alpha} \in \text{Mat}_{n \times 1}, \\ s = 0, \dots, m-1, \quad (s, \alpha) \text{ with } 1 \leq \alpha \leq d_s$$

Action of $g = (g_s) \in \prod_s \text{GL}_n$:

$$g \cdot (X_s, Y_s, W_{s,\alpha}, V_{s,\alpha}) = (g_s X_s g_{s+1}^{-1}, g_{s+1} Y_s g_s^{-1}, g_s W_{s,\alpha}, V_{s,\alpha} g_s^{-1}),$$

Calogero-Moser space $\mathcal{C}_{n,\mathbf{d}}$: $(\prod_s \text{GL}_n)$ -orbits of $[(\lambda_s) \in \mathbb{C}^m \text{ generic}]$

$$X_s Y_s - Y_{s-1} X_{s-1} - \sum_{1 \leq \alpha \leq d_s} W_{s,\alpha} V_{s,\alpha} = \lambda_s \text{Id}_{n_s}, \quad \forall 0 \leq s \leq m-1$$

On open dense subset : $\text{tr}(Y_{\bullet}^{km})$ generalise spin CM system

$$\rightsquigarrow \frac{1}{m} \text{tr}(Y_{\bullet}^{km}) = \sum_{i=1}^n p_i^{mk} + O(p_i^{mk-1})$$

CM spaces from cyclic quivers (3)

By adapting [Chalykh-Silantyev,'17], we can “visualise” the functions that guarantee integrability of the generalised CM systems :

$$y_\bullet^{km}$$

tr(Y_\bullet^{km})

$$v_{s,\alpha} y_\bullet^{km} w_{s,\alpha}$$

$t_{s\alpha; s\alpha}^k$

$$v_{s,\alpha} y_\bullet^{km} y_\bullet^{s-r} w_{r,\beta}$$

$t_{r\beta; s\alpha}^k = \text{tr}(W_{r,\beta} V_{s,\alpha} Y_\bullet^{km+s-r})$

Proposition ([F.-Görbe,'21 / 2101.05520])

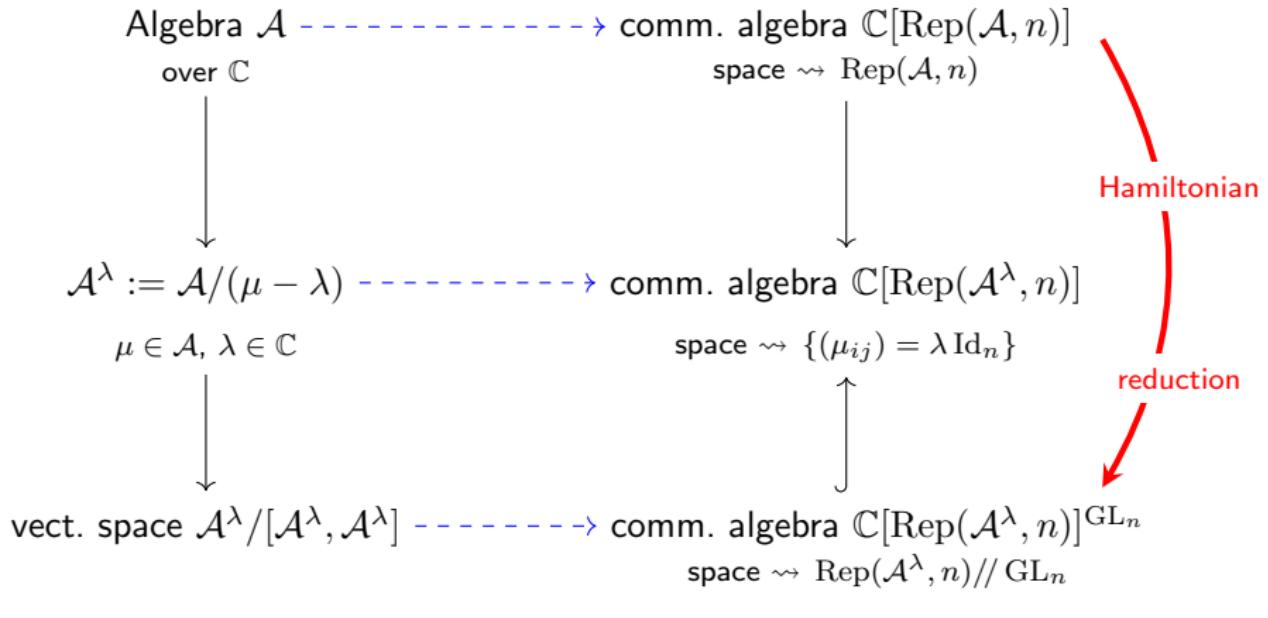
1. The commutative algebra generated by $(\text{tr}(Y_\bullet^{km}), t_{s\alpha; s\alpha}^k)$, is an abelian Poisson algebra of dimension $n|\mathbf{d}|$.
2. The commutative algebra generated by $(\text{tr}(Y_\bullet^{km}), t_{r\beta; s\alpha}^k)$, is a Poisson algebra of dimension $2n|\mathbf{d}| - n$, with centre of dimension n containing the $\text{tr}(Y_\bullet^{km})$.

In fact, maximal degenerate integrability of each $\text{tr}(Y_\bullet^{km})$

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- ② **Method : Noncommutative Poisson geometry**
- ③ Integrable systems from multiplicative quiver varieties

Noncommutative Poisson geometry (1)

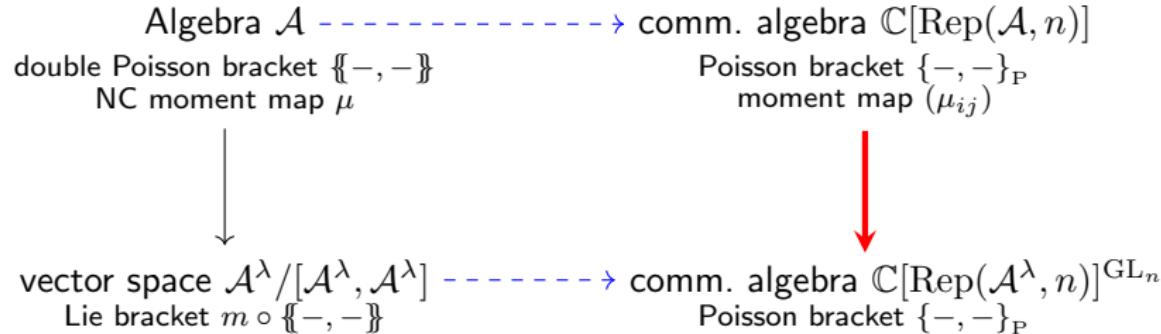


$\mathbb{C}[\text{Rep}(\mathcal{A}, n)]$ generated by symbols $a_{ij}, \forall a \in \mathcal{A}, 1 \leq i, j \leq n.$

Rules : $1_{ij} = \delta_{ij}$, $(a + b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik}b_{kj}.$

$\mathbb{C}[\text{Rep}(\mathcal{A}, n)]^{\text{GL}_n}$ generated by $\text{tr}(a), a \in \mathcal{A}$

Noncommutative Poisson geometry (2)



A **double bracket** is a \mathbb{C} -bilinear map $\{{\cdot}, {\cdot}\} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}^{\otimes 2}$ s.t.

- ① $\{{\cdot}, {\cdot}\} = -\tau_{(12)} \{{\cdot}, {\cdot}\}$ (skewsymmetry)
- ② $\{{\cdot}, bc\} = (b \otimes 1) \{{\cdot}, c\} + \{{\cdot}, b\} (1 \otimes c)$ (outer derivation)
- ③ $\{{\cdot}ad, {\cdot}\} = (1 \otimes a) \{{\cdot}d, {\cdot}\} + \{{\cdot}, {\cdot}\} (d \otimes 1)$ (inner derivation)

A double bracket is **Poisson** if “double Jacobi” [Van den Bergh,'08]

Induced Poisson bracket : $\{a_{ij}, b_{kl}\}_P = \{{\cdot}, {\cdot}\}'_{kj} \{{\cdot}, {\cdot}\}''_{il}$ [Van den Bergh,'08]

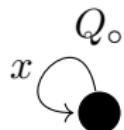
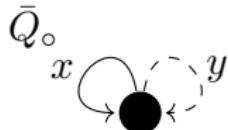
$$\Rightarrow \{\text{tr}(a), \text{tr}(b)\}_P = \text{tr}(\{{\cdot}, {\cdot}\}' \{{\cdot}, {\cdot}\}'')$$

How does it relate to the rest?

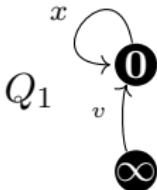
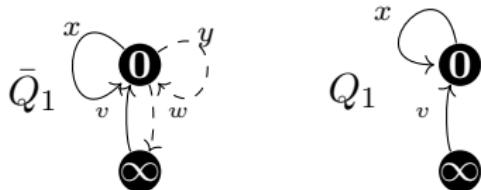
[Van den Bergh, '08] : double Poisson bracket on $\mathcal{A} = \mathbb{C}\bar{Q}$, \forall quiver Q

Example

$$\begin{aligned}\mathcal{A} &= \mathbb{C}\langle x, y \rangle \\ &= \mathbb{C}\bar{Q}_\circ\end{aligned}$$

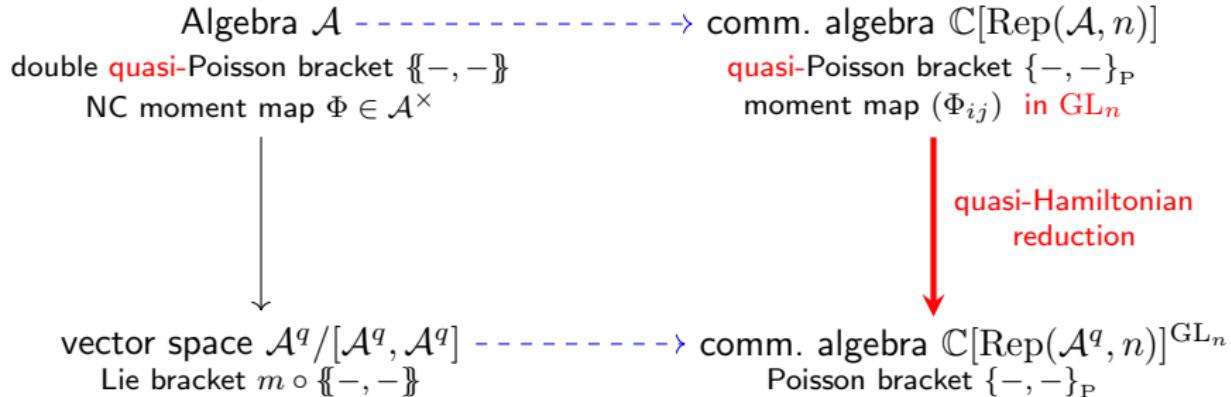


The data $\{x, x\} = 0 = \{y, y\}$, $\{x, y\} = 1 \otimes 1$, $\mu = xy - yx$ are defined on $\mathbb{C}\bar{Q}_\circ$ (from Q_\circ) and induce the Poisson bracket on associated quiver varieties.



Original example from [Wilson, '98]...
Attach \mathbb{C}^n to 0 , \mathbb{C} to ∞
 \Rightarrow Calogero-Moser space
 $(\text{tr}(Y^k) := \text{tr}(y_{ij})^k)_{k=1}^n$ commute

Noncommutative quasi-Poisson geometry



$$\mathcal{A}^q = \mathcal{A}/(\Phi - q) \text{ for } q \in \mathbb{C}^\times$$

[Van den Bergh, '08] : double quasi-Poisson bracket on $\mathcal{A} = \mathbb{C}\bar{Q}_{(1+aa^*)_{a \in \bar{Q}}}$

On rep. spaces : $\{a_{ij}, b_{kl}\}_P = \{\!\{ a, b \}\!}'_{kj} \{\!\{ a, b \}\!}''_{il} \Rightarrow \{\text{tr}(a), \text{tr}(b)\}_P = \text{tr}(\{\!\{ a, b \}\!}' \{\!\{ a, b \}\!}'')$

Plan for the talk

- ① Integrable systems from quiver varieties
- ② Method : Noncommutative Poisson geometry
- ③ **Integrable systems from multiplicative quiver varieties**

How to read a quiver multiplicatively?

Given Q , $\mathbf{n} = (n_s) \in \mathbb{N}^I$, we got $\text{Rep}(\mathbb{C}\bar{Q}, \mathbf{n})$

$$s \xrightarrow{a} t \in \bar{Q} \quad \rightsquigarrow \quad \text{matrix } A \in \text{Mat}(n_s \times n_t, \mathbb{C})$$

Also : natural $\text{GL}(\mathbf{n}) := \prod_{s \in I} \text{GL}(n_s)$ action by conjugation

There is a cumbersome* quasi-Poisson structure [Van den Bergh, '08]
+ a multiplicative moment map $\Phi_{Q,<} := \underbrace{\prod_{a \in Q} (\text{Id} + AA^*)(\text{Id} + A^*A)^{-1}}_{\text{inverted elements + order } <}$

Need quasi-Poisson geometry [Alekseev – Kosmann-Schwarzbach – Meinrenken, '02]

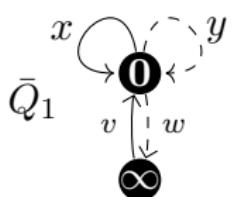
Fix parameters $(q_s) \in (\mathbb{C}^\times)^I$.

Multiplicative quiver variety : $\{\Phi_{Q,<} = \prod_{s \in I} q_s \text{Id}_{\mathbb{C}^{n_s}}\} // \text{GL}(\mathbf{n})$

* Bracket with quadratic terms between matrices A, B of arrows a, b sharing a vertex

Extra constant term for matrices A, A^* of an arrow+double a, a^*

Ruijsenaars-Schneider space (1)



$\bar{Q}_1 \rightsquigarrow$ add inverses for $1 + xy, 1 + yx, 1 + vw, 1 + wv$

e.g. $(1 + xy)^{-1}$ in $\mathbb{C}\bar{Q}_1^{\text{loc}}$ such that

$$(1 + xy)(1 + xy)^{-1} = 1 = (1 + xy)^{-1}(1 + xy)$$

$$\text{Rep}(\mathbb{C}\bar{Q}_1, (1, n)) = \{X, Y \in \mathfrak{gl}_n, V \in \text{Mat}_{1 \times n}, W \in \text{Mat}_{n \times 1}\} = \mathcal{M}$$
$$\cup$$

$$\text{Rep}(\mathbb{C}\bar{Q}_1^{\text{loc}}, (1, n)) = \{\det(\text{Id}_n + XY) \neq 0, 1 + VW \neq 0\} =: \mathcal{M}^\circ$$

Same action of $\text{GL}_n(\mathbb{C})$ on $\mathcal{M}^\circ \subset \mathcal{M}$ which preserves

- a *quasi-Poisson bracket* on \mathcal{M}° ;
- a $\text{GL}_n(\mathbb{C})$ -valued moment map on \mathcal{M}° .

Ruijsenaars-Schneider space (2)

$$\text{Rep}(\mathbb{C}\bar{Q}_1, (1, n)) = \{X, Y \in \mathfrak{gl}_n, V \in \text{Mat}_{1 \times n}, W \in \text{Mat}_{n \times 1}\} = \mathcal{M}$$

$$\text{Rep}(\mathbb{C}\bar{Q}_1^{\text{loc}}, (1, n)) = \{\det(\text{Id}_n + XY) \neq 0, 1 + VW \neq 0\} =: \mathcal{M}^\circ$$

Ruijsenaars-Schneider space : [Chalykh-F.,'17 /1704.05814] (also [Oblomkov,'04])

$$\mathcal{C}_{n,q,1} := \{(\text{Id}_n + XY)(\text{Id}_n + YX)^{-1}(\text{Id}_n + WV)^{-1} = q \text{ Id}_n\} // \text{GL}_n \ (q^k \neq 1)$$

Parametrised on dense subset with X invertible by :

$$X = \text{diag}(x_1, \dots, x_n), \quad V = (1, \dots, 1)$$

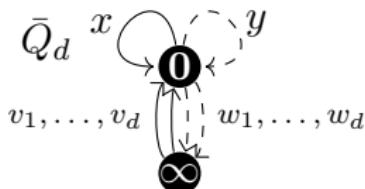
$$\text{from which } (Y + X^{-1})_{ij} = \frac{(1-q)x_j}{x_i - qx_j} \sigma_j \prod_{k \neq j} \frac{x_k - qx_j}{x_k - x_j}$$

Hamiltonian of Ruijsenaars-Schneider ($x_i = e^{q_i}, \sigma_i = e^{p_i}$)

$$\text{tr}(Y + X^{-1}) = \sum_{1 \leq j \leq n} \sigma_j \prod_{k \neq j} \frac{x_k - qx_j}{x_k - x_j}$$

Spin RS space (1)

Case $d \geq 2$ with spin [Chalykh-F.,'20 / 1811.08727]



“add inverses” to $\bar{Q}_d \rightsquigarrow \bar{Q}_d^{\text{loc}}$

$$\mathcal{M}^\circ = \text{Rep}(\bar{Q}_d^{\text{loc}}, (1, n))$$

parametrised by : $X, Y \in \mathfrak{gl}_n$

$$V_\alpha \in \text{Mat}_{1 \times n}, W_\alpha \in \text{Mat}_{n \times 1}$$

$$\mathcal{C}_{n,q,d} := \left\{ (\text{Id}_n + XY)(\text{Id}_n + YX)^{-1} \xrightarrow[1 \leq \alpha \leq d]{} (\text{Id}_n + W_\alpha V_\alpha)^{-1} = q \text{ Id}_n \right\} // \text{GL}_n$$

Ruijsenaars-Schneider space with d spin/degrees of freedom

Spin RS space (2)

$$\mathcal{C}_{n,q,d} := \left\{ (\text{Id}_n + XY)(\text{Id}_n + YX)^{-1} \prod_{1 \leq \alpha \leq d}^{\longrightarrow} (\text{Id}_n + W_\alpha V_\alpha)^{-1} = q \text{ Id}_n \right\} // \text{GL}_n$$

We set $Z := Y + X^{-1} \rightsquigarrow$ (what we needed for $d = 1$)

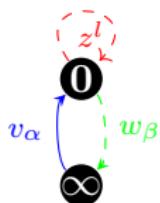
Proposition ([Chalykh-F.,'20])

On open subset of $\mathcal{C}_{n,q,d}$, there exists local coordinates such that

- *the equations of motion associated with $\text{tr}(Z)$ reproduce the eqs of trigonometric RS system with spin from [Krichever-Zabrodin,'95] ;*
- *the matrix Z is the Lax matrix of that system ;*
- *the Poisson bracket written in those coordinates allows to prove a conjecture from [Arutyunov-Frolov,'98].*

Spin RS space (3)

We write $t_{\alpha\beta}^l = V_\alpha Z^l W_\beta$, $Z = Y + X^{-1}$



$$\{\mathrm{tr} Z^k, \mathrm{tr} Z^l\}_P = 0 = \{\mathrm{tr} Z^k, t_{\alpha\beta}^l\}_P$$

$$\{t_{\alpha\beta}^k, t_{\gamma\epsilon}^l\}_P = \text{very complicated !}$$

Proposition ([Chalykh-F.,'20])

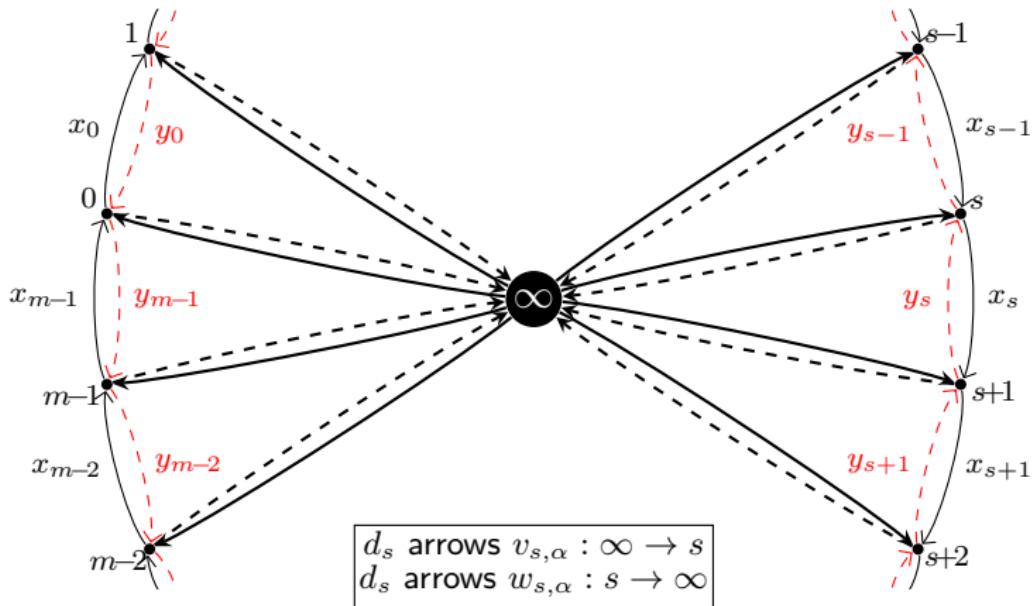
The commutative algebra generated by $(\mathrm{tr} Z^k, t_{\alpha\beta}^k)$, $1 \leq \alpha, \beta \leq d$,
is a Poisson algebra of dimension $2nd - n$,
with centre of dimension n containing the $(\mathrm{tr} Z^k)$.

⇒ degenerate integrability of the $(\mathrm{tr} Z^k)$.

RS spaces from cyclic quivers (1)

General case : [F.,21' /2108.02496] for spin vector $\mathbf{d} \in \mathbb{N}^{\mathbb{Z}_m}$

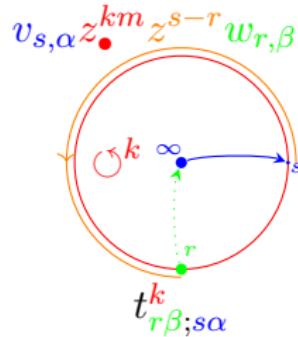
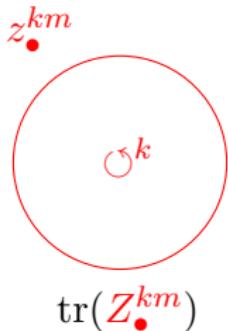
See [F.-Chalykh,'17;F.,'19] for $\mathbf{d} = (1, 0, \dots, 0)$ and $\mathbf{d} = (d, 0, \dots, 0)$



We are interested in $\text{Rep}(\mathbb{C}\bar{Q}_{\mathbf{d}}^{\text{loc}}, (1, n, \dots, n))$ and $z_\bullet = \sum_s (y_s + x_s^{-1})$

RS spaces from cyclic quivers (2)

As in CM case, can “visualise” interesting functions : $(Z_\bullet = \sum_s (Y_s + X_s^{-1}))$



Proposition ([F.,21' /2108.02496])

The commutative algebra generated by $(\text{tr}(Z_\bullet^{km}), t_{r\beta; s\alpha}^k)$,
is a Poisson algebra of dimension $2n|\mathbf{d}| - n$,
with centre of dimension n containing the $\text{tr}(Z_\bullet^{km})$.

⇒ degenerate integrability of the $(\text{tr } Z_\bullet^{km})$ in reduced space $\mathcal{C}_{n,q,d}$

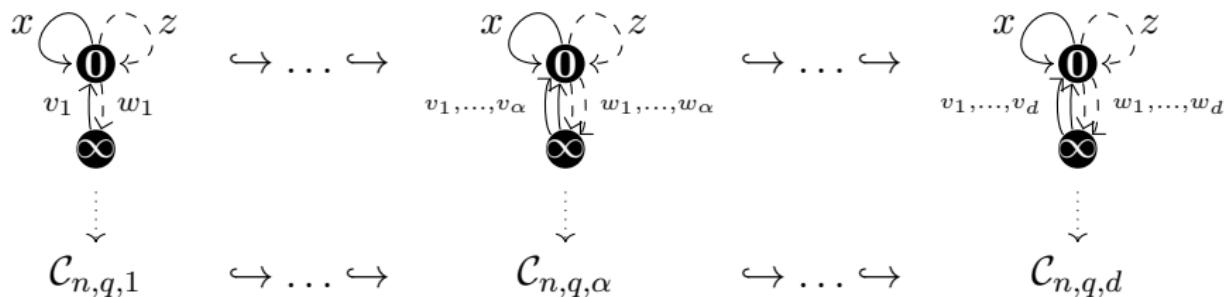
RS spaces from cyclic quivers (3)

In fact, 3 distinct families of degenerately integrable systems for
 $(\text{tr } Z_\bullet^{km} = \text{tr}(Y_\bullet + X_\bullet^{-1})^{km}); \quad (\text{tr } Y_\bullet^{km}); \quad (\text{tr}(\text{Id} + X_\bullet Y_\bullet)^k)$

In local coord., eq. of motion for the 3rd are a *many-spin generalisation* of the eqs of trigonometric RS system with spin from [Krichever-Zabrodin,'95];

Remark (Liouville integrability through embeddings of quivers)

How it works in the simplest case for $(\text{tr } Z^k)$ (1-loop quiver)



$$S_\alpha := (\text{Id}_n + W_\alpha V_\alpha) \dots (\text{Id}_n + W_1 V_1) Z \quad (= q^{-1} X Z X^{-1} \text{ on } \mathcal{C}_{n,q,\alpha})$$

$\Rightarrow \text{tr } S_\alpha^k \in \mathbb{C}[\mathcal{C}_{n,q,d}]$ Poisson commute for all indices

New challenges...

... from the space $\mathcal{C}_{n,q,d}$ for the trigo. spin RS system

- parametrises solutions to $q\text{KP}/2D$ Toda hierarchy (\star)
- geometry for non-generic parameter (q_s) (?)
- quantisation (?)
- irreducible representations of a (cyclotomic) DAHA (?)

... from double brackets :

- phase space of the elliptic CM system (\star)
- phase space of the elliptic RS system (?)
- quasi-version of NC cotangent spaces (?)

(\star) in progress with O. Chalykh

Thank you for your attention !

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<https://www.maths.gla.ac.uk/~mfairon/>