Random partitions and gauge group integrals: recent results

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Overview

- Random matrix integrals naturally appear in gauge theories: various gauge theory partition functions involve matrix integrals over the Haar measure(s) of relevant Lie gauge group(s).
- Have an intricate relation to probabilty measures on *random partitions* – a notion of probability on partitions of positive integers.
- In turn, certain such measures have interpretation as irreducible characters of gauge group representations.
- Hence a very interdisciplinary research area connecting together seemingly disparate topics: high energy statistical physics, integer partitions and representation theory, among others.

$$\text{Hep-th/Statmech} \leftrightarrow \text{Random partitions} \leftrightarrow \text{Rep. theory}$$

Can use these connections to study phase transitions, asymptotic behaviour, integrable systems etc. – Random partitions are very useful tools [Okounkov 2003] .

What will this talk be about?

- Review of the unitary GWW matrix model; asymptotic analysis [Kimura–Zahabi 2021] of its phase-space structure using random partition techniques.
- Generalization [Kimura–Purkayastha 2022] of this model to the special orthogonal and symplectic cases, i.e. $\mathrm{U}(N)$ is replaced by other compact classical Lie groups $\mathrm{SO}(2N)$, $\mathrm{SO}(2N+1)$, $\mathrm{Sp}(N)$.
- Possible other applications of random partitions.

Gross–Witten–Wadia (GWW) unitary matrix model

- Derives from $d = 2$ $\mathrm{U}(N)$ lattice gauge theory [Gross–Witten 1980] . Partition function

$$\mathcal{Z}_{\mathrm{U}(N)}(\beta) = \int_{\mathrm{U}(N)} \mathrm{d}U \, \exp\left(\frac{N\beta}{2}\left(\mathrm{tr}\, U + \mathrm{tr}\, U^{-1}\right)\right), \ \beta \geq 0.$$

- Large N limit: path integral extremized around the 'classical extremum' of distribution of eigenvalues

$$\mathcal{Z}_{\mathrm{U}(N)}(\beta) \xrightarrow{N\to\infty} \int_{\|\rho\|_1=1} \mathcal{D}\rho \, \exp\left(N^2 S_{\mathrm{eff}}[\rho]\right) \approx \exp\left(N^2 S_{\mathrm{eff}}[\rho_0]\right).$$

  Effective action S_{eff} depends on extremal distribution of eigenvalues $\rho_0 : S^1 \to \mathbb{R}_{\geq 0}$.

- Free energy normalized w.r.t. gauge group rank,

$$\mathcal{F}_{\mathrm{U}}(\beta) = \lim_{N\to\infty} \frac{1}{N^2} \ln \mathcal{Z}_{\mathrm{U}(N)}(\beta) \approx S_{\mathrm{eff}}[\rho_0].$$

- Phase transition at $\beta = 1$ of third order:

$$\rho_0(\phi) = \begin{cases} \frac{1}{2\pi}\left(1 + \beta\cos\phi\right) & \beta \leq 1, \\ \frac{\beta}{\pi}\cos\left(\frac{\phi}{2}\right)\sqrt{\frac{1}{\beta} - \sin^2\left(\frac{\phi}{2}\right)} \times \mathbf{1}_{[-\alpha,\alpha]}(\phi) & \beta > 1, \end{cases}$$

$$\mathcal{F}_{\mathrm{U}}(\beta) = \begin{cases} \frac{\beta^2}{4} & \beta \leq 1, \\ \beta - \frac{1}{2}\ln\beta - \frac{3}{4} & \beta > 1. \end{cases}$$

- $\phi \in [-\pi, \pi)$, α is the smallest positive root of $\sin\left(\frac{\phi}{2}\right) = \frac{1}{\beta}$. *Ungapped* $(\beta \leq 1)$ and *gapped* $(\beta > 1)$ phases – gap appears in eigenvalue distribution.

Generalized GWW

- Parametrization by coupling constants $\boldsymbol{\beta} = (\beta_n)_{n\geq 1}, \boldsymbol{\gamma} = (\gamma_n)_{n\geq 1}$, $g_n, \bar{g}_n = \frac{1}{2n}(\beta_n \pm i\gamma_n)$:

$$\mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta},\boldsymbol{\gamma}) = \int_{\mathrm{U}(N)} \mathrm{d}U \,\exp\left(N \sum_{n\geq 1} \left(g_n \,\mathrm{tr}\, U^n + \overline{g}_n \,\mathrm{tr}\, U^{-n}\right)\right).$$

- Ungapped phase – standard derivation e.g. [Mariño 2015]

$$\rho_0(\phi) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n\geq 1}\left(\beta_n \cos n\phi + \gamma_n \sin n\phi\right), \; \mathcal{F}_{\mathrm{U}}(\boldsymbol{\beta},\boldsymbol{\gamma}) = \sum_{n\geq 1} \frac{\beta_n^2 + \gamma_n^2}{4n}.$$

- Remaining phase structure quickly gets complicated to analyze and describe with more coupling constants; random partition formulation to the rescue!

- Schur polynomial in N (possibly infinite) variables [Macdonald 1980] , in terms of complete homogenous polynomials h_k of degree k:[1]

$$s_\lambda(x_1,\ldots,x_N) = \det\left(h_{j-k+\lambda_k}\right)_{j,k=1}^{\ell(\lambda)}.$$

- $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_{\ell(\lambda)})$ is a partition of depth $\ell(\lambda)$. s_λ **identically vanish** for $\ell(\lambda) > N$, leading to standard result using Miwa variables, $\operatorname{tr} X^n = \frac{N(\beta_n - i\gamma_n)}{2}$, $\operatorname{tr} Y^n = \frac{N(\beta_n + i\gamma_n)}{2}$.

$$\mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta},\boldsymbol{\gamma}) = \sum_{\ell(\lambda)\leq N} s_\lambda(X)s_\lambda(Y).$$

- Free energy $\mathcal{F}$ may be divided into 'continuum' component $\mathcal{F}^c$ and 'fluctuation' component $\mathcal{F}^f$,

$$\mathcal{F}_{\mathrm{U}}(\boldsymbol{\beta},\boldsymbol{\gamma}) = \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_\infty(\boldsymbol{\beta},\boldsymbol{\gamma})}_{\mathcal{F}^c_{\mathrm{U}}(\boldsymbol{\beta},\boldsymbol{\gamma})} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\left(\frac{\mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta},\boldsymbol{\gamma})}{\mathcal{Z}_\infty(\boldsymbol{\beta},\boldsymbol{\gamma})}\right)}_{\mathcal{F}^f_{\mathrm{U}}(\boldsymbol{\beta},\boldsymbol{\gamma})}.$$

- Unrestricted Schur sum in terms of plethystic exponential gives ungapped free energy

$$\mathcal{Z}_\infty(\boldsymbol{\beta},\boldsymbol{\gamma}) = \sum_\lambda s_\lambda(X)s_\lambda(Y) = \mathrm{PE}\left[\operatorname{tr} X \operatorname{tr} Y\right], \quad \mathrm{PE}[f(x_i)] = \exp\left(\sum_{n=1}^\infty \frac{1}{n} f(x_i^n)\right).$$

- Fluctuation component is responsible for the phase transitions; asymptotic analysis allows identification of gapped phases [Kimura–Zahabi 2021] .
- Results from [Kimura–Zahabi 2021] : Partition function in terms of Fredholm determinant [Borodin–Okounkov 2000, Okounkov 2001] . Asymptotic behaviour in terms the higher-order Tracy–Widom distribution F_p [Claeys–Krasovsky–Its 2009]

$$\mathcal{F}^{f}_{\mathrm{U}}(\boldsymbol{\beta}, \mathbf{0}) \sim N^{-2} \lim_{s \to \pm\infty} \ln F_p(s), \quad s = \frac{(\beta_c - \beta)N}{(\alpha_p N)^{\frac{1}{p+1}}}.$$

α_k, β defined in terms of couplings; $\alpha_{p'} = 0$ for all $p' < p$ with p, p' positive integers and p even.

- From asymptotics of the F_p, free energy edge behaviour

$$\mathcal{F}^{f}_{\mathrm{U}}(\boldsymbol{\beta}, \mathbf{0}) \sim \begin{cases} \mathcal{O}\left(e^{-cN}\right), & \beta < \beta_c, \\ \alpha_p^{-2/p} |\beta_c - \beta|^{2(p+1)/p} + \mathcal{O}(N^{-2}), & \beta > \beta_c. \end{cases}$$

Classical groups model

- GWW-type matrix model for compact classical groups $\mathrm{SO}(2N)$, $\mathrm{SO}(2N+1)$ and $\mathrm{Sp}(N)$ [Kimura–Purkayastha 2022] , building on known large N results [García-García–Tierz 2020] for the one-coupling case

$$\mathcal{Z}_{\mathrm{G}(N)}(\boldsymbol{\beta}) = \int_{\mathrm{G}(N)} \mathrm{d}X \, \exp\left(N \sum_{n\geq 1} g_n \operatorname{tr} X^n \right),$$

real coupling constants $(g_n)_{n\geq 1}$ with $g_n = \frac{\beta_n}{n}$.
- $\mathrm{G}(N) = \mathrm{SO}(2N), \mathrm{SO}(2N+1), \mathrm{Sp}(N)$, compact classical group of rank N. Canonically represented as $2N \times 2N$ matrices.

- In [Kimura–Purkayastha 2022] : Coulomb gas analysis similar to the $\mathrm{U}(N)$ model. Different maximal tori for each of the three cases.

$$\mathcal{Z}_{\mathrm{G}(N)}(\beta)[\rho] = \int_{\|\rho\|_1=1} \mathcal{D}\rho \ \exp\left(N^2 \underbrace{\left(\mathcal{P}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi \mathrm{d}\varphi \, \Delta(\phi,\varphi)\rho(\phi)\rho(\varphi) + \frac{1}{N}\int_{-\pi}^{\pi} \mathrm{d}\phi \, \Xi(\phi) + \int_{-\pi}^{\pi} \mathrm{d}\phi \, V(\phi)\rho(\phi) \right)}_{S_{\mathrm{eff}}[\rho]} \right).$$

- Effective action $S[\rho]$ depends on Fredholm kernel (identical for the three cases)

$$\Delta(\phi,\varphi) = \frac{1}{2}\left[\ln\left(4\sin^2\left(\frac{\phi+\varphi}{2}\right)\right) + \ln\left(4\sin^2\left(\frac{\phi-\varphi}{2}\right)\right)\right].$$

- Subleading $\mathcal{O}\left(\frac{1}{N}\right)$ Ξ-contributions differ; exact results in the random partition approach.

- Ungapped phase probability distribution identical to the $\mathrm{U}(N)$ model with real coefficients, but **doubling** of free energy upto leading order:

$$\rho(\phi) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n\geq 1}\beta_n \cos n\phi, \ \mathcal{F}_{\mathrm{G}}(\boldsymbol{\beta}) = \sum_{n\geq 1}\frac{\beta_n^2}{2n} = 2\mathcal{F}_{\mathrm{U}}(\boldsymbol{\beta}, \mathbf{0}).$$

- Doubling is a consequence of normalization based on rank – N in all cases – but effective action dependent on matrix dimensions, doubles w.r.t. the unitary case.

- Gapped phase tackled by introducing the resolvent for the Fredholm kernel and a modified Plemelj formula ($\text{singularities} \xrightarrow{N\to\infty} \text{cuts}$)

$$W(\phi) = \frac{1}{N}\sum_{l=1}^{N}\left[\cot\left(\frac{\phi+\phi_l}{2}\right) + \cot\left(\frac{\phi-\phi_l}{2}\right)\right], \rho(\phi) = \frac{1}{8\pi i}\left[W(\phi - i0) - W(\phi + i0)\right].$$

- General solution m-cut solution – defined on higher-genus Riemann surface – is rather involved, but for the 1-cut solution, i.e. one coupling constant β, analytically easy to derive identical probability distribution as $\mathrm{U}(N)$ model, with **doubling** of free energy

$$\rho_0(\phi) = \frac{\beta}{\pi}\cos\left(\frac{\phi}{2}\right)\sqrt{\frac{1}{\beta} - \sin^2\left(\frac{\phi}{2}\right)}, \quad \mathcal{F}_{\mathrm{G}}(\beta) = 2\beta - \ln\beta - \frac{3}{2} = 2\mathcal{F}_{\mathrm{U}}(\beta).$$

- This is consistent with the $2N$ dimensions in consideration rather than N.

- For random partition description, introduce generalized Schur polynomials: $o_\lambda^{\text{even}}, o_\lambda^{\text{odd}}, sp_\lambda$ [Fulton–Harris 2004] . These represent orthogonal characters (generally irreducible) for even and odd special orthogonal, and symplectic groups respectively.
- Cauchy sum formulae [Koike–Terada 1987, García-García–Tierz 2019] . ALso vanish for $\ell(\lambda) > N$.

$$\mathrm{SO}(2N): \quad \sum_\lambda o_\lambda^{\text{even}}(X) s_\lambda(Y) = \mathrm{PE}[\mathrm{tr}\, X\, \mathrm{tr}\, Y]\, \mathrm{PE}\left[\frac{1}{2}\left(-\mathrm{tr}\, Y^2 - (\mathrm{tr}\, Y)^2\right)\right],$$

$$\mathrm{SO}(2N+1): \quad \sum_\lambda o_\lambda^{\text{odd}}(X) s_\lambda(Y) = \mathrm{PE}[\mathrm{tr}\, X\, \mathrm{tr}\, Y]\, \mathrm{PE}\left[\frac{1}{2}\left(-\mathrm{tr}\, Y^2 - (\mathrm{tr}\, Y)^2\right) + \mathrm{tr}\, Y\right],$$

$$\mathrm{Sp}(N): \quad \sum_\lambda sp_\lambda(X) s_\lambda(Y) = \mathrm{PE}[\mathrm{tr}\, X\, \mathrm{tr}\, Y]\, \mathrm{PE}\left[\frac{1}{2}\left(\mathrm{tr}\, Y^2 - (\mathrm{tr}\, Y)^2\right)\right].$$

- Character orthogonality:

$$\int_{\mathrm{U}(N)} \mathrm{d}U\, s_\lambda(U) s_\mu(U^{-1}) = \int_{\mathrm{SO}(2N)} \mathrm{d}X\, o_\lambda^{\text{even}}(X) o_\mu^{\text{even}}(X^{-1})$$
$$= \int_{\mathrm{SO}(2N+1)} \mathrm{d}X\, o_\lambda^{\text{odd}}(X) o_\mu^{\text{odd}}(X^{-1}) = \int_{\mathrm{Sp}(N)} \mathrm{d}X\, sp_\lambda(X) sp_\mu(X^{-1}) = \delta_{\lambda\mu}.$$

- Random partition expressions for the partition functions:

$$\mathcal{Z}_{\mathrm{SO}(2N)}(\boldsymbol{\beta}) = \mathsf{PE}\left[\operatorname{tr} Z^2\right] \mathcal{Z}_{\infty}(\boldsymbol{\beta}, \mathbf{0}) \mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta}, \mathbf{0}),$$
$$\mathcal{Z}_{\mathrm{SO}(2N+1)}(\boldsymbol{\beta}) = \mathsf{PE}\left[\operatorname{tr}\left(Z^2 - 2Z\right)\right] \mathcal{Z}_{\infty}(\boldsymbol{\beta}, \mathbf{0}) \mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta}, \mathbf{0}),$$
$$\mathcal{Z}_{\mathrm{Sp}(N)}(\boldsymbol{\beta}) = \mathsf{PE}\left[-\operatorname{tr} Z^2\right] \mathcal{Z}_{\infty}(\boldsymbol{\beta}, \mathbf{0}) \mathcal{Z}_{\mathrm{U}(N)}(\boldsymbol{\beta}, \mathbf{0}).$$

  Miwa variable parametrization $\operatorname{tr} Z^n = \frac{N\beta_n}{2}$.
- Subleading $\mathcal{O}\left(\frac{1}{N}\right)$ contribution clearly different in the three cases.

- Free energies are calculated to be

$$\mathcal{F}_{\mathrm{SO}}^{\text{even}}(\beta) = \lim_{N\to\infty}\frac{1}{N}\sum_{n\geq 1}\frac{\beta_{2n}}{2n} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_\infty^2(\beta,\mathbf{0})}_{=\mathcal{F}_{\mathrm{SO}}^{c,\text{even}}(\beta)} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\left(\frac{Z_{\mathrm{U}(N)}(\beta,\mathbf{0})}{\mathcal{Z}_\infty(\beta,\mathbf{0})}\right)}_{\mathcal{F}_{\mathrm{SO}}^{f,\text{even}}(\beta)},$$

$$\mathcal{F}_{\mathrm{SO}}^{\text{odd}}(\beta) = -\lim_{N\to\infty}\frac{1}{N}\sum_{n\geq 1}\frac{\beta_{2n-1}}{2n-1} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_\infty^2(\beta,\mathbf{0})}_{=\mathcal{F}_{\mathrm{SO}}^{c,\text{odd}}(\beta)} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\left(\frac{Z_{\mathrm{U}(N)}(\beta,\mathbf{0})}{\mathcal{Z}_\infty(\beta,\mathbf{0})}\right)}_{\mathcal{F}_{\mathrm{SO}}^{f,\text{odd}}(\beta)},$$

$$\mathcal{F}_{\mathrm{Sp}}(\beta) = -\lim_{N\to\infty}\frac{1}{N}\sum_{n\geq 1}\frac{\beta_{2n}}{2n} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_\infty^2(\beta,\mathbf{0})}_{=\mathcal{F}_{\mathrm{Sp}}^{c}(\beta)} + \underbrace{\lim_{N\to\infty}\frac{1}{N^2}\ln\left(\frac{Z_{\mathrm{U}(N)}(\beta,\mathbf{0})}{\mathcal{Z}_\infty(\beta,\mathbf{0})}\right)}_{\mathcal{F}_{\mathrm{Sp}}^{f}(\beta)}.$$

- Continuum parts **double** the unitary model, and fluctuation parts **identical**:

$$\mathcal{F}_{\mathrm{SO}}^{c,\text{even}}(\beta) = \mathcal{F}_{\mathrm{SO}}^{c,\text{odd}}(\beta) = \mathcal{F}_{\mathrm{Sp}}^{c}(\beta) = 2\mathcal{F}_{\mathrm{U}}^{c}(\beta,\mathbf{0}),$$
$$\mathcal{F}_{\mathrm{SO}}^{f,\text{even}}(\beta) = \mathcal{F}_{\mathrm{SO}}^{f,\text{odd}}(\beta) = \mathcal{F}_{\mathrm{Sp}}^{f}(\beta) = \mathcal{F}_{\mathrm{U}}^{f}(\beta,\mathbf{0}).$$

- Hence asymptotic analysis of fluctuation part can be directly ported from [Kimura–Zahabi, 2021] . Phase space structure is formally identical for unitary, special orthogonal and symplectic cases.
- $\mathcal{O}\left(\frac{1}{N}\right)$ subleading terms correspond to Ξ-terms of the Coulomb gas formalism; corroborates with results due to Szegö-Johansson theorem. [Johansson 1997] .

Other applications

- Supersymmetric indices, e.g. $\mathcal{N} = 1$ superconformal index for supersymmetric gauge theories constructed out of compactified superstring theory.
- Hubbard-Stratonovich transform [Álvarez-Gaumé–Basu–Mariño–Wadia 2006] relates building blocks of the $\mathcal{N} = 1$ superconformal index to GWW-type matrix models,

$$\exp\left(f \operatorname{tr} U \operatorname{tr} U^{\dagger}\right) = \iint_{\mathbb{R}^2} \frac{\mathrm{d}x\,\mathrm{d}y}{\pi f} \exp\left(-\frac{t\bar{t}}{f} + t \operatorname{tr} U + \bar{t} \operatorname{tr} U^{\dagger}\right).$$

  LHS of this equation is template building-block for the index – chiral multiplets in bifundamental representation or vector multiplets in adjoint representation; RHS contains the GWW form.
- Of relevance to work in progress [Melczer–Purkayastha–Qu–Zahabi 202x] about asymptotic analysis of the large N superconformal index for toric quivers.