Repeated and Continuous Quantum Interactions, Quantum Noises

Conférence IHES

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1) Repeated Quantum Interactions

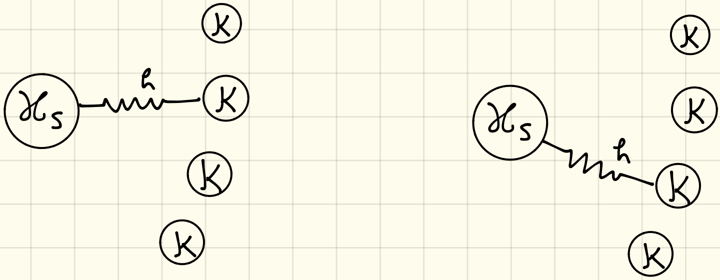
It is a rather simple model of interaction between a small quantum system and a large quantum environment. It is not so naive as it corresponds to true experiments, typically the ones of Haroche's team; it also corresponds to the imaginary experiment described by Froelich some weeks ago. It is not naive, as we will see, for it contains already all the setup of quantum noises, classical noises and Quantum Langevin Equations in discrete time.

The continuous time limits of this model are really non obvious and allow to recover well-known objects, such as:

Fock spaces, quantum noises, quantum Langevin equations, quantum trajectories, Lindblad quantum master equations....

The presentation we give here is the one introduced by S. A. and Yan Pautrat in AHP 2006.

The "small" quantum system $\mathcal{H}_S$ interacts with an environment made of a chain of identical subsystems K, one after the other, for a length time h for each interaction.



The state space is thus $\mathcal{H}_S \otimes \bigotimes_{\mathbb{N}^*} K$.

We put $T\Phi = \bigotimes_{\mathbb{N}^*} K$.

The main ingredient is the Hamiltonian of a single interaction:

on $\mathcal{H}_S \otimes K$

$$H_{tot} = H_S \otimes I + I \otimes H_K + \lambda H_{int}$$

and the associated unitary operator:

$$U = e^{-ih H_{tot}}, \quad \text{on } \mathcal{H}_S \otimes K \text{ also.}$$

We fix an o.n.b. of K: $|0\rangle, |1\rangle \dots$ and put

$$a^i_j = |j\rangle\langle i|.$$

U can always be decomposed as

$$U = \sum_{i,j} U^i_j \otimes a^i_j \text{, for some bounded operators } U^i_j \text{ on } \mathcal{H}_S.$$

Note that

$$tr_K\left[U(\rho \otimes \omega) U^* \right] = L(\rho)$$

for a certain C.P. map L on $\mathcal{H}_S$.

On $T\Phi$ we put $a^i_j(n)$ to be the operator a^i_j but acting on $K_{(n)}$ only. The n-th interaction is described by

$$U_n = \sum_{i,j} U^i_j \otimes a^i_j(n).$$

The result of the n first interactions is given by

$$V_n = U_n \cdots U_1 .$$

So that

$$\begin{aligned} V_{n+1} &= U_{n+1} V_n \\ &= \sum_{i,j} U^i_j \otimes a^i_j(n+1)\, V_n \\ &= \sum_{i,j} U^i_j\, V_n \otimes a^i_j(n+1) \quad \text{for } V_n \text{ and } a^i_j(n+1) \text{ commute.} \end{aligned}$$

$$\boxed{V_{n+1} = \sum_{i,j} U^i_j\, V_n \otimes a^i_j(n+1).}$$

$$\boxed{V_{n+1} - V_n = \sum_{i,j} \left(U^i_j - \delta_{ij} I\right) V_n \otimes a^i_j(n+1)}$$

$$\boxed{V_{(n+1)h} - V_{nh} = \sum_{i,j} \left(U^i_j(h) - \delta_{ij} I\right) V_{nh} \otimes a^i_j((n+1)h)}$$

Note that we have

$$\boxed{\mathrm{Tr}_{T\Phi}\left[V_n \left(\rho \otimes \bigotimes_{\mathbb{N}^*} \omega\right) V_n^*\right] = \mathcal{L}^n(\rho)}$$

2) Discrete time quantum noises

The $a^i_j(n)$ here interprets as (increments of) quantum noises.

- They form a basis of local operations on $T\Phi$
- They describe all the possible actions of the bath
- They contain all the classical noises (i.e. random walks).

Let see that last point more closely here.

Put $Q = a^0_1 + a^1_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q_n = a^0_1(n) + a^1_0(n)$. The Q_n's form a commuting family of s.a. operators, hence multiplication operators of a classical stochastic process. The law of Q_n is $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ under the state $|0\rangle\langle 0|$.

If we put $X_{nh} = \sum_{k=0}^{n} \sqrt{h}\, Q_{kh}$, we have a classical random walk which we know converges to the B.M. (W_t).

An easy computation gives:

$$X_{nh}^2 = 2\sum_{k=0}^{n} X_{kh}\left(X_{(k+1)h} - X_{kh}\right) + nh$$

which is the exact discrete analog of

$$W_t^2 = 2\int_0^t W_s\, dW_s + t.$$

If $U = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ then

$$V_{n+1} = A V_n \otimes I_{n+1} + B V_n \otimes Q_{n+1}$$

$$= A V_n + B V_n X_{n+1}$$

A random walk on $\mathcal{U}(\mathcal{H}_S)$.

Poisson process is also there:

$$N_{nh} = \sum_{k=1}^{n} \left(\sqrt{h}\, a_0^1(kh) + \sqrt{h}\, a_1^0(kh) + a_1^1(kh) + h\, a_0^0(kh) \right)$$

converges to a Poisson Process.

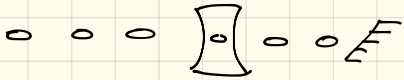
Putting $N = \sqrt{h}\, a_0^1(kh) + \sqrt{h}\, a_1^0(kh) + a_1^1(kh) + h\, a_0^0(kh)$

$$= \begin{pmatrix} h & \sqrt{h} \\ \sqrt{h} & 1 \end{pmatrix}$$

we get $N^2 = (1+h) N$, typical of $(dN_t)^2 = dN_t$ which characterizes the Poisson process.

3) Quantum Trajectories

I won't speak much of quantum trajectories here, but they are not so far. After each interaction, you make a measurement of an observable of K.



$$\rho_0 \otimes \omega \mapsto U_1 (\rho_0 \otimes \omega) U_1^*$$

measurement of A with spectral projections $P_1, \dots, P_n$ gives, with probability $p_k = \mathrm{Tr}(P_k U_1 (\rho_0 \otimes \omega) U_1^* P_k)$, the state

$$\frac{1}{p_k} P_k U_1 (\rho_0 \otimes \omega) U_1^* P_k$$

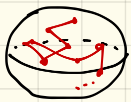
On $\mathcal{H}_S$ we have a new random state:

$$\rho_1(k) = \mathrm{Tr}_{K_1} \left(\frac{1}{p_k} P_k U_1 (\rho_0 \otimes \omega) U_1^* P_k \right)$$

And so on: $\rho_1(k) \otimes \omega \mapsto U_2 (\rho_1(k) \otimes \omega) U_2^*$

+ measurement etc.

At the end we get a classical Markov chain in the set of density matrices of $\mathcal{H}_S$:



It is a simulation of the quantum master equation, for

$$\mathbb{E}[\rho_{n+1} / \rho_n = \rho] = \mathcal{L}(\rho).$$

4) Continuous Time Limit

We have $T\Phi(h) = \bigotimes_{h\mathbb{N}^*} K$, the evolution equation

$$V_{(n+1)h} - V_{nh} = \sum_{i,j} \left(U_j^i(h) - \delta_{ij} \cdot I \right) V_{nh} \; a_j^i((n+1)h).$$

We wish to go to the limit $h \to 0$.

First of all, the space: $\bigotimes_{h\mathbb{N}^*} K \longrightarrow \bigotimes_{\mathbb{R}^+} K$. Which space is that?

Claim: it is $\boxed{\Gamma_s(L^2(\mathbb{R}^+; K - 1d))}$

Why? This a long story (S. A., Sem. de Proba. XXXVI, 2003)

An idea: $K = \mathbb{C}^2$, with the operators $a_0^0, a_1^0, a_0^1, a_1^1$.

A typical element of the 1-particle space (or 1-excitation space) is

$$\varphi = \sum_n f(nh) \, X_{nh}, \quad \text{where } X_{nh} = |1\rangle_{nh}, \text{ with}$$

$$\|\varphi\|^2 = \sum_n |f(nh)|^2$$

If one wants this to have a limit $h \to 0$ it can only be an integral of the form $\int_0^\infty |g(t)|^2 dt$, that is, we write

$$\|\varphi\|^2 = \sum_n \left|\frac{f(nh)}{\sqrt{h}}\right|^2 h = \sum_n |g(nh)|^2 h \longrightarrow \int_0^\infty |g(t)|^2 dt$$

This means: $\varphi = \sum_n \frac{f(nh)}{\sqrt{h}} \sqrt{h}\, X_{nh} = \sum_n g(nh)\, dX_{nh}$

with $\|dX_{nh}\|^2 = h$.

If we want an action like below to have a limit:

$$\left(\sum_n \varphi(nh)\, a_0'(nh)\right)\left(\sum_m g(mh)\, dX_{mh}\right)$$

$$= \sum_{n,m} \varphi(nh)\, g(mh)\, a_0'(nh)\, dX_{mh}$$

$$= \sum_n \varphi(nh)\, g(nh)\, \sqrt{h}\, |0\rangle$$

$$= \sum_n \frac{\varphi(nh)}{\sqrt{h}}\, g(nh)\, h \longrightarrow \int_0^\infty \psi(t)\, g(t)\, dt$$

This means

$$\sum_n \frac{\varphi(nh)}{\sqrt{h}} \sqrt{h}\, a_0'(nh) = \sum_n \psi(nh)\, da(nh)$$

is the correct form.

At the end we get

	$\lvert 0\rangle$	dX
$da = \sqrt{h}\, a_0^1$	0	$h \lvert 0\rangle$
$da^* = \sqrt{h}\, a_1^0$	dX	0
$d\Lambda = a_1^1$	0	dX
$dJ = h a_0^0$	$h\lvert 0\rangle$	0

It is easy to see that we are constructing the Fock space $\Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}))$, with its usual creation and annihilation operators, $d\Lambda$ is a differential second quantization operator, dJ is just $dt\, I$.

In the reference above, I constructed the spaces $T\Phi(h)$ as concrete subspaces of $\Phi = \Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}))$, with $P_h : \Phi \to T\Phi_h$, such that $P_h \xrightarrow[h\to 0]{} I$ and

$$P_h \, \frac{a((n+1)h) - a(nh)}{\sqrt{h}} \, P_h = a_1^0(nh) \quad \text{etc.}$$

We end up with 4 quantum noises on Φ: $da_0^0(t)$, $da_1^0(t)$, $da_0^1(t)$, $da_1^1(t)$.

$da_0^0(t)$ just happens to be $dt\, I$, $da_1^0(t)$ and $da_0^1(t)$ are the usual creation, annihilation, $da_1^1(t)$ is a certain diff. second. quant.

So in fact there are only 3 true quantum noises.

Another way to see them naturally is the following.

On $\Phi = \Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}))$ we have a continuous tensor product structure: $\Phi \simeq \Phi_{[0,t_1]} \otimes \Phi_{[t_1,t_2]} \otimes \cdots \otimes \Phi_{[t_n,+\infty[}$.

We are looking for operators A_t on Φ having the property to be "local", that is, $A_{t+h} - A_t = I_{[0,t]} \otimes (A_{t+h} - A_t) \otimes I_{[t+h,+\infty[}$.

A difficult theorem, proved by A. Coquio, using the informal ideas above, prove that there are only 4 such families, the 4 noises introduced above.

Coming back to repeated interactions, we had

$$V_{(n+1)h} - V_{nh} = \sum_{ij} \left(U_j^i(h) - \delta_{ij} I\right) V_{nh}\, a_j^i((n+1)h)$$

$$= \sum_{ij} \frac{U_j^i(h) - \delta_{ij} I}{h^{\varepsilon_{ij}}} V_{nh}\, h^{\varepsilon_{ij}}\, a_j^i((n+1)h)$$

with $\varepsilon_0^0 = 1$, $\varepsilon_1^0 = \varepsilon_0^1 = 1/2$, $\varepsilon_1^1 = 0$.

So if $\lim_{h \to 0} \dfrac{U_j^i(h) - \delta_{ij} I}{h^{\varepsilon_j^i}} = L_j^i$ $\qquad (*)$

we get in the limit

$$dV_t = \sum_{ij} L_j^i V_t \, da_j^i(t)$$

$$dV_t = L_0^0 V_t \, dt + \sum_{i,j \neq (0,0)} L_j^i V_t \, da_j^i(t)$$

a perturbation of the Schrödinger equation with quantum noise terms.

If $L_1^0 = L_0^1 = L$ and $L_1^1 = 0$ we get

$$dV_t = L_0^0 V_t \, dt + L V_t \, dW_t$$

a Schrödinger equation with Brownian term.

We could get the same with Poisson process terms.

We also have

$$\mathrm{Tr}_{\Phi}\left[V_t (\rho \otimes \Omega) V_t^* \right] = \mathcal{P}_t(\rho)$$

where $\mathcal{P}_t = e^{t\mathcal{L}}$ is a Lindblad semigroup. This semigroup is the continuous time limit of the discrete time one: $\mathcal{L}^{nh}$.

A typical Hamiltonian which gives rise to condition $(*)$ is:

$$H_{tot} = H_S \otimes I + I \otimes H_{\mathcal{H}} + \frac{1}{\sqrt{h}} \sum_i W_i \otimes a_i^0 + W_i^* \otimes a_0^i$$
$$+ \frac{1}{h} \sum_{ij} D_{ij} \otimes a_j^i \, .$$