Gromov–Witten theory of complete intersections

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Plan

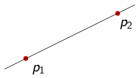
- Gromov–Witten invariants
  - Properties and computational techniques
- Gromov–Witten invariants of complete intersections
- Gromov–Witten classes of complete intersections

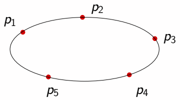
An algorithm computing Gromov–Witten invariants of all smooth complete intersections of hypersurfaces in projective space.

A.–Bousseau–Pandharipande–Zvonkine, arxiv:2109.13323.

In the complex projective space $\mathbb{P}^2$

- How many lines are there passing through two distinct points p_1, p_2?
- How many conics are there passing through five general points $p_1, \ldots, p_5$?
- How many degree d curves of genus zero are there passing through $3d-1$ general points $p_1, \ldots, p_{3d-1}$?





Degree	1	2	3	4	5
Number of curves	1	1	12	640	84000

For all degrees these numbers are computed by a recursive formula due to Kontsevich.

Gromov-Witten Invariants

Correct conditions

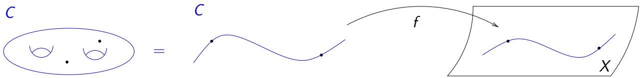
Finitely many of curves

Gromov-Witten Invariant

Definition

- X: smooth projective variety over $\mathbb{C}$
- $g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$
- Insertions: $\alpha_1, \ldots, \alpha_n \in H^*(X, \mathbb{Q})$

Gromov–Witten invariants of X count of **genus** g **curves in** X **of class** β **with** n **marked points** passing through the submanifolds realizing $\mathrm{PD}(\alpha_i)$.



The following definition is due to Kontsevich (1994):

An n-pointed genus g stable map to X of class β is a morphism

$$f : (C, x_1, \ldots, x_n) \longrightarrow X,$$

where

- C: nodal projective curve of arithmetic genus g.
- $x_1, \ldots, x_n$: n (ordered) smooth marked points on C.
- $f_*[C] = \beta \in H_2(X, \mathbb{Z})$.
- (stability) there are finitely automorphisms of $(C, x_1, \ldots, x_n)$ commuting with f.

- $\overline{\mathcal{M}}_{g,n,\beta}(X)$: moduli space of n-pointed genus g stable maps to X of class β.
- $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is a proper Deligne-Mumford stack.
- We have evaluation maps

$$\begin{aligned} ev_i : \overline{\mathcal{M}}_{g,n,\beta}(X) &\longrightarrow X \\ (f : (C, x_1, \ldots, x_n) \to X) &\longmapsto f(x_i) \end{aligned}$$

- There is a natural way to construct a homology class $[\overline{\mathcal{M}}_{g,n,\beta}(X)]^{virt}$, called the virtual fundamental class, which is invariant under deformations of the complex structure.
  - If $\overline{\mathcal{M}}_{g,n,\beta}(X)$ is smooth and of the expected dimension, then $[\overline{\mathcal{M}}_{g,n,\beta}(X)]^{virt}$ agrees with the usual fundamental class.

Gromov–Witten Invariants

- Fix $g, n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$, $\alpha_1, \ldots, \alpha_n \in H^\star(X, \mathbb{Q})$.

Gromov–Witten invariants of X:

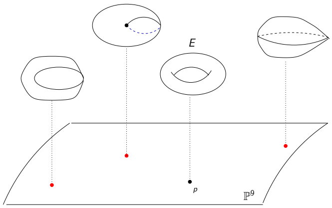
$$\deg \left(\prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in \mathbb{Q} \, .$$

- (virtual) counts of genus g curves in X of class β with n marked points passing through $\mathrm{PD}(\alpha_i)$.

Let E be an elliptic curve in $\mathbb{P}^2_{[X_0:X_1:X_2]}$ given by the equation

$$c_1X_0^3 + c_2X_1^3 + c_3X_2^3 + c_4X_0^2X_1 + c_5X_0X_1^2 + c_6X_1^2X_2 + c_7X_1X_2^2 + c_8X_2^2X_0 + c_9X_2X_0^2 + c_{10}X_0X_1X_2$$

- Changing the coefficients $c_1, \dots c_{10}$ changes the complex structure.
- As long as E remains smooth we want all of the Gromov–Witten invariants to be invariant, but we can not ensure this if we naively count curves.
- Virtual counts of curves are deformation invariant by construction.



Gromov–Witten invariants

Problem

Given a smooth projective variety X, "compute" all Gromov–Witten invariants of X.

Known cases:

- X: point (Kontsevich, Witten's conjecture, 1992)
- X: projective space, or more generally an homogeneous variety (Graber–Pandharipande, 1999)
- X: curve (Okounkov–Pandharipande, 2003)
- X: quintic 3-fold hypersurface in $\mathbb{P}^4$ (Maulik-Pandharipande, 2006)
- X: complete intersections in projective space (A.–Bousseau–Pandharipande–Zvonkine, 2021).

How do we compute Gromov–Witten Invariants?

- How do we compute Gromov–Witten invariants

- So far we know two major techniques to compute Gromov–Witten invariants:
  - Localization (Graber–Pandharipande)
  - Degeneration (Jun Li)

Computing Gromov–Witten invariants via localization

- If X has a torus action, then the Gromov–Witten invariants $GW(X)$ of X can be computed from the Gromov–Witten invariants of the fixed locus.
  - $GW(\mathbb{P}^n)$ can be computed from the Gromov–Witten invariants of a point.
- Each time we use localization we need additional *psi* class insertions.
- L_i: line bundle on $\overline{\mathcal{M}}_{g,n,\beta}(X)$, whose fiber over $(f : (C, x_1 \ldots, x_n) \to X)$ is the cotangent line of C at the i-th marked point,

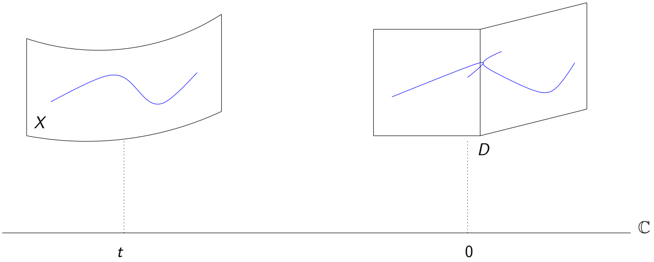
$$\psi_i := c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n,\beta}(X), \mathbb{Q}) \,.$$

- $g, n, k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X, \mathbb{Z})$, $\alpha_1, \ldots, \alpha_n \in H^*(X, \mathbb{Q})$
- Gromov–Witten invariants of X are:

$$\deg \left(\prod_{i=1}^{n} \psi_i^{k_i} \, \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in \mathbb{Q} \,.$$

Computing Gromov–Witten invariants via degeneration

- Jun Li's degeneration formula expresses $GW(X)$ in terms of **relative Gromov–Witten invariants** of the components of the central fiber of a degeneration of X.
  - Relative Gromov–Witten invariants are counts of complex curves with additional tangency conditions with respect to a divisor D.
  - This degeneration formula works under restrictive assumptions on the insertions.



Example: vanishing cycles

- Denote by W the total space of a degeneration of X.
- Jun Li's degeneration formula applies if the cohomology insertions α_i are in the image of the restriction map

$$H^\star(W) \to H^\star(X)$$

- Not surjective in general!
  - Dually, $H_\star(X) \to H_\star(W)$ not injective (there exist vanishing cycles)

Example

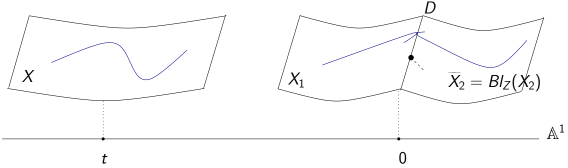
Degeneration of a smooth elliptic curve E to a nodal elliptic curve E_0.

- We want to use the degeneration formula to compute $GW(X)$ when X is a complete intersection, with arbitrary insertions.
  - Will explain how go around the issue with vanishing cycles.

- X: m-dim'l smooth complete intersection of r hypersurfaces in $\mathbb{P}^{m+r}$,

$$f_1 = \cdots = f_r = 0,$$

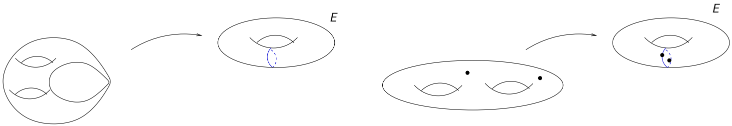
  of degrees $(d_1, \ldots, d_r)$.
- Study Gromov–Witten invariants of X using degeneration.
  - $d_r = d_{r,1} + d_{r,2}$, pick general $f_{r,1}$ and $f_{r,2}$ of degree $d_{r,1}$ and $d_{r,2}$.
  - $f_1 = \cdots = tf_r + f_{r,1}f_{r,2} = 0$: one-parameter family: $W \to \mathbb{A}^1$.



- **GOAL**: $GW(X) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z)$. To do this, we need to
  - Overcome the restrictive assumptions of Jun Li's degeneration formula
  - Express relative invariants $GW(X_1, D), GW(\widetilde{X}_2, D)$ in terms of absolute invariants (Maulik-Pandharipande).

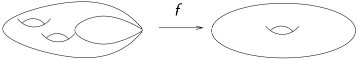
Vanishing cycles / primitive cohomology

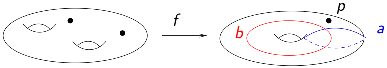
- X: complete intersection of dimension m.
- $H^\star(X) = H^{simple}(X) \oplus H^{prim}(X)$
- $H^{simple}(X) = \langle 1, H, H^2, \ldots, H^m \rangle$
- Lefschetz hyperplane theorem $\implies$ $H^{prim}(X) \subset H^m(X)$
  - H^{prim} contains all vanishing cycles.
- We want to compute Gromov–Witten invariants with also primitive insertions.
  - Key idea: trade primitive insertions against nodes.
  - Compute nodal Gromov–Witten invariants with simple insertions.



All $GW(X)$ from simple and nodal simple $GW(X)$

- X: elliptic curve E. Fix $g = 2$ and $n = 2$.
- There are 4 Gromov–Witten invariants to compute with primitive insertions: $\langle a, a\rangle$, $\langle b, b\rangle$, $\langle a, b\rangle$, $\langle b, a\rangle$ where a, b generate $H^1(E)$.





Insertion of the diagonal class $\Delta \subset E \times E$

Simple nodal $sNGW(X)$ $\longleftrightarrow$ splitting formula $\langle p, 1\rangle + \langle 1, p\rangle + \langle a, b\rangle - \langle b, a\rangle$

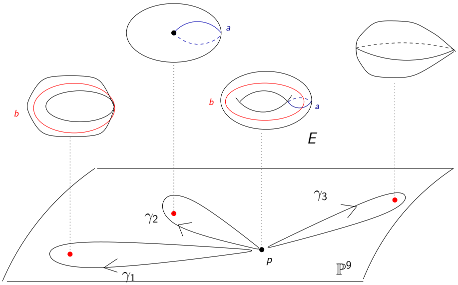
$\langle a, a\rangle = ?$ $\quad \langle b, b\rangle = ?$ $\quad \langle a, b\rangle = ?$ $\quad \langle b, a\rangle = ?$

Deformation invariance $\longleftrightarrow$ $\langle a, a\rangle = 0$, $\langle b, b\rangle = 0$ $\quad \langle a, b\rangle = -\langle b, a\rangle$

- Hence, we obtain $2\langle a, b\rangle = sNGW(X) - \langle p, 1\rangle - \langle 1, p\rangle$

Monodromy invariance of $GW(X)$

- We consider a family of X given by varying the coefficients of $f_i's$.
  - Deformation invariance $\implies$ monodromy invariance of $GW(X)$



- Around γ_1: $\langle a, b\rangle = \langle a+b, b\rangle = \langle a, b\rangle + \langle b, b\rangle \implies \langle b, b\rangle = 0$
- Around γ_2: $\langle a, b+a\rangle = \langle a, b+a\rangle = \langle a, b\rangle + \langle a, a\rangle \implies \langle a, a\rangle = 0$
- Around γ_3: $\langle a, b\rangle = \langle b, -a\rangle =$ - $\langle b, a\rangle$

Monodromy action

- X: complete intersection, $f_1 = \ldots = f_r = 0$
- Monodromy action on $H^\star(X)$:
  - $U = \{\text{coefficients of } f_i\}$,
  - $U_0 = \{X \text{ singular}\} \subset U$ closed subset,
  - $\pi_1(U \setminus U_0, p)$ acts on $H^\star(X)$

Theorem (Deligne)

Let G: Zariski closure of the image of $\pi_1(U \setminus U_0, p)$ in $GL(H^{prim})$. Then, $G = O(k)$ if m even, $G = Sp(k)$ if m odd ($k = \dim H^{prim}$), except if:

- X cubic surface, $G = W(E_6)$ (finite group),
- or X even dimensional complete intersection of two quadrics, $G = W(D_{m+3})$ (also finite group).

Main idea: trading primitive insertions against nodes

Theorem (A.-Bousseau-Pandharipande-Zvonkine, 2021)

Let X be a complete intersection in projective space. Then, the Gromov–Witten invariants of X can be effectively reconstructed from the nodal Gromov–Witten invariants of X with only insertions of simple cohomology classes.

- In the case monodromy is finite: local monodromy theorem for semi-stable degenerations ensures it is identity. In this case, there are no vanishing cycles, and we can apply the classical degeneration formula.
- In the case monodromy is $O(k)$ or $Sp(k)$ the proof uses invariance theory of symplectic and orthogonal groups.

Trading primitive insertions against nodes: proof

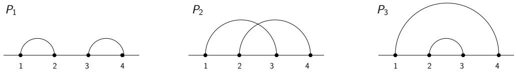
- $V := H(X)^{prim}$
- We will study the Gromov–Witten invariants of X with $2n$ primitive insertions. The data of these invariants is given by a multi-linear form

$$GW_{2n} \colon V^{\otimes 2n} \longrightarrow \mathbb{Q}$$
$$\alpha_1 \otimes \cdots \otimes \alpha_{2n} \longmapsto \deg\Big(\prod_{i=1}^{2n} \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,2n,\beta}(X)]^{\mathrm{virt}}\Big)$$

- Monodromy invariance $\implies$ that this multi-linear form is invariant under the action of $G = O(k)$ or $Sp(k)$ on V.
  - If we would have an odd number of insertions, since $-Id \in G = O, Sp$, this multi-linear form would be zero, so the Gromov–Witten invariants would be zero.
- **Goal**: Describe GW_{2n}.
  - To do this we will study "n-pairings" of $2n$.

Defining multi-linear forms using n-pairings of $2n$ objects

- An n-pairing of $2n$ is given by an arc diagram



- There are $(2n-1)!! = 1 \cdot 3 \cdots (2n-3) \cdot (2n-1)$ pairings of $2n$.
- For each n-pairing P_i, there is a natural multilinear form $\alpha_{P_i} : V^{\otimes 2n} \to \mathbb{Q}$ which is $O(k)$ or $Sp(k)$ invariant.

Example

For the pairing P_1 above

$$\begin{aligned} \alpha_{P_1} : V^{\otimes 4} &\longrightarrow \mathbb{Q} \\ v_1 \otimes \cdots \otimes v_4 &\longmapsto (v_1, v_2)(v_3, v_4) \end{aligned}$$

where $(-,-)$ is the intersection form on V, which is invariant under $O(k)$ or $Sp(2k)$.

Creating nodes using data of n-pairings of $2n$ objects

- Fundamental theorem of invariance theory: the forms α_{P_i} generate the space of invariant multilinear forms.

$$GW_{2n} = \sum_{i=1}^{(2n-1)!!} c_{P_i} \alpha_{P_i}$$

- We need to determine the coefficients c_{P_i}.
- Observation: any n-pairing P_i of $2n$ also defines a way to create n nodes out of $2n$ marked points.
- For each pairing, using the splitting formula, we obtain an equation involving primitive Gromov–Witten invariants.
  - We obtain a system of $(2n-1)!!$ equations with unknowns c_{P_i} (we have as many equations as unknowns, which are indexed by pairings).

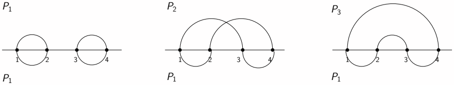
Trading primitive insertions against nodes: proof

- The matrix of the system of equations obtained from the splitting formula is a $(2n-1)!! \times (2n-1)!!$ matrix with ij'th entry

$$M_{ij} = x^{L(P_i,P_j)}$$

  - $L(P_i, P_j)$: loop number of the n-pairings P_i and P_j.
  - $x = \dim V$ when m even, $x = -\dim V$ when m odd.

Example



$$M = \begin{pmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{pmatrix},$$

- We show that M has exactly the correct rank, so we can solve for all c_{P_i}'s. Hence, the result follows.

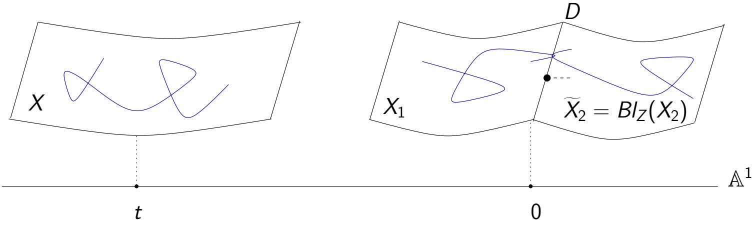
Computing simple nodal Gromov–Witten invariants?

Theorem (A.-Bousseau-Pandharipande-Zvonkine, 2021)

- There is a nodal degeneration formula:

$$sNGW(X) \leftarrow NGW(X_1, D), NGW(\widetilde{X}_2, D)$$

where $NGW(X_1, D), NGW(\widetilde{X}_2, D)$ are "nodal relative invariants". [1]



[1] This requires "carefully" defining **nodal relative Gromov–Witten invariants**!

How to compute simple nodal Gromov–Witten invariants sNG(X)?

Theorem (A.-Bousseau-Pandharipande-Zvonkine, 2021)

- There is a splitting formula for nodal relative invariants

$$NGW(X_1, D), NGW(\widetilde{X}_2, D) \leftarrow GW(X_1, D), GW(\widetilde{X}_2, D)$$

- This requires describing the "virtual fundamental class" for the moduli space of nodal relative stable maps.
  - Uses ideas coming from log geometry (requires working with an Artin stack $\mathbb{A}^1/\mathbb{C}^*$ associated to X, and studying the moduli space of stable maps to $\mathbb{A}^1/\mathbb{C}^*$ which is equi-dimensional and admits a usual fundamental class)!

Step by step

- Goal:

$$GW(X) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z),$$

  where X_1, X_2, D, Z are complete intersections of either smaller degree or smaller dimension.

- Step 1: trade primitive insertions for nodes:

$$GW(X) \leftarrow sNGW(X)$$

- Step 2: apply the nodal degeneration formula to compute simple nodal Gromov–Witten invariants:

$$sNGW(X) \leftarrow NGW(X_1, D), NGW(\widetilde{X}_2, D)$$

- Step 3: apply the splitting formula to reduce nodal relative Gromov–Witten invariants to relative Gromov–Witten invariants

$$NGX(X_1, D), NGW(\widetilde{X}_2, D) \leftarrow GW(X_1, D), GW(\widetilde{X}_2, D)$$

- Step 4: apply previous results of Maulik-Pandharipande

$$GW(X_1, D), GW(\widetilde{X}_2, D) \leftarrow GW(X_1), GW(X_2), GW(D), GW(Z)$$

Let X be an m-dimensional smooth complete intersection in $\mathbb{P}^{m+r}$ of degrees $(d_1, \ldots, d_r)$. Then, for every decomposition

$$d_r = d_{r,1} + d_{r,2} \quad \text{with} \quad d_{r,1}, d_{r,2} \in \mathbb{Z}_{\geq 1}\,,$$

then $GW(X)$ can be effectively reconstructed from:

(i) $GW(X_1)$, where $X_1 \subset \mathbb{P}^{m+r}$ is an m-dimensional smooth complete intersection $X_1 \subset \mathbb{P}^{m+r}$ of degrees $(d_1, \ldots, d_{r-1}, d_{r,1})$.

(ii) $GW(X_2)$, where $X_2 \subset \mathbb{P}^{m+r}$ is an m-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_{r,2})$.

(iii) $GW(D)$, where $D \subset \mathbb{P}^{m+r}$ is an $(m-1)$-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_{r,1}, d_{r,2})$.

(iv) $GW(Z)$, where $Z \subset \mathbb{P}^{m+r}$ is an $(m-2)$-dimensional smooth complete intersection of degrees $(d_1, \ldots, d_{r-1}, d_r, d_{r,1}, d_{r,2})$.

Upgrading to Gromov–Witten classes

- Forgetful morphism $\pi\colon \overline{\mathcal{M}}_{g,n,\beta}(X) \to \overline{\mathcal{M}}_{g,n}$.
- Gromov–Witten classes

$$\pi_* \left(\prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) \cap [\overline{\mathcal{M}}_{g,n,\beta}(X)]^{\mathrm{virt}} \right) \in H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\,.$$

Conjecture

For every smooth projective variety X, the Gromov–Witten classes of X are tautological.

Tautological ring $RH^\star(\overline{\mathcal{M}}_{g,n}) \subset H^\star(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Set of tautological rings is the smallest system of subrings containing 1 and preserved by pullback-pushforward along the natural maps $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$.

[1] Kontsevich–Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, **Communications in Mathematical Physics**, 1994

Gromov–Witten classes of complete intersections

Known cases when Gromov–Witten classes are tautological:

- X a projective space, or more generally an homogeneous variety (Graber-Pandharipande, 1999)
- X a curve (Janda, 2013)

Theorem (A.-Bousseau-Pandharipande-Zvonkine, 2021)

All Gromov–Witten classes of all complete intersections in projective space are tautological.

In progress

- In progress (ABPZ): Gromov–Witten theory of complete intersections in some toric varieties and homogeneous spaces.
- Long term goal (ABPZ): Virasoro conjecture for complete intersections.

Thank you for your attention!