Planar Carrollean dynamics

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and w.i.p with P. Horvathy and PM. Zhang

The curious physics of the plane

There are non trivial effective phenomena in the plane, for instance the recent discovery of anyons (see H. Bartolomei, et. al., Science **368** (2020)).

Anyons are quasi particles living in a plane hypothesized in the 1980s.

They feature an infinite amount of spin statistics: their spin can be any real number, not only half integers.

These results/studies are mostly for condensed matter.
Question: are there similar effective phenomena for relativistic mechanics in the plane?

The Carroll group

The Carroll group, cousin of the Galilei group, was originally introduced as a new contraction ($c \to 0$) of the Poincaré group [Lévy-Leblond, '65].

Both the Galilei and Carroll groups act on spacetime:

- For the Galilei group,

$$\begin{pmatrix} \boldsymbol{x} \\ t \end{pmatrix} \mapsto \begin{pmatrix} R\boldsymbol{x} + \boldsymbol{b}t + \boldsymbol{c} \\ t + e \end{pmatrix}$$

- For the Carroll group,

$$\begin{pmatrix} \boldsymbol{x} \\ s \end{pmatrix} \mapsto \begin{pmatrix} R\boldsymbol{x} + \boldsymbol{c} \\ s - \langle \boldsymbol{b}, R\boldsymbol{x} \rangle + f \end{pmatrix}$$

$\Rightarrow$ They differ by the way boosts act.

Contractions of the Poincaré algebra

Let us compare the Poincaré, Galilei, and Carroll algebras,

Poincaré	Galilei	Carroll
$[J_i, J_j] = \epsilon_{ijk} J_k,$	$[J_i, J_j] = \epsilon_{ijk} J_k,$	$[J_i, J_j] = \epsilon_{ijk} J_k,$
$[J_i, K_j] = \epsilon_{ijk} K_k,$	$[J_i, K_j] = \epsilon_{ijk} K_k,$	$[J_i, K_j] = \epsilon_{ijk} K_k,$
$[K_i, K_j] = -\epsilon_{ijk} J_k,$	$[K_i, K_j] = 0,$	$[K_i, K_j] = 0,$
$[J_i, P_j] = \epsilon_{ijk} P_k,$	$[J_i, P_j] = \epsilon_{ijk} P_k,$	$[J_i, P_j] = \epsilon_{ijk} P_k,$
$[K_i, P_j] = \delta_{ij} P_0,$	$[K_i, P_j] = 0,$	$[K_i, P_j] = \delta_{ij} P_0,$
$[J_i, P_0] = 0,$	$[J_i, P_0] = 0,$	$[J_i, P_0] = 0,$
$[K_i, P_0] = P_i,$	$[K_i, P_0] = P_i,$	$[K_i, P_0] = 0,$
$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$
$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$

Physical applications for the Carroll group

The Carroll group was originally dismissed as non physical.

Physical applications for the Carroll group were found much later, in General Relativity, typically at boundaries. See related topics,

- BMS symmetry
- Flat holography
- Black hole horizons
- Gravitational waves
- ...

Carroll structures

A Carroll structure [Henneaux, '79 ; Duval et. al., '91] is a triple (M, g, ξ) with $\dim M = d + 1$, a degenerate "metric" g with $\dim \ker g = 1$, a nowhere vanishing vector field $\xi \in \ker g$, (and $L_\xi g = 0$). Locally, coordinates (x^i, s) with $g = g_{ij} dx^i \otimes dx^j$, and $\xi = \partial_s$.

It is possible to add a (non unique connection) to obtain a strong Carroll structure (M, g, ξ, ∇), with $\nabla g = \nabla \xi = 0$.

Isometries of a flat strong Carroll structure: the Carroll group.

Carroll structures are often present in General Relativity:

- Conformal null infinity
- Black holes horizon [L. Donnay, C. Marteau, '19]
- Any null hypersurface in a Lorentzian spacetime [L. Ciambelli, et. al. '19]

Note that all these examples are $2 + 1$ dimensional.

What we know so far about Carrollean dynamics

Two main references on Carrollean dynamics:

- C. Duval, G. Gibbons, P. Horváthy, PM. Zhang, Class. Quant. Grav 31 (2014)
- E. Bergshoeff, J. Gomis, G. Longhi, Class. Quant. Grav 31 (2014)

$\Rightarrow$ A Carrollean elementary particle does not move

These results apply to general elementary Carroll particles...
but only when the spatial dimension d is 3 or higher.

$\Rightarrow$ There might be non trivial motions on a $2+1$ plane.

The subtlety of Carroll planar dynamics: central extensions

The dynamics of an elementary particle can be obtained by studying the associated group *or* a central extension of this group.

Example: the Galilei group only describes massless particles. The description of massive systems requires its non-trivial central extension.

	Dimension of non-trivial central extensions	
spatial dimension	Galilei	Carroll
$d \geq 3$	1	0
$d = 2$	2	2

$\Rightarrow$ The Carroll group potentially has a richer structure in $d = 2$.

Contractions of the Poincaré algebra, $2+1$ dimensions

The extended Galilei algebra can be found in [Lévy-Leblond, '72], and the extended Carroll algebra was computed in [de Azcarraga, et. al. '98; Ngendakumana, et. al. '14].

Poincaré	Extended Galilei	Extended Carroll
$[J_3, K_i] = \epsilon_{ij} K_j,$	$[J_3, K_i] = \epsilon_{ij} K_j,$	$[J_3, K_i] = \epsilon_{ij} K_j,$
$[K_i, K_j] = -\epsilon_{ij} J_3,$	$[K_i, K_j] = \epsilon_{ij} A_{exo},$	$[K_i, K_j] = \epsilon_{ij} A_{exo},$
$[J_3, P_i] = \epsilon_{ij} P_j,$	$[J_3, P_i] = \epsilon_{ij} P_j,$	$[J_3, P_i] = \epsilon_{ij} P_j,$
$[K_i, P_j] = \delta_{ij} P_0,$	$[K_i, P_j] = \delta_{ij} M,$	$[K_i, P_j] = \delta_{ij} P_0,$
$[J_3, P_0] = 0,$	$[J_3, P_0] = 0,$	$[J_3, P_0] = 0,$
$[K_i, P_0] = P_i,$	$[K_i, P_0] = P_i,$	$[K_i, P_0] = 0,$
$[P_i, P_j] = 0,$	$[P_i, P_j] = 0,$	$[P_i, P_j] = \epsilon_{ij} A_{mag},$
$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$	$[P_i, P_0] = 0.$

Carroll's double central extension in 2+1 dimensions

Elements of the extended Carroll group can be represented as [L. M. '21],

$$\begin{pmatrix} R & 0 & \boldsymbol{c} & 0 & \epsilon\boldsymbol{b} \\ -\overline{\boldsymbol{b}}R & 1 & f & 0 & a_{exo} \\ 0 & 0 & 1 & 0 & 0 \\ -\overline{\epsilon\boldsymbol{c}}R & 0 & a_{mag} & 1 & -f-\langle\boldsymbol{b},\boldsymbol{c}\rangle \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \widetilde{\mathrm{Carr}}(3)$$

$R \in \mathrm{O}(2)$

$\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2$

$f, a_{mag}, a_{exo} \in \mathbb{R}$

$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

This is the group one should consider when working out Carrollean dynamics in the plane.

The physical quantities dual to the elements of the algebra:

$$J = (\ell, \boldsymbol{g}, \boldsymbol{p}, m, \kappa_{mag}, \kappa_{exo}) \in \widetilde{\mathfrak{carr}}(3)^*$$

Coadjoint action of the group: $\mathrm{Coad}(a)J = (\ell', \boldsymbol{g}', \boldsymbol{p}', m, \kappa_{mag}, \kappa_{exo})$, with,

$$\begin{aligned}
\ell' &= \ell + \boldsymbol{b} \times A\boldsymbol{g} - \boldsymbol{c} \times A\boldsymbol{p} + m\boldsymbol{b} \times \boldsymbol{c} - \frac{1}{2}\kappa_{mag}\boldsymbol{c}^2 - \frac{1}{2}\kappa_{exo}\boldsymbol{b}^2 \\
\boldsymbol{g}' &= A\boldsymbol{g} + m\boldsymbol{c} + \kappa_{exo}\epsilon\boldsymbol{b} \\
\boldsymbol{p}' &= A\boldsymbol{p} + m\boldsymbol{b} - \kappa_{mag}\epsilon\boldsymbol{c}
\end{aligned}$$

$\Rightarrow$ 4 Casimir invariants ($m \neq 0$):

$$\begin{aligned}
C_1 &= m \qquad \text{(convention: mass, not energy)} \\
C_2 &= \left(1 - \frac{\kappa_{mag}\kappa_{exo}}{m^2}\right)\ell + \frac{\boldsymbol{g} \times \boldsymbol{p}}{m} - \frac{\kappa_{mag}}{2m^2}\boldsymbol{g}^2 - \frac{\kappa_{exo}}{2m^2}\boldsymbol{p}^2 \\
C_3 &= \kappa_{mag} \quad \Rightarrow \quad [\kappa_{mag}] = MT^{-1} \\
C_4 &= \kappa_{exo} \quad \Rightarrow \quad [\kappa_{exo}] = MT
\end{aligned}$$

Dynamics out of a symplectic model

If a dynamical system is G-invariant, one can (locally) build its phase space as a coadjoint orbit of G (or of a central extension of G).

1. Build the "evolution space" V out of parameters of the group (*e.g.* spatial translations $\sim$ position, boosts $\sim$ momentum, etc)
2. Choose the Casimir invariants that describe the elementary particle, and pick a $J_0 \in \mathfrak{g}^*$ with such invariants
3. Endow V of a presymplectic 2-form σ out of the Maurer-Cartan form Θ on G:
$$\sigma = d\left(J_0 \cdot \Theta\right)$$
4. The equations of motion are then spanned by the kernel of σ

A free massive Carroll particle in 3+1 dimensions

Consider a massive spinless elementary particle: $J_0 = (0, 0, 0, m)$.

The evolution space

$$V = \mathrm{Carr}(3+1)/\mathrm{SO}(3) \ni y = (\boldsymbol{x}, \boldsymbol{v}, s)$$

is endowed with the left-invariant 1-form $\varpi := J_0 \cdot \Theta$,

$$\varpi = m\langle \boldsymbol{v}, d\boldsymbol{x}\rangle + mds,$$

whose exterior derivative defines the presymplectic 2-form on V,

$$\sigma = md\overline{\boldsymbol{v}} \wedge d\boldsymbol{x} \quad \Rightarrow \quad H = 0$$

The equations of motions are then spanned by the kernel of σ,

$$\frac{d\boldsymbol{x}}{ds} = 0, \qquad \text{(Carrollean velocity)}$$

$$m\frac{d\boldsymbol{v}}{ds} = 0.$$

$\Rightarrow$ We recover the well known property that Carroll particles do not move.

Free, planar, massive, and spinless Carroll particles

The evolution space $V = \widetilde{\mathrm{Carr}}(3)/\mathrm{SO}(2) \ni y = (\boldsymbol{x}, \boldsymbol{v}, s, w, z)$ is endowed with the 2-form,

$$\sigma = m d\overline{\boldsymbol{v}} \wedge d\boldsymbol{x} + \frac{1}{2}\kappa_{mag}\epsilon_{ij} dx^i \wedge dx^j + \frac{1}{2}\kappa_{exo}\epsilon_{ij} dv^i \wedge dv^j$$

The equations of motions depend on an *effective mass*

$$\widetilde{m}^2 = m^2 - \kappa_{mag}\kappa_{exo}$$

$$\boxed{\widetilde{m}^2 \neq 0}$$

Free planar particles do not move:

$$\frac{d\boldsymbol{x}}{ds} = 0$$
$$m\frac{d\boldsymbol{v}}{ds} = 0$$

$$\boxed{\widetilde{m}^2 = 0}$$

The equations degenerate:

$$\frac{d\boldsymbol{x}}{ds} = -\frac{\kappa_{exo}}{m}\epsilon\frac{d\boldsymbol{v}}{ds}$$

$\Rightarrow$ not localized?

3+1 Carroll particles with spin in an EM field

For a massive particle with spin and electric charge e, the equations of motion are,

$$\frac{d\boldsymbol{x}}{ds} = 0,$$
$$m\frac{d\boldsymbol{v}}{ds} = e\boldsymbol{E} + \mu\boldsymbol{\nabla}_{\boldsymbol{x}}\langle\boldsymbol{u}, \boldsymbol{B}\rangle,$$
$$\frac{d\boldsymbol{u}}{ds} = \mu\,\boldsymbol{u} \times \boldsymbol{B},$$

where $\boldsymbol{u} \in S^2$ represents the direction of the particle's spin.

$\Rightarrow$ no actual motion, but precession of the spin around the magnetic field

Planar Carroll particles in an EM field – anyons

A particle described by the Casimirs: $m \neq 0$, $\ell \neq 0$, $\kappa_{mag} \neq 0$, $\kappa_{exo} \neq 0$.

There is again an effective mass,

$$\widetilde{m}^2 := m^2 - (\kappa_{mag} + eB)\,\kappa_{exo},$$

There equations of motion are, for $\widetilde{m}^2 \neq 0$,

$$\begin{aligned} \frac{d\boldsymbol{x}}{ds} &= -\frac{\kappa_{exo}}{\widetilde{m}^2}\epsilon\left(e\boldsymbol{E} + \mu\ell\boldsymbol{\nabla}_{\boldsymbol{x}}B\right), \\ m\frac{d\boldsymbol{v}}{ds} &= \frac{m^2}{\widetilde{m}^2}\left(e\boldsymbol{E} + \mu\ell\boldsymbol{\nabla}_{\boldsymbol{x}}B\right). \end{aligned}$$

$\Rightarrow$ We see actual motion!

Planar massless Carroll particles in an EM field

For a massless ($m = 0$), chargeless particle with anyonic spin and non vanishing magnetic moment $\mu \neq 0$,

$$
\begin{aligned}
\frac{d\boldsymbol{x}}{ds} &= \frac{\mu\ell}{\kappa_{mag}} \epsilon \, \boldsymbol{\nabla}_{\boldsymbol{x}} B, \\
\kappa_{exo} \frac{d\boldsymbol{v}}{ds} &= 0.
\end{aligned}
$$

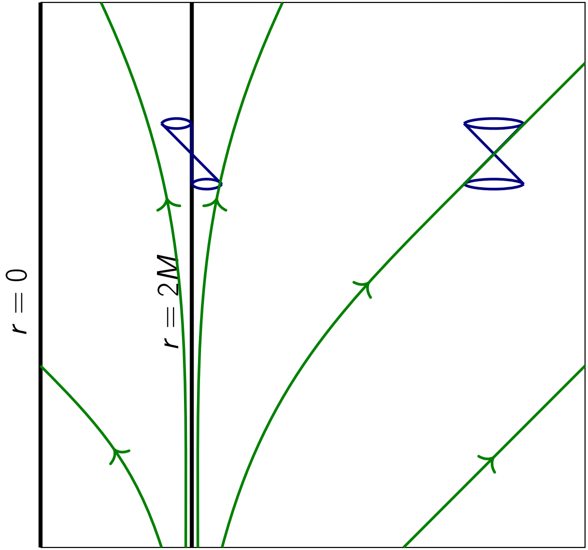
$\Rightarrow$ We have a velocity transverse to the gradient of the magnetic field.

Mathematical motion

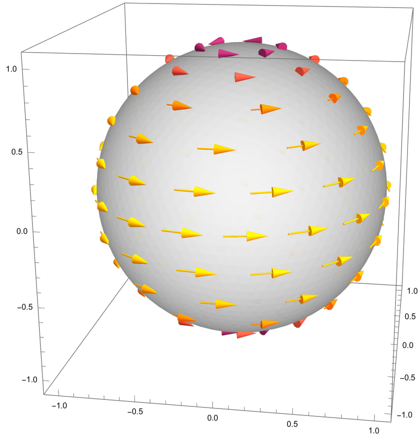
- Mathematically, Carrollian dynamics allow for non trivial motion
  $\Rightarrow$ are these motions actually realized in Nature?
- To understand what could these motion look like, we will apply these equations to an hypothetical photon with non vanishing magnetic moment μ on the horizon of a black hole.

Trapping a photon on an horizon

Eddington-Finkelstein diagram of a Schwarzschild black hole:



Velocity drift on the horizon of a Kerr-Newmann BH



Electromagnetic potential in a KN spacetime:

$$A = Qr\frac{a\sin^2\theta d\varphi - dt}{r^2 + a^2\cos^2\theta}.$$

The dr component of the magnetic field is induced on $r = \text{const}$ hypersurfaces:

$$B_r = \frac{2aQr\left(r^2 + a^2\right)\cos\theta}{\left(r^2 + a^2\cos^2\theta\right)^3}.$$

$$\dot{x}^\theta = 0\,, \quad \dot{x}^\varphi = 2aQr\frac{\mu\ell}{\kappa_{mag}}\frac{(r^2 + a^2)(r^2 - 5a^2\cos^2\theta)}{(r^2 + a^2\cos^2\theta)^4}\sin\theta.$$

Coupling to a gravitational field – The idea

Coupling to the gravitational field is achieved by altering the geometry and considering a curved evolution space V.

Presymplectic spaces considered here can be seen as a group quotient $V = G/H$ equipped with a 1-form that is derived from the Maurer-Cartan form, which can be seen as the flat connection of the Klein geometry (G, H).

The coupling to curved space is done by considering V to be a Cartan geometry based on (G, H), but equipped with a (non flat) Cartan connection ω that will replace the Maurer-Cartan form in the definition of the presymplectic potential.

$$\varpi := J_0 \cdot \omega$$

Coupling to a gravitational field – Equations of motion

We find the following equations of motion,

$$\frac{dx^\mu}{d\tau} = \lambda \xi^\mu,$$
$$m\frac{Dv_\mu}{d\tau} = 0,$$

where $\mu = 0, 1, 2$.

Using coordinates such that $\xi = \partial_s$ and a suitable λ,

$$\frac{d\boldsymbol{x}}{ds} = 0,$$
$$m\frac{D\boldsymbol{v}}{ds} = 0.$$

$\Rightarrow$ We find the same trivial motions as in the flat case.

Conclusion

- Carrollean dynamics are important in General Relativity
- In $2 + 1$ dimensions, one needs to consider the double central extension of the Carroll group to be the most general
- Two new Casimirs to describe elementary particles: κ_{mag} and κ_{exo}
  $\Rightarrow$ Physical interpretation?
- These two exotic charges couple to the EM field to bring actual motion in Carrollean dynamics
- No exotic coupling to the gravitational field
- Mathematically, Carrollian dynamics allow for non trivial motion
  $\Rightarrow$ are these motions actually realized in Nature?

Backup slides

Coupling to a gravitational field – Cartan geometry

Define the Cartan geometry based on $(\widetilde{\mathrm{Carr}}(3), \mathrm{SO}(2))$. A Cartan connection is parametrized as, with spatial indices $A, B = 1, 2$,

$$\widetilde{\omega} = \begin{pmatrix} \widetilde{\omega}^A{}_B & 0 & \theta^A & 0 & \epsilon^A{}_C \delta^{CD} \widetilde{\omega}^0{}_D \\ -\widetilde{\omega}^0{}_B & 0 & \theta^0 & 0 & \widetilde{\omega}_2 \\ 0 & 0 & 0 & 0 & 0 \\ \theta^C \delta_{CD} \epsilon^D{}_B & 0 & \widetilde{\omega}_1 & 0 & -\theta^0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with structure equations,

$$\Omega^A{}_B = d\widetilde{\omega}^A{}_B + \widetilde{\omega}^A{}_C \wedge \widetilde{\omega}^C{}_B,$$

$$\Omega^0{}_B = d\widetilde{\omega}^0{}_B + \widetilde{\omega}^0{}_C \wedge \widetilde{\omega}^C{}_B,$$

$$\Omega^A = d\theta^A + \widetilde{\omega}^A{}_C \wedge \theta^C,$$

$$\Omega^0 = d\theta^0 - \widetilde{\omega}^0{}_C \wedge \theta^C,$$

$$\Omega_1 = d\widetilde{\omega}_1 + \theta^C \wedge \theta^D \epsilon_{CD},$$

$$\Omega_2 = d\widetilde{\omega}_2 - \widetilde{\omega}^0{}_C \wedge \widetilde{\omega}^0_D \epsilon^{CD}.$$

Coupling to a gravitational field – Presymplectic potential

We defined the 1-form potential as $\varpi := J_0 \cdot \widetilde{\omega}$. For a massive spinless particles with charges q_1 and q_2,

$$\varpi = m\theta^0 + q_1\widetilde{\omega}_1 + q_2\widetilde{\omega}_2.$$

Upon using the structure equations of the connection we find for $\sigma = d\varpi$, with vanishing torsion and exotic curvatures,

$$\sigma = m\widetilde{\omega}^0{}_C \wedge \theta^C - q_1\theta^C \wedge \theta^D\epsilon_{CD} + q_2\widetilde{\omega}^0{}_C \wedge \widetilde{\omega}^0{}_D\epsilon^{CD}.$$

Both of these forms are defined on $\widetilde{V}$, but σ has the added property of projecting down to $H(M)$, the Carroll frame bundle over the spacetime M.

Coupling to a gravitational field – The Carroll frame bundle

Consider the Carroll frame bundle $H(M)$ above spacetime M with local coordinates $(x^\mu, e^\mu{}_a)$. The tetrad is linked to the Carroll metric,

$$\delta_{AB} = g_{\mu\nu} e^\mu{}_A e^\nu{}_B,$$
$$\xi^\mu = e^\mu{}_0,$$

The soldering form θ and the linear connection $\widetilde{\omega}^a{}_b$ are given by,

$$\theta^a = \theta^a{}_\mu dx^\mu,$$
$$\widetilde{\omega}^a{}_b = \theta^a{}_\mu \left(de^\mu{}_b + \Gamma^\mu_{\nu\lambda} e^\nu{}_b dx^\lambda \right),$$

where we have $\theta^a{}_\mu e^\mu{}_b = \delta^a_b$ and $e^\mu{}_c \theta^c{}_\nu = \delta^\mu_\nu$.

Coupling to a gravitational field – Equations of motion

We need to compute the kernel of σ. Define a vector field $X \in T_{(x,e)}(H(M))$,

$$X = \frac{dx^\mu}{d\tau}\partial_\mu + \frac{de^\mu{}_a}{d\tau}\partial_{e^\mu{}_a},$$

and $\dot{x}^\mu := \frac{dx^\mu}{d\tau}$ and $\dot{e}^\mu{}_a := \frac{de^\mu{}_a}{d\tau} + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu e^\lambda{}_a$. We find that,

$$\sigma(X) = \left(m\theta^0{}_\mu\dot{e}^\mu{}_C - 2q_1\theta^B{}_\mu\dot{x}^\mu\epsilon_{BC}\right)\theta^C - \left(m\theta^D{}_\mu\dot{x}^\mu - 2q_2\theta^0{}_\mu\dot{e}^\mu{}_C\epsilon^{CD}\right)\widetilde{\omega}^0{}_D$$

vanishes if,

$$\dot{x}^\mu = \lambda\xi^\mu,$$
$$m\dot{v}_\mu = 0,$$

where $\theta^0{}_\mu := v_\mu$ and (recall) $\xi^\mu := e^\mu{}_0$.

$\Rightarrow$ We find the same trivial motions as in the flat case.

Link between Galilei and Carroll groups

The Carroll group is a subgroup of the Bargmann group, which is the central extension of the Galilei group:

$$\underset{\text{Bargmann}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & e \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}} \longrightarrow \underset{\text{Galilei}}{\begin{pmatrix} R & \boldsymbol{b} & \boldsymbol{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\uparrow$$

$$\underset{\text{Carroll}}{\begin{pmatrix} R & \boldsymbol{b} & 0 & \boldsymbol{c} \\ 0 & 1 & 0 & 0 \\ -\overline{\boldsymbol{b}}R & -\|\boldsymbol{b}\|^2/2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}} \cong \underset{\text{Carroll}}{\begin{pmatrix} R & 0 & \boldsymbol{c} \\ -\overline{\boldsymbol{b}}R & 1 & f \\ 0 & 0 & 1 \end{pmatrix}}$$

Explicit example: the horizon of Kerr-Newman black holes

The Kerr-Newman metric,

$$g = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\varphi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\, dt - (r^2+a^2)d\varphi\right)^2 + \Sigma d\theta^2 + \frac{\Sigma}{\Delta}dr^2$$

The induced metric on this horizon, at $\Delta = 0$ with $r = \text{const}$, is,

$$\widetilde{g} = \frac{\sin^2\theta}{\Sigma}\left(a\, dt - (r^2+a^2)d\varphi\right)^2 + \Sigma d\theta^2$$

The vector field ξ such that $g(\xi) = 0$ and $L_\xi g = 0$ is,

$$\xi = \partial_t + \frac{a}{r^2+a^2}\partial_\varphi$$

Change of coordinates $(\theta, \varphi, t) \mapsto (\theta, \widetilde{\varphi} = \varphi - \frac{a}{r^2+a^2}s, s = t)$,

$$\widetilde{g} = \frac{(r^2+a^2)\sin^2\theta}{\Sigma}d\widetilde{\varphi}^2 + \Sigma d\theta^2 \quad \& \quad \xi = \partial_s$$