Explicit computations for Spin Chain models and the remarkable operator Co-derivative

Helal H.M ALDarak   LEURENT Sébastien

Department of Mathematics
Université de Bourgogne-Franche Comté

July 2, 2022

Outline

1. Reminder and what to come
2. Co-derivative in the GL context
   - Defination and some properties.
   - Advantage of using Co-derivative in defining the T,Q operators
3. Co-derivative in the SO context
   - Defination and different representations
   - Conserved charges in different representations
4. For now and future research

The Q-function and it's power

- Bethe equations.

The Q-function and it's power

- Bethe equations:
  - The Wronskian expression for the Q-functions GL(3):

$$u^N \propto \begin{vmatrix} Q_1(u) & Q_2(u) & Q_3(u) \\ x_1 Q_1^{[-]}(u) & x_2 Q_2^{[-]}(u) & x_3 Q_3^{[-]}(u) \\ x_1^2 Q_1^{[-2]}(u) & x_2^2 Q_2^{[-2]}(u) & x_3^2 Q_3^{[-2]}(u) \end{vmatrix}$$

$Q_3\left(u-\frac{1}{2}\right)$

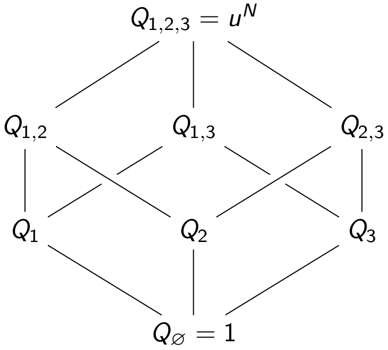
eigenvalues of a twist

$Q_2\left(u-1\right)$

The Q-function and it's power

- Bethe equations
  - The Wronskian expression for the Q-functions GL(3):

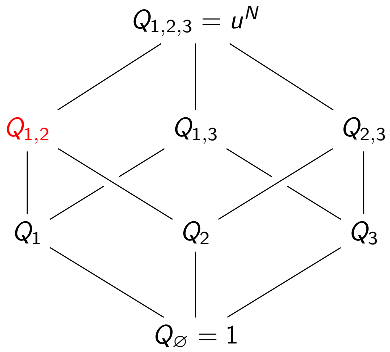
$$u^N \propto \begin{vmatrix} Q_1(u) & Q_2(u) & Q_3(u) \\ x_1 Q_1^{[-]}(u) & x_2 Q_2^{[-]}(u) & x_3 Q_3^{[-]}(u) \\ x_1^2 Q_1^{[-2]}(u) & x_2^2 Q_2^{[-2]}(u) & x_3^2 Q_3^{[-2]}(u) \end{vmatrix}$$



The Q-function and it's power

- Bethe equations
  - The Wronskian expression for the Q-functions GL(3):

$$u^N \propto \begin{vmatrix} Q_1(u) & Q_2(u) & Q_3(u) \\ x_1 Q_1^{[-]}(u) & x_2 Q_2^{[-]}(u) & x_3 Q_3^{[-]}(u) \\ x_1^2 Q_1^{[-2]}(u) & x_2^2 Q_2^{[-2]}(u) & x_3^2 Q_3^{[-2]}(u) \end{vmatrix}$$



The Q-function and it's power

- Bethe equations
  - The Wronskian expression for the Q-functions GL(n):

$$u^N \propto \left| \left(x_j^{1-i} Q_j \left(u + 1 - i \right) \right)_{1 \leq i,j \leq n} \right|$$

  - The polynomiality of the Q-functions.

$$Q_j(u) = \alpha_j \prod_{k=1}^{d_j} (u - u^{(k)})$$

number of excitations

Bethe roots

The Q-function and it's power

- Bethe equations
  - The Wronskian expression for the Q-functions GL(n):

$$u^N \propto \left| \left(x_j^{1-i} Q_j\,(u+1-i) \right)_{1\le i,j\le n} \right|$$

  - The polynomiality of the Q-functions.

$$Q_j(u) = \alpha_j \prod_{k=1}^{d_j} (u - u^{(k)})$$

$$\text{Bethe eq.} \;\; \Rightarrow \frac{Q_m(u_m+1)Q_{m+1}(u_m)Q_{m-1}(u_m-1)}{Q_m(u_{m-1})Q_{m+1}(u_m+1)Q_{m-1}(u_m)} = -\frac{x_{m+1}}{x_m}$$

$m \in [[1, n]]$

The Q-function and it's power

- Bethe equations
  - The Wronskian expression for the Q-functions GL(n):

$$u^N \propto \left| \left(x_j^{1-i} Q_j\,(u+1-i) \right)_{1\le i,j\le n} \right|$$

  - The polynomiality of the Q-functions.

$$Q_j(u) = \alpha_j \prod_{k=1}^{d_j} (u - u^{(k)})$$

$$\text{Bethe eq.} \;\; \Rightarrow \frac{Q_m(u_m+1)Q_{m+1}(u_m)Q_{m-1}(u_m-1)}{Q_m(u_{m-1})Q_{m+1}(u_m+1)Q_{m-1}(u_m)} = -\frac{x_{m+1}}{x_m}$$

- Q functions give us QSC.

Defination and some properties

- Definition of Co-derivative in the GL(n) case:

group element

generators of gl

$$\hat{D}^{i_0}_{j_0} f(g) = \lim_{\epsilon \to 0} \frac{f\left(e^{\epsilon E_{j_0,i_0}} g\right) - f(g)}{\epsilon}$$

$$\to \hat{D}^{i_0}_{j_0} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(g) = \frac{\partial}{\partial \epsilon} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}\left(e^{\epsilon E_{j_0,i_0}} g\right)\Big|_{\epsilon=0}$$

Defination and some properties

- Definition of Co-derivative in the GL(n) case:

$$\hat{D}^{i_0}_{j_0} f(g) = \lim_{\epsilon \to 0} \frac{f\left(e^{\epsilon E_{j_0,i_0}} g\right) - f(g)}{\epsilon}$$

$$\to \hat{D}^{i_0}_{j_0} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(g) = \frac{\partial}{\partial \epsilon} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}\left(e^{\epsilon E_{j_0,i_0}} g\right)\Bigg|_{\epsilon=0}$$

- Some of it's properties:

$$\hat{D} \otimes \pi_\lambda(g) = \mathcal{P}_{1,\lambda}(1 \otimes \pi_\lambda(g))$$

$$\hat{D} \otimes (A \cdot B) = \left(\hat{D} \otimes A\right) \cdot (1 \otimes B) + (1 \otimes A) \cdot \left(\hat{D} \otimes B\right)$$

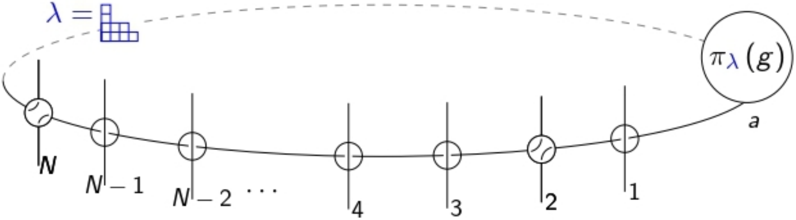
generalized representation

Where $\mathcal{P}_{1,\lambda} = \sum_{1 \leq k,l \leq n} (E_{k,l} \otimes \pi_\lambda(E_{l,k}))$.

fundamental representation

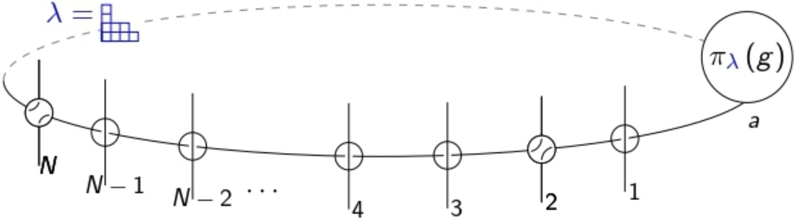
generalized permutation operator

T-Operators



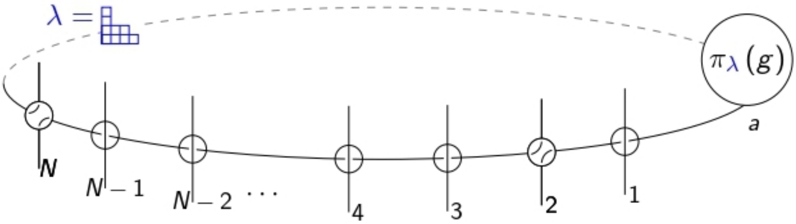
- $T^{\lambda}(u) =$
  $$tr_a\Big((u_N\mathbb{I}+\mathcal{P}_{N,a})\otimes(u_{N-1}\mathbb{I}+\mathcal{P}_{N-1,a})\otimes\cdots\otimes(\overset{u+\zeta(1)}{\overset{\downarrow}{u_1}}\mathbb{I}+\mathcal{P}_{1,a})\ \pi_{\lambda}(g)\Big)$$

T-Operators



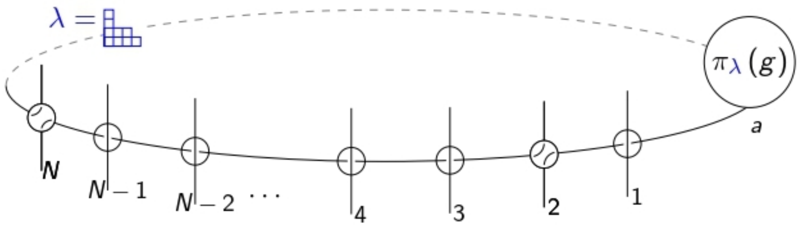
- $T^{\lambda}(u) =$
$$tr_a\Big((u_N\mathbb{I} + \mathcal{P}_{N,a}) \otimes (u_{N-1}\mathbb{I} + \mathcal{P}_{N-1,a}) \otimes \cdots \otimes (u_1\mathbb{I} + \mathcal{P}_{1,a})\ \pi_{\lambda}(g)\Big)$$
- Form the property: $\boxed{\hat{D} \otimes \pi_{\lambda}(g) = \mathcal{P}_{1,\lambda}(1 \otimes \pi_{\lambda}(g))}$

T-Operators



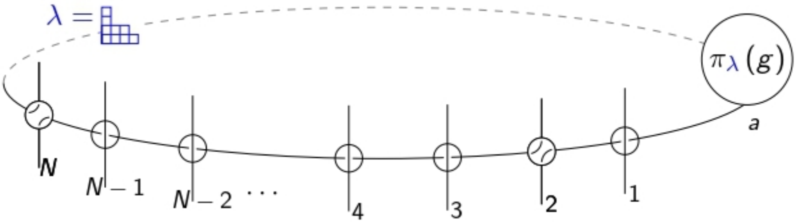
- $T^{\lambda}(u) =$
  $$tr_a\Big((u_N\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N,a}}}) \otimes (u_{N-1}\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N-1,a}}}) \otimes \cdots \otimes (u_1\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{1,a}}})\ \pi_{\lambda}(g)\Big)$$
- Form the property: $\boxed{\hat{D} \otimes \pi_{\lambda}(g) = \mathcal{P}_{1,\lambda}(1 \otimes \pi_{\lambda}(g))}$

T-Operators



- $T^{\lambda}(u) =$
  $$tr_a\Big((u_N\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N,a}}}) \otimes (u_{N-1}\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N-1,a}}}) \otimes \cdots \otimes (u_1\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{1,a}}})\ \pi_\lambda(g)\Big)$$
- Form the property: $\boxed{\hat{D} \otimes \pi_\lambda(g) = \mathcal{P}_{1,\lambda}(1 \otimes \pi_\lambda(g))}$
- $T^{\lambda}(u) = \Big\{\Big(u_1\mathbb{I} + \hat{D}\Big) \otimes \Big(u_2\mathbb{I} + \hat{D}\Big) \otimes \cdots \otimes \Big(u_N\mathbb{I} + \hat{D}\Big)\Big\} tr_a\Big(\pi_\lambda(g)\Big)$

T-Operators



- $T^{\lambda}(u) =$
$$tr_a\Big((u_N\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N,a}}}) \otimes (u_{N-1}\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{N-1,a}}}) \otimes \cdots \otimes (u_1\mathbb{I} + \overset{\hat{D}}{\cancel{\mathcal{P}_{1,a}}})\ \pi_{\lambda}(g)\Big)$$
- Form the property: $\boxed{\hat{D} \otimes \pi_\lambda(g) = \mathcal{P}_{1,\lambda}(1 \otimes \pi_\lambda(g))}$
- $T^{\lambda}(u) = \Big\{\Big(u_1\mathbb{I} + \hat{D}\Big) \otimes \Big(u_2\mathbb{I} + \hat{D}\Big) \otimes \cdots \otimes \Big(u_N\mathbb{I} + \hat{D}\Big)\Big\} tr_a\Big(\pi_\lambda(g)\Big)$
- $\boxed{T^{\lambda}(u) = \Big\{\Big(u_1\mathbb{I} + \hat{D}\Big) \otimes \Big(u_2\mathbb{I} + \hat{D}\Big) \otimes \cdots \otimes \Big(u_N\mathbb{I} + \hat{D}\Big)\Big\} \chi_\lambda(g)}$

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_{b} \left(Q_{\underset{\uparrow}{\bar{b}}}^{[1-n]} Q_{b}^{[-s]} \right)$$

$$[[1,n]] \backslash \{b\}$$

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_{b} \left(Q_{\bar{b}}^{[1-n]} Q_{b}^{[-s]} \right) \Rightarrow \sum_{s \geq 0} T_{1,s}^{[s]} t^{s} \propto \sum_{b} \left(\sum_{s \geq 0} \left(Q_{\bar{b}}^{[1-n+s]} t^{s} \right) Q_{b} \right)$$

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_b \left(Q_{\bar{b}}^{[1-n]} Q_b^{[-s]} \right) \Rightarrow \sum_{s\geq 0} T_{1,s}^{[s]} t^s \propto \sum_b \left(\sum_{s\geq 0} \left(Q_{\bar{b}}^{[1-n+s]} t^s \right) Q_b \right)$$

$$\sum_{s\geq 0} \chi_s\left(g\right) t^s$$

normalization

$$\rightarrow Q_j\left(u\right) = \lim_{t\rightarrow \frac{1}{x_j}} \mathcal{N}\left\{ \bigotimes_{i=1}^{N} (u_i + \frac{1}{2} t \frac{\partial}{\partial t} + \hat{D}) w(t) \right\}$$

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_{b} \left(Q_{\bar{b}}^{[1-n]} Q_{b}^{[-s]} \right) \Rightarrow \sum_{s\geq 0} T_{1,s}^{[s]} t^{s} \propto \sum_{b} \left(\sum_{s\geq 0} \left(Q_{\bar{b}}^{[1-n+s]} t^{s} \right) Q_{b} \right)$$

$$\sum_{s\geq 0} \chi_s \left(g \right) t^{s}$$

$$\rightarrow Q_j \left(u \right) = \lim_{t \rightarrow \frac{1}{x_j}} \mathcal{N} \left\{ \bigotimes_{i=1}^{N} \left(u_i + \frac{1}{2} t \frac{\partial}{\partial t} + \hat{D} \right) w(t) \right\}$$

- By understanding the action of $\hat{D}$ on $w(t)$ as combinatorics:

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_{b} \left(Q_{\bar{b}}^{[1-n]} Q_{b}^{[-s]} \right) \Rightarrow \sum_{s \geq 0} T_{1,s}^{[s]} t^{s} \propto \sum_{b} \left(\sum_{s \geq 0} \left(Q_{\bar{b}}^{[1-n+s]} t^{s} \right) Q_{b} \right)$$

$$\sum_{s \geq 0} \chi_{s}\left(g\right) t^{s}$$

$$\to Q_{j}\left(u\right) = \lim_{t \to \frac{1}{x_j}} \mathcal{N} \left\{ \bigotimes_{i=1}^{N} \left(u_i + \frac{1}{2} t \frac{\partial}{\partial t} + \hat{D}\right) w(t) \right\}$$

- By understanding the action of $\hat{D}$ on $w(t)$ as combinatorics:

$$\hat{D}_{j_1}^{i_1} w(t) = \overset{i_1}{\underset{j_1}{\Big|}} \, w(t) \qquad , \hat{D}_{j_1}^{i_1} \otimes \hat{D}_{j_2}^{i_2} w(t) = \overset{i_1 \; i_2}{\underset{j_1 \; j_2}{\Big| \; \Big|}} + \overset{i_1 \; i_2}{\underset{j_1 \; j_2}{\times}} \, w(t)$$

Q-Operators

combinatorial Relations

- From the relation between T and Q operators:

$$T_{1,s} \propto \sum_{b} \left(Q_{\bar{b}}^{[1-n]} Q_{b}^{[-s]} \right) \Rightarrow \sum_{s\geq 0} T_{1,s}^{[s]} t^{s} \propto \sum_{b} \left(\sum_{s\geq 0} \left(Q_{\bar{b}}^{[1-n+s]} t^{s} \right) Q_{b} \right)$$

$$\sum_{s\geq 0} \chi_s (g)\, t^{s}$$

$$\to Q_j(u) = \lim_{t\to \frac{1}{x_j}} \mathcal{N}\left\{ \bigotimes_{i=1}^{N} (u_i + \frac{1}{2} t \frac{\partial}{\partial t} + \hat{D}) w(t) \right\}$$

- By understanding the action of $\hat{D}$ on $w(t)$ as combinatorics:

$$\hat{D}_{j_1}^{i_1} w(t) = \Big|_{j_1}^{i_1}\, w(t) \qquad , \hat{D}_{j_1}^{i_1} \otimes \hat{D}_{j_2}^{i_2} w(t) = \Big|_{j_1}^{i_1} \Big|_{j_2}^{i_2} + \times_{j_1\, j_2}^{i_1\, i_2}\, w(t)$$

$$\text{Where } \Big|_{j_1}^{i_1} = \left(\frac{gt}{1-gt} \right)_{j_1}^{i_1},\ \vdots_{j_1}^{i_1} = \left(\frac{1}{1-gt} \right)_{j_1}^{i_1} \text{ and a cross } \times_{j_1\, j_2}^{i_1\, i_2} = \left(\mathcal{P}_{1,2} \left(\frac{1}{1-gt} \otimes \frac{gt}{1-gt} \right) \right)_{j_1, j_2}^{i_1, i_2}$$

Co-derivative In the SO Case

- Definition of Co-derivative in the SO(2n) case :

group element

generators of so

$$\hat{D}^{i_0}_{j_0} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(g) = \frac{\partial}{\partial \epsilon} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(e^{\epsilon F_{j_0, i_0}} g)\Big|_{\epsilon=0}$$

Co-derivative In the SO Case

- Definition of Co-derivative in the SO(2n) case :

$$\hat{D}^{i_0}_{j_0} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(g) = \frac{\partial}{\partial \epsilon} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(e^{\epsilon F_{j_0, i_0}} g) \Big|_{\epsilon = 0}$$

- One of the properties of the Co-derivative in the SO Case:

$$\hat{D}_{so} \otimes \pi_\lambda(g) = \Big(\sum_{i,j} \underbrace{E_{i,j}}_{\text{Fundamental repp.}} \otimes \overbrace{\pi_\lambda(F_{j,i})}^{\text{generalized repp.}} \Big) (1 \otimes \pi_\lambda(g)) \, ,$$

Co-derivative In the SO Case

- Definition of Co-derivative in the SO(2n) case :

$$\hat{D}^{i_0}_{j_0} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(g) = \frac{\partial}{\partial \epsilon} f^{i_1 i_2 \dots i_N}_{j_1 j_2 \dots j_N}(e^{\epsilon F_{j_0,i_0}} g)\bigg|_{\epsilon=0}$$

- One of the properties of the Co-derivative in the SO Case:

$$\hat{D}_{so} \otimes \pi_\lambda(g) = \left(\sum_{i,j} E_{i,j} \otimes \pi_\lambda(F_{j,i}) \right) (1 \otimes \pi_\lambda(g)) \ ,$$

- We have two different representation in the SO(2n) case:

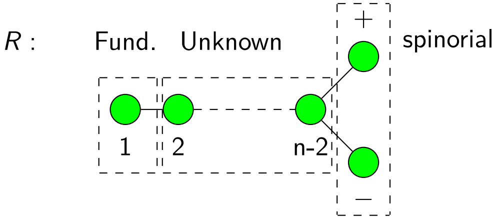


Figure: $SO(2n)$ Dynkin Diagram

Spinorial case

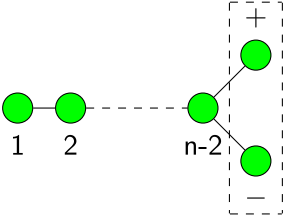


Figure: $SO(2n)$ Dynkin Diagram

- The R matrix in this representation:

$$R(u) = u\mathbb{I} + \sum_{i,j} E_{i,j} \otimes F_{j,i}$$

Spinorial case

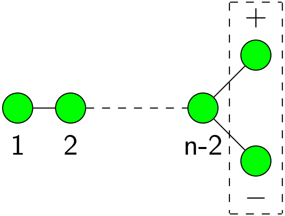


Figure: $SO(2n)$ Dynkin Diagram

- The R matrix in this representation:

$$R(u) = u\mathbb{I} + \sum_{i,j} E_{i,j} \otimes F_{j,i}$$

Which changes after using the Co-derivative to:

$$R(u) = u\mathbb{I} + \hat{D}_{so}$$

Fundamental case

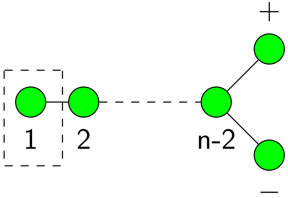


Figure: $SO(2n)$ Dynkin Diagram

- The R matrix in this representation is quadratic in the spectral parameter:

$$R = u^2 - \frac{1}{4}((\kappa - 1)^2 + 2\kappa s + s^2) + (u + \frac{\kappa}{2}) \sum_{i,j} E_{i,j} \otimes F_{j,i} + \frac{1}{2} \sum_{i,j} E_{i,j} \otimes \sum_k F_{k,j} F_{i,k}$$

Fundamental case

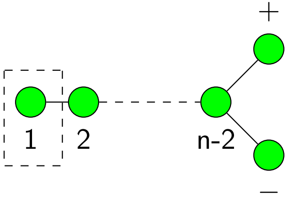


Figure: $SO(2n)$ Dynkin Diagram

- The R matrix in this representation is quadratic in the spectral parameter:

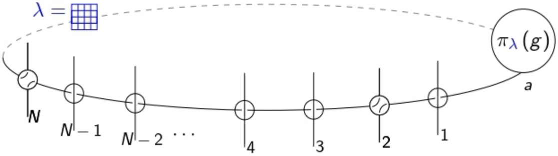
$$R = u^2 - \frac{1}{4}((\kappa-1)^2 + 2\kappa s + s^2) + (u + \frac{\kappa}{2}) \sum_{i,j} E_{i,j} \otimes F_{j,i} + \frac{1}{2} \sum_{i,j} E_{i,j} \otimes \sum_k F_{k,j} F_{i,k}$$

Using the def. of Co-derivative

$$= \left(u^2 - \frac{1}{4}((\kappa-1)^2 + 2\kappa s + s^2) + (u + \frac{\kappa}{2})(\hat{D}_{so}) + \frac{1}{2}(\hat{D}_{so})^2 \right)$$

For Now and Future Research

- We are testing to find a formula for the T-operators in the case of rectangular representation in the auxiliary space.



- Proving the Wronskian expression for our Q operators from the Co-derivative.

Thank You