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Correlations for the XYZ spin chain and Painlevé VI

Hjalmar Rosengren

Chalmers University of Technology
and University of Gothenburg

with Christian Hagendorf, Université Catholique de Louvain

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Goal: Compute certain correlations for XYZ spin chain exactly for finite systems.

Exact result for finite systems are rare.
In our case the reason seems to be supersymmetry.

XYZ spin chain

Chain of L spin $1/2$ particles.
Hilbert space $V^{\otimes L}$, where $V = \mathbb{C}|\uparrow\rangle + \mathbb{C}|\downarrow\rangle$.

Hamiltonian

$$H^{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\Big(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z\Big)\,.$$

J_x, J_y, J_z (real) anisotropy parameters.

Pauli matrices σ_j^x etc. act on j-th tensor factor.

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\sigma_{L+1}^x = \sigma_1^x$ etc. Periodic boundary conditions.

Combinatorial/supersymmetric case

$$J_x J_y + J_x J_z + J_y J_z = 0$$

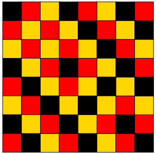
Why combinatorial?

- Contains XXZ model with $\Delta = -1/2$

$$J_x = J_y = 1, \qquad J_z = -\frac{1}{2}.$$

Deep connections to combinatorics of alternating sign matrices and plane partitions (Razumov–Stroganov etc.).

- General XYZ case has connections to three-colourings (R. 2011, Hietala 2020).



Combinatorial/supersymmetric case

$$J_x J_y + J_x J_z + J_y J_z = 0$$

Why supersymmetric?

- Scaling limit to massive sine-Gordon QFT.
  Under condition above it has $\mathcal{N} = 2$ supersymmetry
  (Saleur & Warner 1993).
- Supersymmetry on finite lattice
  (Fendley & Hagendorf 2012):

$$H^{\mathsf{XYZ}} = \mathsf{Const} + QQ^\dagger + Q^\dagger Q$$

(on subspace of $V^{\otimes L}$) where $Q : V^{\otimes L} \to V^{\otimes (L+1)}$.

The importance of being odd

Baxter (1972) computed the ground state energy (lowest eigenvalue of H^{XYZ}) as $L \to \infty$.

When $J_x J_y + J_x J_z + J_y J_z = 0$ (and $J_x + J_y + J_z > 0$) it takes the simple form

$$E_0 \sim -\frac{L}{2}(J_x + J_y + J_z), \qquad L \to \infty.$$

Stroganov (2001) conjectured that if L is odd then

$$E_0 = -\frac{L}{2}(J_x + J_y + J_z).$$

Proved by Hagendorf and Liénardy (2018) using supersymmetry.

$$H^{\mathsf{XYZ}} = E_0 + QQ^\dagger + Q^\dagger Q.$$

Correlation functions

We will assume

- Periodic boundary
- $J_x J_y + J_x J_z + J_y J_z = 0$
- $L = 2n + 1$ odd

$|\Psi\rangle$ ground state with even number of up spins.
Nearest neighbour correlations (for ground state)

$$C^x = \frac{\langle \Psi | \sigma_j^x \sigma_{j+1}^x | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \quad C^y = \cdots , \quad C^z = \cdots .$$

Computed for XXZ chain ($J_x = J_y = 1$, $J_z = -1/2$)
by Stroganov (2001):

$$C^x = C^y = \frac{5}{8} + \frac{3}{8L^2}, \qquad C^z = -\frac{1}{2} + \frac{3}{2L^2}.$$

Preliminary result

For $a \in \{x, y, z\}$ we can write

$$C^a = 1 + \frac{J_x J_y J_z}{J_a^2 (J_x + J_y + J_z)} f_n$$

where f_n is a rational function of $Z = (J_x + J_y + J_z)^3 / J_x J_y J_z$.

$$f_0 = 0, \qquad f_1 = 1, \qquad f_2 = \frac{Z+27}{Z+25},$$
$$f_3 = \frac{(Z+24)(Z+27)}{(Z+21)(Z+28)},$$
$$f_4 = \frac{Z^3 + 74Z^2 + 1807Z + 14520}{Z^3 + 72Z^2 + 1701Z + 13068}, \ \dots$$

Polynomials s_n and $\bar{s}_n$

Bazhanov and Mangazeev (2005, 2010) introduced two families of polynomials s_n and $\bar{s}_n$ ($n \in \mathbb{Z}$).

Tau functions of Painlevé VI, related to Q-operator eigenvalue (see below).

Toda-type recursions

$$8(2n+1)^2 s_{n+1}s_{n-1}+2z(z-1)(9z-1)^2(s_n''s_n-(s_n')^2)+2(3z-1)^2(9z-1)s_ns_n' \\ -\big(4(3n+1)(3n+2)+n(5n+3)(9z-1)\big)s_n^2=0.$$

$$\dots,\ s_{-2}=\frac{3+9z}{4},\quad s_{-1}=1,\quad s_0=0,$$
$$s_1=1,\quad s_2=1+z,\quad s_3=1+3z+4z^2,\ \dots\ .$$

Main result

Parametrize the chain as

$$J_x = 1 + \zeta, \qquad J_y = 1 - \zeta, \qquad J_z = \frac{\zeta^2 - 1}{2}.$$

The function f_n is

$$f_n = \frac{(\zeta^2+3)(\zeta^2-3)}{(\zeta^2-1)^2} - \frac{2\zeta^2(\zeta^2+3)}{(2n+1)^2(\zeta^2-1)^2}\frac{\bar{s}_n(\zeta^{-2})\bar{s}_{-n-1}(\zeta^{-2})}{s_n(\zeta^{-2})s_{-n-1}(\zeta^{-2})}.$$

We prove this assuming a technical condition (see below).

Infinite lattice limit

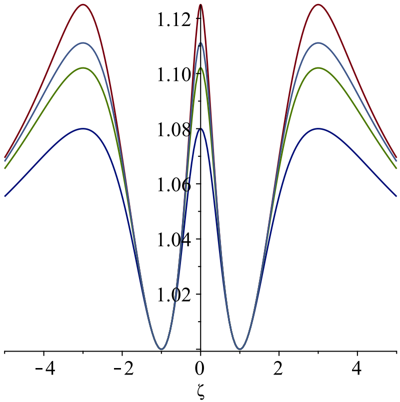
Using Baxter's formula for the free energy, one can show that

$$f_\infty = \lim_{n\to\infty} f_n = \begin{cases} \dfrac{(\zeta^2+3)(\zeta^2-3)}{(\zeta^2-1)^2}, & |\zeta| \geq 3, \\ -\dfrac{(\zeta^2+3)(\zeta^2+6\zeta-3)}{8(\zeta-1)^2}, & -3 < \zeta < 0, \\ -\dfrac{(\zeta^2+3)(\zeta^2-6\zeta-3)}{8(\zeta+1)^2}, & 0 < \zeta < 3. \end{cases}$$

The three regimes are related by permuting the anisotropy parameters.

The function f_∞ is twice differentiable but $f_\infty^{(3)}$ jumps at the XXZ points $\zeta = 0$ and $\zeta = \pm 3$.

Finite versus infinite chain



From bottom to top: f_2, f_3, f_4 (length 5, 7, 9), f_∞.

The points $\zeta = 0$ and $\zeta = \pm 3$ correspond to XXZ chains (e.g. $J_x = J_y$).

The points $\zeta = \pm 1$ and $\zeta = \infty$ correspond to X00 chains (e.g. $J_y = J_z = 0$).

Painlevé VI

PVI is the most general 2nd order ODE, all of whose movable singularities are poles.

Elliptic form:

$$\frac{d^2q}{dt^2} = \sum_{j=0}^{3} \alpha_j \wp'(q - \gamma_j | 2\pi \mathrm{i} t),$$

α_j parameters, $\wp$ Weierstrass' function with half-periods γ_j.

Algebraic form:

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt}$$
$$+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha_0 - \alpha_1 \frac{t}{q^2} + \alpha_2 \frac{t-1}{(q-1)^2} + \left(\frac{1}{2} - \alpha_3\right)\frac{t(t-1)}{(q-t)^2}\right).$$

Hamiltonian form of PVI

$$\frac{d^2q}{dt^2} = \sum_{j=0}^{3} \alpha_j \wp'(q - \gamma_j | 2\pi \mathrm{i} t).$$

Introducing $p = dq/dt$, this is equivalent to the non-stationary Hamiltonian system (Manin 1998)

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H(p, q, t) = \frac{p^2}{2} - V(q, t)$$

with Darboux–Treibich–Verdier potential

$$V(q, t) = \sum_{j=1}^{4} \alpha_j \wp(q - \gamma_j | 2\pi \mathrm{i} t).$$

Picard solutions

$$\frac{d^2q}{dt^2} = \sum_{j=0}^{3} \alpha_j \wp'(q - \gamma_j | 2\pi \mathrm{i} t).$$

When $\alpha_0 = \cdots = \alpha_3 = 0$, the solution is $q = C_1 t + C_2$.

Applying so called Bäcklund transformations

$$(q, p, \alpha_0, \ldots, \alpha_3) \mapsto (\tilde{q}, \tilde{p}, \tilde{\alpha}_0, \ldots, \tilde{\alpha}_3)$$

gives Picard class solutions with $\alpha_j = n_j^2/2$, $n_j \in \mathbb{Z}$.

The polynomials s_n and $\bar{s}_n$ as tau functions

A tau function is a solution to

$$\frac{\tau'(t)}{\tau(t)} = H(p(t), q(t), t)$$

where (p, q) solves PVI.

The polynomials s_n and $\bar{s}_n$ can be identified with tau functions, obtained from the Picard solution $q = 2\pi\mathrm{i} + 2\pi\mathrm{i}t/3$ through sequences of Bäcklund transformations.

The parameters α_j are

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right) \quad \text{for } s_n, \qquad \left(\frac{n^2}{2}, \frac{n^2}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \text{for } \bar{s}_n.$$

Correlation functions and Painlevé VI

What does our expression

$$f_n = \frac{(\zeta^2+3)(\zeta^2-3)}{(\zeta^2-1)^2} - \frac{2\zeta^2(\zeta^2+3)}{(2n+1)^2(\zeta^2-1)^2}\frac{\bar{s}_n(\zeta^{-2})\bar{s}_{-n-1}(\zeta^{-2})}{s_n(\zeta^{-2})s_{-n-1}(\zeta^{-2})}$$

mean for PVI?

It is essentially the PVI Hamiltonian with parameters

$$\left(\frac{(n+1/2)^2}{2}, \frac{(n+1/2)^2}{2}, 0, 0\right)$$

evaluated at a solution to PVI with parameters

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right).$$

Transfer matrix

Based on Baxter's Q-operator method.

Parametrize $H^{\mathsf{XYZ}} = H^{\mathsf{XYZ}}(\eta, \tau)$ by elliptic functions.
Supersymmetric case is $\eta = \pi/3$.

One-parameter family $\mathbf{T}(u) = \mathbf{T}(u, \eta, \tau)$ commuting with H^{XYZ}.
Transfer matrices of eight-vertex model.
H^{XYZ} is essentially $\mathbf{T}^{-1}(u)\mathbf{T}'(u)\big|_{u=\eta}$.

Can extend Ψ to η near $\pi/3$ and write

$$\mathbf{T}(u)\Psi = t(u)\Psi, \qquad H^{\mathsf{XYZ}}\Psi = \varepsilon\Psi,$$

where ε is essentially $t'(u)/t(u)\big|_{u=\eta}$.

Correlation functions and transfer matrix eigenvalue

$$H^{\mathsf{XYZ}} = -\frac{1}{2}\sum_{j=1}^{L}\Big(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z\Big)\,.$$

Gives

$$\varepsilon = \langle H^{\mathsf{XYZ}}\rangle = -\frac{L}{2}\left(J_xC^x + J_yC^y + J_zC^z\right).$$

Taking derivatives in η and τ, these will not hit C^x, C^y, C^z (Hellmann–Feynman theorem).
Gives a system of three equations for the three correlations.

Solution can be expressed in terms of the quantity $t\,t_{u\eta} - t_u t_\eta\big|_{u=\eta=\pi/3}$.

The η-derivatives are problematic!

Q-operator

A Q-operator $\mathbf{Q}(u) = \mathbf{Q}(u, \eta, \tau)$ should satisfy $[\mathbf{Q}(u), \mathbf{T}(v)] = 0$,

$$\mathbf{T}(u)\mathbf{Q}(u) = \phi(u - \eta)\mathbf{Q}(u + 2\eta) + \phi(u + \eta)\mathbf{Q}(u - 2\eta),$$

where $\phi(u) = \theta_1(u|\tau)^L$ (Jacobi theta function).

If $\mathbf{Q}(u)\Psi = q(u)\Psi$ we get

$$t(u)q(u) = \phi(u - \eta)q(u + 2\eta) + \phi(u + \eta)q(u - 2\eta).$$

Using this tq-relation we can express correlations in terms of q.

Problem

We need to differentiate

$$t(u)q(u) = \phi(u-\eta)q(u+\eta) + \phi(u+\eta)q(u-2\eta)$$

in η at $\eta = \pi/3$.

Problem: Known constructions of Q-operators either work for $\eta \neq \pi/3$ (Baxter) or $\eta = \pi/3$ (Fabricius, Roan).

Assumption: The tq-relation has a solution q that is analytic in η near $\pi/3$.

Our results are derived rigorously using this assumption.

Connection to s_n, $\bar{s}_n$

After a computation, all η-derivatives cancel.
Can express correlations in terms of

$$q(0)\left(q''\left(\frac{\pi}{3}\right)+\frac{\phi'}{\phi}\left(\frac{\pi}{3}\right)q'\left(\frac{\pi}{3}\right)\right)-q''(0)q\left(\frac{\pi}{3}\right)$$

at $\eta=\pi/3$.

The polynomials s_n and $\bar{s}_n$ can also be expressed in terms of $q(u)$.
For instance, $\bar{s}_n$ is essentally $q'(\pi+\pi\tau/2)$.

To relate these expressions we use a new differential-difference equation for $q(u)\big|_{\eta=\pi/3}$.

Difference-differential equation

Let

$$\psi = \frac{\theta_1(u|\tau)^n}{\theta_1(3u|3\tau)\theta_3(3u/2|3\tau/2)}\, q(u).$$

Then

$$\psi_{uu} - V\psi = \alpha\psi + \beta\frac{\theta_4(3u/2|3\tau/2)^2}{\theta_3(3u/2|3\tau/2)^2}\,\psi(u+\pi),$$

α and β are independent of u,
V is Darboux–Treibich–Verdier potential with parameters

$$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 1, 1\right).$$

This is the main tool to “shift points” in $q(u)$ and complete the proof.

Three special cases

Manin (1998):

respectively twistor geometry and Frobenius manifolds.

Our last remark concerns some similarity between the (generalized) Lamé potentials in the theory of KdV–type equations and our classically integrable potentials of the non–linear equation (2.2). According to [TV], the former are of the form

$$\sum_{j=0}^{3} \frac{n_j(n_j+1)}{2} \wp(z + \frac{T_j}{2}, \tau),$$

whereas according to our discussion the latter have coefficients (proportional to) $(n_j^2)/2$ or $(n_j + \frac{1}{2})^2/2$. Is there a direct connection between the two phenomena?

References

[D] B. Dubrovin. *Geometry of 2D topological field theories.* In: Springer LNM

Triangular numbers

Q-operator eigenvalue satisfies differential-difference equation with parameters

$$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 1, 1\right).$$

Bazhanov and Mangazeev found that it satisfies the QPVI (quantum Painlevé VI) equation

$$\psi_t = \frac{1}{2}\psi_{xx} - V\psi$$

with parameters

$$\left(\frac{n(n+1)}{2}, \frac{n(n+1)}{2}, 0, 0\right).$$

Half squares

Our expression for correlation functions can be interpreted as the Hamiltonian of PVI with parameters

$$\left(\frac{(n+1/2)^2}{2}, \frac{(n+1/2)^2}{2}, 0, 0\right)$$

evaluated at a solution to PVI with parameters

$$\left(\frac{n^2}{2}, \frac{n^2}{2}, 0, 0\right).$$

Manin's three cases appear together!

Questions

- Other correlations? One-point correlations, emptiness formation probability.
- Scaling limit $n \to \infty$, $\tau \to \mathrm{i}\infty$, $e^{\pi \mathrm{i} \tau} \sim n^{-2/3}$.
  Should be related to SUSY sine-Gordon and Painlevé III.
- Conceptual explanation for the relations between XYZ model, QPVI equation and PVI equation?
  Compare "quantum Painlevé–Calogero correspondence" of Zabrodin & Zotov (2012).
- What is the eigenvector Ψ? For the XXZ model, there are explicit integral formulas (Razumov, Stroganov & Zinn-Justin 2007).
  Analogous formulas for XYZ would settle several open problems.