Bethe ansatz solution for a new integrable open quantum system

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+ work in progress

with M. de Leeuw, B. Pozsgay, A. Pribytok, A. L. Retore, P. Ryan

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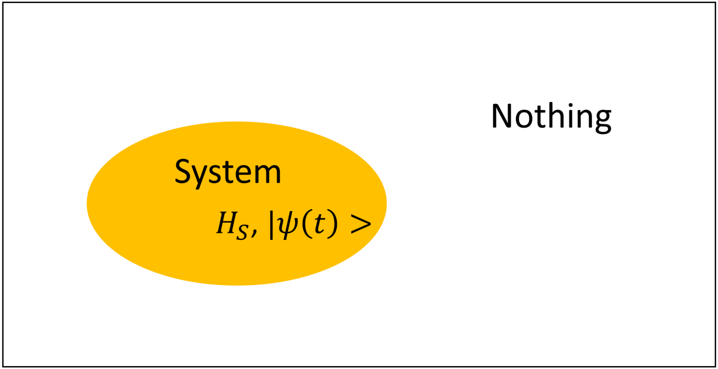
What is an open quantum system?

Closed systems are an idealization of the real ones.

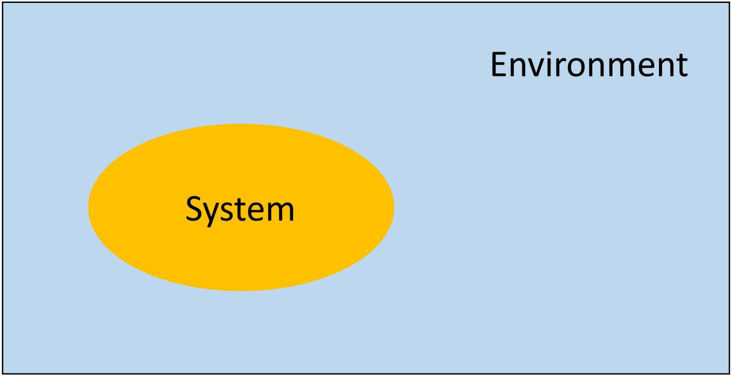


State: pure $|\psi(t)\rangle$
Evolution: Schrödinger equation

$$\frac{d|\psi(t)\rangle}{dt} = -i\, H_S |\psi(t)\rangle$$

What is an open quantum system?

To give a more accurate description of real worlds we need:
Open quantum systems

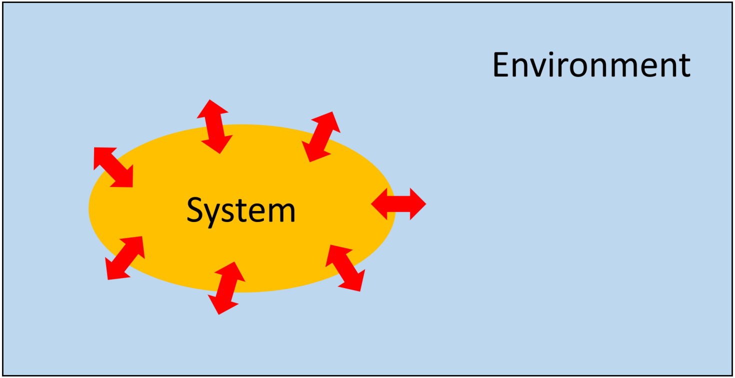


[Manzano, 2020; Manzano, Hurtado, 2018; Petruccione, Breuer, 2002; Medvedyeva, Essler, Prosen, 2016; de Vega (lectures), 2019]

What is an open quantum system?

To give a more accurate description of real worlds we need:

Open quantum systems

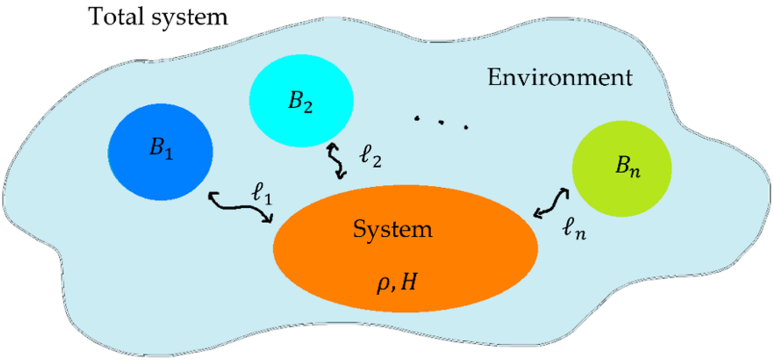


$$\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E, \quad H_T = H_S \otimes 1_E + 1_S \otimes H_E + \alpha H_I.$$

Systems+Environment dynamics:

$$\dot{\rho}(t) = i[H_T, \rho(t)]$$

Lindblad Master equation



$$\dot{\rho}_T(t) = i[H_T, \rho_T(t)]$$

$$\downarrow$$

$$\underbrace{\dot{\rho} = i\,[\rho, H]}_{\text{Liouville equation}} + \underbrace{\sum_{a=1}^{n} \left[\ell_a \rho \ell_a^\dagger - \frac{1}{2}\{\ell_a^\dagger \ell_a, \rho\} \right]}_{\text{Dissipator}}$$

How do we solve the Lindblad master equation?

Hard to solve.

- Numerical methods
- Perturbative methods

Does an exactly solvable model exist?

Different meaning of solvability, we focus on:
Yang Baxter Integrable Lindblad systems

Reasons: The out of equilibrium dynamics can be studied:

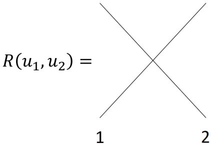
- the Non-Equilibrium steady states can be constructed with exact methods,
- the generator of the dynamics can be diagonalized.

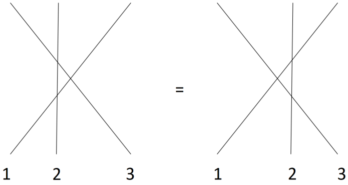
What do we mean by: Yang-Baxter integrable system?

Models with a high amount of symmetry and a tower of conserved charges

$$[\mathbb{H}, \mathbb{Q}_r] = [\mathbb{Q}_r, \mathbb{Q}_s] = 0, \quad r, s = 1, 2, \ldots, \infty$$

generated by the R-matrix.





Yang-Baxter equation

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2)$$

What do we mean by: Yang-Baxter integrable Lindblad system?

Super-operator formalism:

$$\underbrace{\dot{\rho} = i\,[\rho, H]}_{\text{Liouville equation}} + \underbrace{\sum_{a=1}^{n} \left[\ell_a \rho \ell_a^\dagger - \frac{1}{2}\{\ell_a^\dagger \ell_a, \rho\} \right]}_{\text{Dissipator}}$$

$$\downarrow$$

$$\dot{\rho} \equiv \mathcal{L}\rho$$

Lindblad superoperator, $\mathcal{L} = \sum_j \mathcal{L}_{j,j+1}$

$$\mathcal{L}_{i,j} = -i\, h_{i,j}^{(1)} + i h_{i,j}^{(2)*} + \ell_{i,j}^{(1)} \ell_{i,j}^{(2)*} - \tfrac{1}{2} \ell_{i,j}^{(1)\dagger} \ell_{i,j}^{(1)} - \tfrac{1}{2} \ell_{i,j}^{(2)\,T} \ell_{i,j}^{(2)*}$$

We require $\mathcal{L}$ to be one of the conserved charge of an integrable model.

Yang-Baxter integrable superoperator

Set up: Spin $1/2$ chain of length L

Idea: Identify $\mathcal{L}$ as a (non-Hermitian) Hamiltonian

$$H^{SL}_{i,j} = \mathcal{L}_{i,j} = -i\, h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j}\ell^{(2)*}_{i,j} - \frac{1}{2}\ell^{(1)\dagger}_{i,j}\ell^{(1)}_{i,j} - \frac{1}{2}\ell^{(2)\,T}_{i,j}\ell^{(2)*}_{i,j}$$

Require that $\mathcal{L}$ is one of the conserved charge of the integrable model

$$Q_2 = \mathcal{L}$$

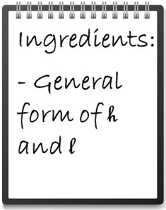
We focus on:

Construction of new integrable models.

Construction of new integrable models

$$\mathcal{L}_{i,j} = -i\, h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j}\ell^{(2)*}_{i,j} - \frac{1}{2}\ell^{(1)\dagger}_{i,j}\ell^{(1)}_{i,j} - \frac{1}{2}\ell^{(2)T}_{i,j}\ell^{(2)*}_{i,j}$$

[de Leeuw, CP, Pribytok, Retore, Ryan]



Boost
automorphism
mechanism

New integrable
superoperators!

How do we find **integrable** $\mathcal{L}$?

Regular solutions $R(u,u) = P$ of the Yang-Baxter equation

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2)$$

R generates a tower of commuting conserved charges $Q_1, Q_2, Q_3 \ldots$

Bottom-up approach:

Boost automorphism mechanism

de Leeuw, Pribytok, Ryan 2019
de Leeuw, Pribytok, Retore, Ryan 2020
de Leeuw, CP, Pribytok, Retore, Ryan 2020

Steps:

1) Traditionally Q_2 is the Hamiltonian, in our case $Q_2 = \mathcal{L}$ (depends on the entries of ℓ and h).
2) Use the boost operator $B[Q_2]$ to construct Q_3.
3) Solve a set of differential equations derived from $[Q_2, Q_3] = 0$ for the components of $\mathcal{L}$ (h and ℓ). *Potentially*, $\mathcal{L}$ is integrable.
4) Compute the R from $\mathcal{L}$ by solving (YBE)' with boundaries $R(u,u) = P$, $P\dot{R}(u,u) = \mathcal{L}(u)$.

New model: Model B3

$$h = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \ell = \sqrt{\frac{\gamma}{2}}\begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & i(\gamma-1)e^{i\phi} & 0 \\ 0 & -i(\gamma+1)e^{-i\phi} & -1 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$

$\phi \in \mathbb{R}, \quad \gamma \in \mathbb{R}^+$ coupling constant

$h_{j,j+1} = \frac{1}{2}\left[e^{i\phi}\sigma_j^+\sigma_{j+1}^- + e^{-i\phi}\sigma_j^-\sigma_{j+1}^+\right]$: *free-fermion* hopping model

$$\ell_{j,j+1} = \sqrt{\frac{\gamma}{2}}[i(\gamma-1)e^{i\phi}\sigma_j^+\sigma_{j+1}^- - i(\gamma+1)e^{-i\phi}\sigma_j^-\sigma_{j+1}^+ + \frac{\gamma}{2}\left(\sigma_{zj}\sigma_{zj+1}+1\right) + \frac{1}{2}(\sigma_{zj}-\sigma_{zj+1})]$$

$$\mathcal{L}_{i,j} = -i\, h_{i,j}^{(1)} + i h_{i,j}^{(2)*} + \ell_{i,j}^{(1)}\ell_{i,j}^{(2)*} - \frac{1}{2}\ell_{i,j}^{(1)\dagger}\ell_{i,j}^{(1)} - \frac{1}{2}\ell_{i,j}^{(2)T}\ell_{i,j}^{(2)*}$$

It can be shown that $[\mathcal{L}, Q_3] = [\mathcal{L}, Q_4] = \cdots = 0$

Can we be sure that the model is integrable?

The R-matrix is: R_i^j j-row and i-column

$$
\begin{aligned}
&R_1^1 = R_6^6 = R_{11}^{11} = R_{16}^{16} = 1,\\
&\frac{R_2^2}{e^{i\phi}} = -e^{i\phi}R_3^3 = -e^{i\phi}R_8^8 = \frac{R_{12}^{12}}{e^{i\phi}} = \frac{i(\tanh(\psi)+1)}{\coth(u-\psi)+\tanh(\psi)},\\
&-e^{i\phi}R_5^5 = \frac{R_9^9}{e^{i\phi}} = \frac{R_{14}^{14}}{e^{i\phi}} = -e^{i\phi}R_{15}^{15} = \frac{i(\tanh(\psi)-1)}{\coth(u-\psi)+\tanh(\psi)},\\
&R_5^2 = R_2^5 = R_{12}^{15} = R_{15}^{12} = R_{14}^{8} = R_8^{14} = R_3^9 = R_9^3 = \mathrm{sech}(u)\cosh(\psi),\\
&e^{i\phi}R_7^4 = -\frac{R_{10}^4}{e^{i\phi}} = \frac{R_{10}^{13}}{e^{i\phi}} = -e^{i\phi}R_7^{13} = -i\,\mathrm{sech}(u)\cosh(\psi)\sinh(u-\psi)\mathrm{sech}(u+\psi),\\
&-e^{i\phi}R_4^7 = \frac{R_4^{10}}{e^{i\phi}} = ie^{2\psi}\mathrm{sech}(u)\cosh(\psi)\sinh(u-\psi)\mathrm{sech}(u+\psi),\\
&e^{i\phi}R_{13}^7 = -\frac{R_{13}^{10}}{e^{i\phi}} = ie^{-2\psi}\mathrm{sech}(u)\cosh(\psi)\sinh(u-\psi)\mathrm{sech}(u+\psi),\\
&R_7^{10} = R_{10}^7 = \frac{1}{2}\mathrm{sech}(u)(\cosh(\psi)+\cosh(3\psi))\mathrm{sech}(u+\psi),\\
&e^{2i\phi}R_7^7 = \frac{R_{10}^{10}}{e^{2i\phi}} = \frac{R_4^4}{e^{2i\phi}} = \frac{R_{13}^{13}}{e^{-2i\phi}} = \tanh(u)\sinh(u-\psi)\mathrm{sech}(u+\psi),\\
&R_{13}^4 = -e^{u-\psi}\mathrm{sech}(u)\cosh(\psi)(\sinh(u-\psi)\mathrm{sech}(u+\psi)-1),\\
&R_4^{13} = e^{\psi-u}\mathrm{sech}(u)\cosh(\psi)(\sinh(u-\psi)\mathrm{sech}(u+\psi)+1).
\end{aligned}
$$

where $u \to \frac{u}{\frac{1}{2}(\gamma^2+1)}$ and $\gamma = \tanh(\psi)$

How do we solve this model?

The R-matrix is a 16×16 matrix. [Martins, Ramos, '97; Arutyunov, de Leeuw, Suzuki, Torrielli, '09]

Basis of the Hilbert space:

$$|0\rangle \qquad |\uparrow\rangle \qquad |\downarrow\rangle \qquad |\uparrow\downarrow\rangle$$

We use: Nested algebraic Bethe ansatz technique.

[Levkovich-Maslyuk '16]

Starting point: chain of length L

$$T_a(u) = \prod_{i=1}^{L} R_{ai}(u_i - u) = \begin{pmatrix} T_{00} & B_1 & B_2 & B_3 \\ C_1 & T_{11} & T_{12} & T_{13} \\ C_2 & T_{21} & T_{22} & T_{23} \\ C_3 & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Integrable model: RTT relation

$$R_{12}(v-u)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(v-u)$$

$$T(u) = \begin{pmatrix} T_{00}(u) & B_1(u) & B_2(u) & B_3(u) \\ C_1(u) & T_{11}(u) & T_{12}(u) & T_{13}(u) \\ C_2(u) & T_{21}(u) & T_{22}(u) & T_{23}(u) \\ C_3(u) & T_{31}(u) & T_{32}(u) & T_{33}(u) \end{pmatrix}$$

From the RTT relation:

$$[B_i(u), B_i(v)] = 0, \quad i = 1, 2, 3,$$

B_i are creation operators

but...

$$B_\alpha(u) B_\beta(v) = B_\delta(v) B_\gamma(u) r_{\alpha\,\beta}^{\gamma\,\delta} - \epsilon_{\alpha,\beta}\, \eta\, (B_3(v) T_{00}(u) - B_3(u) T_{00}(v))$$

Reference state: $\underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{L \text{ times}}$

$$B_1|0\rangle = |\uparrow\rangle, \qquad B_2|0\rangle = |\downarrow\rangle, \qquad B_3|0\rangle = |\uparrow\downarrow\rangle$$

Main idea of the Algebraic Bethe Ansatz

$$T_a(u) = \begin{pmatrix} T_{00} & B_1 & B_2 & B_3 \\ C_1 & T_{11} & T_{12} & T_{13} \\ C_2 & T_{21} & T_{22} & T_{23} \\ C_3 & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$t(u) = tr_a \, T_a(u) = \sum_{i=0}^{3} T_{ii}(u)$$

$t(u)$ generates all the charges: $Q_\alpha = \partial_u^{\alpha-1} \log t(u)|_{u\to 0}$

Eigenvalues of $t(u) \to$ eigenvalues of Qs

How do we construct the eigenvalues of $t(u)$?

How do we construct the eigenvalues of $t(u)$?

One particle state:

$$|\psi\rangle_{1p} = F^a B_a(v_a)|0\rangle = \#|\uparrow\rangle + \#|\downarrow\rangle$$

We want to find $\lambda(u)$

$$t(u)|\psi\rangle_{1p} = \sum_{i=0}^{3} T_{ii}(u)|\psi\rangle_{1p} = \lambda(u)|\psi\rangle_{1p}$$

We need to know:

1. How $T_{ii}(u)$ commute with $B_a(v_a)$?
2. What is the action of $T_{ii}(u)$ on $|0\rangle \otimes \cdots \otimes |0\rangle$?

2. $T_{ii}(u)|0\rangle = \prod_{i=1}^{L} f_i(u)|0\rangle$

How $T_{ii}(u)$ commute with $B_a(v_a)$?

$$T_{00}(u)B_\alpha(v) = \theta_\alpha(u,v)B_\alpha(v)T_{00}(u) + \dots$$
$$T_{\alpha\alpha'}(u)B_\beta(v) = \alpha_\alpha(u,v)B_\gamma(v)T_{\alpha\tau}(u)r^{\tau\gamma}_{\alpha'\beta}(v-u) + \dots$$
$$T_{33}(u)B_\alpha(v) = \zeta_{1,\alpha}(u,v)B_\alpha(v)T_{33}(u) + \dots$$

"$\dots$" depends on operators and r is a the R-matrix of an integrable model.

We want

$$t(u)|\psi\rangle_{1p} = \lambda(u)|\psi\rangle_{1p}$$

But we got:

$$t(u)|\psi\rangle_{1p} = \lambda(u)|\psi\rangle_{1p} + \dots$$

$|\psi\rangle_{1p} = F^a B_a(v_a)|0\rangle$ is an eigenstate only for some values of v_a!

How do we find the values of the rapidities?

$$t(u)|\psi\rangle_{1p} = \lambda(u)|\psi\rangle_{1p} + \underbrace{\ldots}_{=0}$$

$$\downarrow$$

$$t(u)|\psi(v_1)\rangle_{1p} = \lambda(u)|\psi(v_1)\rangle_{1p}$$

$v_1 \;\rightarrow\;$ Bethe root solution of

$\underbrace{\ldots}_{=0} \;\rightarrow\;$ Bethe equations

We need to remember that:

$$T_{\alpha\alpha'}(u)B_\beta(v) = \alpha_\alpha(u,v)B_\gamma(v)\,T_{\alpha\tau}(u)r^{\tau\gamma}_{\alpha'\beta}(v-u) + \ldots \tag{1}$$

r gives the nested structure.

What happens if we consider M particles?

$M = \#$ particles, $N = \#$ particles of type B_1

$$|\psi\{v_i\}\rangle_{M,N} = \underbrace{\underbrace{B_1(v_1)\dots B_1(v_N)}_{N} B_2(v_{N+1})\dots B_2(v_M)}_{M} |0\rangle$$

commuting $T_{11}(u) + T_{22}(u)$ through all the Bs, using

$$T_{\alpha\alpha'}(u) B_\beta(v) = \alpha_\alpha(u,v) B_\gamma(v) T_{\alpha\tau}(u) r^{\tau\gamma}_{\alpha'\beta}(v-u) + \dots$$

reintroducing explicit matrix products, one find

$$G_a r_{a1}(v_1 - u)\dots r_{aN}(v_n - u) \sim T_{6V}(u)$$

twisted monodromy matrix!

To find all the Bethe equations, one needs to solve the "nested" model.

$$G_a \sim \begin{pmatrix} \left(-e^{2i\phi}\right)^{\frac{M-L}{2}} & 0 \\ 0 & \left(-e^{2i\phi}\right)^{\frac{L-M}{2}} \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2i\phi} b(u) & a(u) & 0 \\ 0 & a(u) & e^{2i\phi} b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $a(u) \to \sinh(2\psi)\operatorname{csch}(2(u+\psi)),\ b(u) \to \sinh(2u)\operatorname{csch}(2(u+\psi))$

Putting all the results together:

$$t(u)|\psi\rangle_{M,N} = \lambda(u)|\psi\rangle_{N,M} + \underbrace{\dots}_{=0}$$

Eigenvalue

$$\frac{\lambda(u)}{(-1)^{M-N} e^{M\psi + i\phi(M-2N)}} = \prod_{i=1}^{M} \frac{i\cosh\left(u - v_i + \psi\right)}{\sinh\left(v_i - u\right)} +$$
$$\prod_{i=1}^{M} i\cosh(\psi)\left(\coth\left(u - v_i\right) - \tanh(\psi)\right) \prod_{j=1}^{L} - \frac{ie^{-\psi - i\phi}\sinh(u+\psi)}{\cosh(u)}\, \Lambda_{6-V}(u) +$$
$$\prod_{i=1}^{M} \frac{i\cosh\left(u - v_i - 2\psi\right)}{\sinh\left(u - v_i - \psi\right)} \prod_{i=1}^{L} \frac{e^{-2\psi}\tanh(u)\sinh(u+\psi)}{\cosh(u-\psi)},$$

$$\Lambda_{6-V}(u) = \prod_{i=1}^{N} \frac{\sinh\left(2\left(w_i - u - \psi\right)\right)}{\sinh\left(2\left(u - w_i\right)\right)} + \left(-e^{2i\phi}\right)^{L} \prod_{i=1}^{N} \frac{\sinh\left(2\left(u - w_i - \psi\right)\right)}{\sinh\left(2\left(w_i - u\right)\right)} \prod_{i=1}^{M} \frac{\sinh\left(2\left(v_i - u\right)\right)}{\sinh\left(2\left(u - v_i - \psi\right)\right)}$$

Bethe equations:

For the rapidities $\{v\}$ in the main chain:

$$\prod_{i=1,i\neq j}^{M} -\frac{\cosh(v_j - v_i + \psi)}{\cosh(v_j - v_i - \psi)} =$$

$$\prod_{i=1}^{L} -\frac{ie^{-\psi - i\phi}\sinh(v_j + \psi)}{\cosh(v_j)} \prod_{i=1}^{N} \frac{\sinh(2(w_i - v_j - \psi))}{\sinh(2(v_j - w_i))}$$

and for the rapidities $\{w\}$ of the nested chain

$$\prod_{i=1,i\neq j}^{N} \frac{\sinh(2(w_i - w_j))\sinh(2(w_i - w_j - \psi))}{\sinh(2(w_j - w_i))\sinh(2(w_j - w_i - \psi))} = \left(-e^{2i\phi}\right)^{L} \prod_{i=1}^{M} \frac{\sinh(2(v_i - w_j))}{\sinh(2(w_j - v_i - \psi))}$$

Conclusions and future work

What we saw today:

- Boost method to classify new integrable models: classify R-matrices (possible to do ABA)
- Classify new integrable models of Lindblad type
- Take the most physically interesting model and find the eigenvalue and the Bethe equations

Some of the possible future directions:

- Analyse the possible Bethe/Gauge correspondence of Nekrasov-Shatashvili for model B3
- Can the boost automorphism mechanism be applied to partially classify Bethe/Gauge correspondence?
- Numerical study of leading decay modes describing the relaxation at late times

Conclusions and future work

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Thank you!

Back up slides:

Example: How do we use the Boost automorphism mechanism?

Step 1. Initial ansatz:

$$h_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h_1 \sigma^+ \otimes \sigma^- + h_2 \sigma^- \otimes \sigma^+$$

$$\ell_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sigma^- \otimes \sigma^+ + \frac{\lambda}{4} (1 - \sigma^z) \otimes (1 + \sigma^z)$$

Superoperator $\mathcal{L}$

$$\mathcal{L}_{i,j} = -i\, h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j} \ell^{(2)*}_{i,j} - \frac{1}{2} \ell^{(1)\dagger}_{i,j} \ell^{(1)}_{i,j} - \frac{1}{2} \ell^{(2)T}_{i,j} \ell^{(2)*}_{i,j}$$

Identify $\mathbb{Q}_2 = \sum_{i=1}^4 \mathcal{L}_{i,i+1},\ \mathcal{L}_{L,L+1} = \mathcal{L}_{L,1}$.

Step 2 Construct $\mathbb{Q}_3$

$$\mathbb{Q}_3 = \sum_{i=1}^{4} [\mathcal{L}_{i-1,i}, \mathcal{L}_{i,i+1}] = \sum_{i=1}^{4} \mathcal{Q}_{i,i+1,i+2}$$

Straightforward but cumbersome! $\mathbb{Q}_3$ depends on h_1, h_2 and λ.

Step 3 Plugging $\mathbb{Q}_2$ and $\mathbb{Q}_3$ into the integrability condition

$$[\mathbb{Q}_2, \mathbb{Q}_3] = 0$$

we get:

$$\lambda + 2ih_1 = 0, \qquad \lambda^* - 2ih_2 = 0, \qquad |\lambda|^2 = 1$$

so

$$h_{i,i+1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i\,e^{i\phi} & 0 \\ 0 & -i\,e^{-i\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \ell_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Potentially integrable $\mathcal{L}$!
We prove integrability by finding the R.

Step 4

Reasonable ansatz for the R (understand non-zero entries and some relations between them)
using

$$[R_{13}R_{23}, \mathcal{L}_{12}(u)] = \partial_u R_{13}R_{23} - R_{13}\partial_u R_{23}\,,$$

$R_{i,j} = R_{i,j}(u)$.

We found that the non-zero entries of the R-matrix are:

$r_a^b = r_{\text{row}}^{\text{column}}$

$$
\begin{aligned}
& r_1^1 = r_6^6 = r_{11}^{11} = r_{16}^{16} = r_4^{13} = 1, \\
& e^{-i\phi} r_2^2 = e^{i\phi} r_3^3 = e^{i\phi} r_8^8 = e^{-i\phi} r_{12}^{12} = 2\, i\, e^{\frac{v-u}{2}} \sin \frac{v-u}{2}, \\
& r_2^5 = r_5^2 = r_3^9 = r_9^3 = r_{14}^8 = r_8^{14} = r_{15}^{12} = r_{12}^{15} = e^{\frac{v-u}{2}}, \\
& r_{13}^4 = r_7^{10} = r_{10}^7 = 1 - r_4^4 = e^{v-u}, \\
& e^{i\phi} r_7^4 = e^{-i\phi} r_{10}^4 = 2\, i\, e^{v-u} \sin \frac{v-u}{2}
\end{aligned}
$$

This is actually a new model!

Can we do the Bethe Ansatz?