Quadratic irrelevant deformations

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June 28, 2022
IMB, Dijon

2d story based on 1903.07606 and 1907.02516
with Márk Mezei (SCGP, Stony Brook)

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[...] it is not clear now whether any RFT exists with another type of UV behavior

Zamolodchikov, *From tricritical Ising to critical Ising by thermodynamic Bethe ansatz,* Nucl.Phys.B 358 (1991) 524-546

Renormalizable QFT:

- free fields (or CFT) in the UV
- relevant interactions
- RG flow towards the IR

  $\rightarrow$ gapped TQFT
  $\rightarrow$ free fields
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- 2d action $S = \int \left(D\Phi\overline{D}\Phi + \Phi^3 \right) d^2x\, d^2\theta$ flows to $\mathcal{M}_4$
  adding $g \int \Phi d^2x\, d^2\theta$ breaks supersymmetry, flows to free fermion ($\mathcal{M}_3$)
  [Kastor, Martinec, Shenker]

$$S_{\text{eff}} = \int \left[g^2 + \psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi} + 8g^{-2} \underbrace{\psi\partial\psi}_{T} \underbrace{\overline{\psi\partial\psi}}_{\overline{T}} + \ldots \right] d^2x$$

Effective field theory:

$$S_{\text{eff}} = S_{\text{ren.}} + \underbrace{\sum_{\mathcal{O}_i \text{ irrelevant}} \lambda_i \int \mathcal{O}_i(x)\, d^2x}$$

$\rightarrow$ UV divergences
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However, sometimes there is a nice UV completion

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1. Generalities on $T\overline{T}$
2. Deformations by current bilinears
   - How energy levels vary
   - Deformed conserved currents: operators A_s^t
   - How charges vary: main evolution equation
3. Two studies
   - Study I: KdV charges under $T\overline{T}$ flow
   - Study II: super-Hagedorn in Lorentz-breaking flow
4. Work in progress: $d > 2$

$T\overline{T}$ operator [Zamolodchikov, 2004]

Universal irrelevant operator (in translation-invariant 2d QFTs)

$$\text{“}T\overline{T}\text{”} = \det T = T_{00}T_{11} - T_{01}T_{10} = T\overline{T} - \Theta\overline{\Theta} \quad (\times 2?)$$

More precisely, $\epsilon^{\mu\nu}T_{0\mu}(x)T_{1\nu}(y) = (T\overline{T})(y) +$ **derivatives**.

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Factorization of matrix elements on $S^1 \times \mathbb{R}$ of circumference L,

$$\langle n|T\overline{T}|n\rangle = \epsilon^{\mu\nu}\langle n|T_{0\mu}|n\rangle\langle n|T_{1\nu}|n\rangle$$

$T\overline{T}$ deformation

[Smirnov–Zamolodchikov, 2016]

Deforming by $\partial_{\lambda_{T\overline{T}}} S = \int d^2x\, T\overline{T}$ namely $\partial_{\lambda_{T\overline{T}}} H = \int \mathrm{d}x\, T\overline{T}(x)$

- preserves symmetries
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Related to Jackiw–Teitelboim gravity (Dubovsky, Gorbenko, ...), 2d random geometry (Cardy), AdS_3 holography (McGough, Mezei, Verlinde, Giveon, Kutasov, Guica, ...)

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Derivation of Burgers equation

$$\partial_{\lambda_{T\overline{T}}} E_n = \langle n|\partial_{\lambda_{T\overline{T}}} H|n\rangle = L \underbrace{\langle n|T_{00}|n\rangle}_{-E_n/L} \underbrace{\langle n|T_{11}|n\rangle}_{-\partial_L E_n} - L \underbrace{\langle n|T_{01}|n\rangle}_{iP_n/L} \underbrace{\langle n|T_{10}|n\rangle}_{iP_n/L}$$

In a relativistic theory: $-E_n/L$, $-\partial_L E_n$, iP_n/L, iP_n/L

$$\boxed{\partial_{\lambda_{T\overline{T}}} E_n = E_n \partial_L E_n + \frac{P_n^2}{L}}$$

(needs either Lorentz-invariance or $P_n = 0$)

Current bilinears

Generalize $T\overline{T}$, $J\overline{T}$, $J\overline{J}$

$X_{ab} := \epsilon_{\mu\nu} J_a^\mu J_b^\nu$ (point-split) defined modulo derivatives

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Proof.

$$\frac{\partial}{\partial x^\rho} \epsilon_{\mu\nu} J_a^\mu(x) J_b^\nu(y) = \left(\frac{\partial}{\partial x^\nu} + \frac{\partial}{\partial y^\nu}\right) \epsilon_{\mu\rho} J_a^\mu(x) J_b^\nu(y)$$

use OPE

$$\epsilon_{\mu\nu} \sum_i \partial_\rho c_i(x-y) O_i^{\mu\nu}(y) = \epsilon_{\mu\rho} \sum_i c_i(x-y) \partial_\nu O_i^{\mu\nu}(y)$$

so any O_i with non-constant $c(x-y)$ must be a total derivative $\partial_\nu(\dots)$ □

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$$\partial_{\lambda^{ab}} S = \int \mathrm{d}^2 x\, X_{ab} \text{ deformation}$$

Only makes sense if J_a and J_b are still conserved at order $O(\lambda)$ etc.

This happens if and only if $[Q_a, Q_b] = 0$ (see later for "if" direction)

Evolution of energies under deformation by current bilinears

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On $S^1 \times \mathbb{R}$ of circumference L, **factorization**

$$\langle n | X_{ab} | n \rangle = \epsilon_{\mu\nu} \langle n | J^\mu_a | n \rangle \langle n | J^\nu_b | n \rangle$$

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$$\partial_{\lambda^{ab}} E_n = 2 \underbrace{L \langle n | J_{[a}^0 | n \rangle}_{(Q_a)_n} \langle n | J_{b]}^1 | n \rangle$$

- Compact flavour symmetry $\implies$ Q_n quantized
- Spatial translation $\implies$ $Q_n = iP_n \in (2\pi i / L)\mathbb{Z}$
- Time translation $\implies$ $Q_n = -E_n$
- KdV charges $\implies$ $\boxed{\text{need } \partial_\lambda Q_n \text{ equation}}$

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Strategy to study $\partial_{\lambda^{ab}} E_n = 2\langle n|Q_{[a}|n\rangle\langle n|J^1_{b]}|n\rangle$

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Then, for $\langle n|J^1_c|n\rangle$, two case studies (much shorter)

Study I: $T\overline{T}$ deformation of Lorentz-invariant theory,
KdV charges "ride the Burgers flow"

Study II: $T_{1\bullet}J_\bullet$ deformation of zero-momentum sector
super-Hagedorn density of states $\exp(E^{(>1)})$

Cartan subalgebra: KdV charges P_s

Focus on **commuting subset** $\{P_s\}$ of all charges $\{Q_a\}$:
translations, Cartan of flavour symmetries, KdV charges

Conserved currents $\overline{\partial} T_{s+1} = \partial \Theta_{s-1}$ of spin $s \in \mathbb{Z}$, charges

$$P_s = \frac{1}{2\pi} \oint \big(T_{s+1} \mathrm{d}z + \Theta_{s-1} \mathrm{d}\overline{z} \big)$$

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$[P_1, \mathcal{O}] = -i\partial\mathcal{O}$ and $[P_{-1}, \mathcal{O}] = i\overline{\partial}\mathcal{O}$ with $P_{\pm 1} = -\frac{1}{2}(H \pm P)$

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Example (CFT): $T_2 = T$, $T_4 = {:}T^2{:}$, $T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}, \dots$ $\Theta_{-2k} = \overline{T_{2k}}$,
$\Theta_0 = \Theta_2 = \Theta_4 = \cdots = T_0 = T_{-2} = T_{-4} = \cdots = 0$

KdV currents fixed (up to improvements) by spin and $[P_s, P_t] = 0$

The operators A_s^t

Integrating $[P_s, T_{t+1}\mathrm{d}z+\Theta_{t-1}\mathrm{d}\bar{z}]$ on a contour $\mathcal{C}$ gives $[P_s, P_t^{\mathcal{C}}]=0$ so the one-form is exact:

$$[P_s, T_{t+1}] = -i\partial A_s^t = [P_1, A_s^t]$$
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In particular $\boxed{A_1^t = T_{t+1}}$ and $\boxed{A_{-1}^t = -\Theta_{t-1}}$ (up to shifts by identity)

Generic A_s^t are **not** in conserved currents

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$$A_1^1 = T_2 \qquad A_1^3 = T_4 \qquad A_1^5 = T_6$$
$$A_3^1 = 3T_4 + \partial(\dots) \qquad A_3^3 = 4{:}T^3{:} - \frac{c+2}{2}{:}(\partial T)^2{:}$$
$$A_5^1 = 5T_6 + \partial(\dots) \qquad A_5^3 = \frac{15{:}T^4{:}}{2} - \frac{5(13+2c){:}T(\partial T)^2{:}}{3} + \frac{5(-47+4c+c^2){:}(\partial^2 T)^2{:}}{72}$$

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Proof.

$$\begin{aligned} & [P_1, [P_{[s}, A_{t]}^u]] \\ &= [P_{[s|}, [P_1, A_{|t]}^u]] && \text{(Jacobi)} \\ &= [P_{[s}, [P_{t]}, A_1^u]] && \text{(definition of } A) \\ &= 0 && \text{(Jacobi)} \end{aligned}$$

Likewise $[P_{-1}, [P_{[s}, A_{t]}^u]] = 0$

so $[P_{[s}, A_{t]}^u] =$ multiple of identity $= 0$ (because traceless) □

Deforming by current bilinears preserves symmetries

For two spins u, v consider $\delta H = \int \mathrm{d}x\, X^{u,v}$

with $X^{u,v} = (T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1})_{\text{reg}}$ current bilinear

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Proof. $[P_s, X^{u,v}] = [P_s, T_{u+1}\Theta_{v-1} - \Theta_{u-1}T_{v+1}]$

$$= [P_1, A_s^u]\Theta_{v-1} + [P_{-1}, A_s^u]T_{v+1} - (u \leftrightarrow v)$$

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$= [P_1, A_s^u]\Theta_{v-1} + [P_{-1}, A_s^u]T_{v+1} - (u \leftrightarrow v)$
$= [P_1, A_s^u\Theta_{v-1}] + [P_{-1}, A_s^u T_{v+1}] - (u \leftrightarrow v)$ □

$$\boxed{\delta P_s = \frac{1}{2}\int \mathrm{d}x\,(X_{s,1}^{u,v} + X_{-1,s}^{u,v})}$$ where $X_{s,t}^{u,v} = (A_s^u A_t^v - A_t^u A_s^v)_{\text{reg}}$

Toward an evolution equation

Goal: $\partial_\lambda \langle n|P_s|n\rangle = \langle n|\partial_\lambda P_s|n\rangle = \dots$ for states $|n\rangle$ on $S^1 \times \mathbb{R}$

We've just seen $\partial_\lambda P_s = \frac{1}{2}\int \mathrm{d}x\,(X^{u,v}_{s,1} + X^{u,v}_{-1,s})$ so we compute

$\langle n|X^{u,v}_{s,t}|n\rangle =$

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Proof summary. Insert complete set of states (eigenstates of all $P_\bullet$)

$$\langle n|X^{u,v}_{s,t}|n\rangle = \sum_{|m\rangle}\Big(\langle n|A^u_s|m\rangle\langle m|A^v_t|n\rangle - \langle n|A^u_t|m\rangle\langle m|A^v_s|n\rangle\Big)$$

For any spin r, compute a bit to show

$$\langle n|[P_r, A^u_s]|m\rangle\langle m|A^v_t|n\rangle - \langle n|[P_r, A^u_t]|m\rangle\langle m|A^v_s|n\rangle = 0$$

This is $\langle m|P_r|m\rangle - \langle n|P_r|n\rangle$ times the summand,
so summand $= 0$ except for $|m\rangle = |n\rangle$ **(assumes nondegenerate spectrum)** □

Side comment on collisions

In fact we can define more **general collisions**

$$k! A^{t_1}_{[s_1}(x_1) \dots A^{t_k}_{s_k]}(x_k) = X^{t_1,\dots,t_k}_{s_1,\dots,s_k}(x) + \sum_i [P_{s_i}, \dots]$$

- defined up to commutators $\sum_i [P_{s_i}, \dots]$ (like $X^{u,v}$ is defined up to derivatives)
- obey factorization

$$\langle n | X^{t_1,\dots,t_k}_{s_1,\dots,s_k} | n \rangle = k! \langle n | A^{t_1}_{[s_1} | n \rangle \dots \langle n | A^{t_k}_{s_k]} | n \rangle$$

- obey

$$[P_{[s_0}, X^{t_1,\dots,t_k}_{s_1,\dots,s_k]}] = 0$$

(but deforming by these operators breaks all symmetries, so they are most likely not that useful)

Main evolution equation

Denoting $\langle \mathcal{O} \rangle := \langle n | \mathcal{O} | n \rangle$, we end up with

$$\boxed{2\partial_{\lambda_{u,v}} \langle P_s \rangle = \langle P_u \rangle \langle A_s^v \rangle - \langle P_v \rangle \langle A_s^u \rangle}$$

Sadly, $\partial_{\lambda_{u,v}} \langle A_s^t \rangle =$ nothing in general

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Study I: $T\overline{T}$ deformation $(u, v) = (1, -1)$
Lorentz-invariance relates $\langle A_s^1 \rangle \sim \langle A_1^s \rangle \sim \langle P_s \rangle$
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Study II: $T_{1\bullet}J_\bullet$ deformation (difference of $u=\pm 1$, arbitrary v)
In zero-momentum sector $\langle A_s^v\rangle$ drops out
Get **super-Hagedorn density of states** $\exp(\gg E)$

1. Generalities on $T\overline{T}$
2. Deformations by current bilinears
   - How energy levels vary
   - Deformed conserved currents: operators A_s^t
   - How charges vary: main evolution equation
3. Two studies
   - Study I: KdV charges under $T\overline{T}$ flow
   - Study II: super-Hagedorn in Lorentz-breaking flow
4. Work in progress: $d > 2$

Study I: Deforming by $T\overline{T}$

$$2\partial_{\lambda_{T\overline{T}}}\langle P_s\rangle = \langle P_1\rangle\langle A_s^{-1}\rangle - \langle P_{-1}\rangle\langle A_s^{1}\rangle$$

Need to understand $A_s^{\pm 1}$. Two steps.

- Understand ∂_L
- Relate $A_s^{\pm 1}$ to $A_{\pm 1}^{s}$ in Lorentz-invariant theories

Changing the length

We know $\partial_L H = \int \mathrm{d}x\, T_{xx} = \frac{1}{2\pi} \int \mathrm{d}x \left(A^1_1 - A^{-1}_1 + A^1_{-1} - A^{-1}_{-1}\right)$

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Use conservation $[H, \partial_L P_s] = [P_s, \partial_L H]$ to deduce

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In fact, for zero momentum ($\langle P_1 - P_{-1}\rangle = 0$),

$$\partial_\lambda \langle P_s\rangle = \langle Q\rangle \partial_L \langle P_s\rangle \quad \text{under } \epsilon^{\mu\nu} J_\mu T_{x\nu} \text{ deformation}$$

The deformation "scales space according to $\langle Q\rangle$"

Relating A^t_s and A^s_t

Relating A_s^t and A_t^s

Example (CFT): $T_2 = T$, $T_4 = {:}T^2{:}$, $T_6 = {:}T^3{:} + \frac{c+2}{12}{:}(\partial T)^2{:}$

$$\begin{array}{lll} A_1^1 = T_2 & A_1^3 = T_4 & A_1^5 = T_6 \\ A_3^1 = 3T_4 + \partial(\dots) & A_3^3 = \dots & A_3^5 = \frac{3}{5}A_5^3 + \dots \\ A_5^1 = 5T_6 + \partial(\dots) & A_5^3 = \dots & \end{array}$$

Relating A_s^t and A_t^s

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Observe $t\,A_s^t = s\,A_t^s$ up to improvements of currents $T_4,\ T_6,\ \dots$
This selects preferred improvements of higher-spin currents:
$T_{s+1} = \frac{1}{s}A_s^1$ is uniquely defined (up to shifts by the identity)

More generally true in Lorentz-invariant theories

Evolution of KdV charges under $T\overline{T}$ deformation

Combining (up to factors)

$$\langle n|A_s^1 - A_s^{-1}|n\rangle = \partial_L \langle n|P_s|n\rangle$$
$$\langle n|A_s^1 + A_s^{-1}|n\rangle = \frac{s}{L}\langle n|P_s|n\rangle$$

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we get the linear equation

$$\partial_\lambda \langle P_s\rangle = \langle H\rangle \partial_L \langle P_s\rangle + \frac{s}{L}\langle P\rangle\langle P_s\rangle$$

All charges propagate along the same characteristics

Starting from a CFT we can solve

$$\langle P_s\rangle = \begin{cases} \#\langle P_1\rangle^s & \text{for holomorphic currents} \\ \#\langle P_1\rangle^{-s} & \text{for antiholomorphic currents} \end{cases}$$

Study II: zero-momentum sector

In the $T\overline{T}$ deformation, for zero-momentum states, [Cardy]

$$\partial_{\lambda_{T\overline{T}}} E_n = \langle n|\partial_{\lambda_{T\overline{T}}} H|n\rangle = L\underbrace{\langle n|T_{00}|n\rangle}_{-E_n/L}\underbrace{\langle n|T_{11}|n\rangle}_{-\partial_L E_n} - L\underbrace{\langle n|T_{01}|n\rangle}_{=0}\underbrace{\langle n|T_{10}|n\rangle}_{\text{who cares?}}$$

Lorentz-invariance not used!

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Our variant: deform by $X^{1,u} - X^{-1,u}$ so

$$\frac{1}{\pi}\partial_\lambda \langle P_s\rangle = \langle P_1 - P_{-1}\rangle\langle A_s^u\rangle - \langle P_u\rangle\langle A_s^1 - A_s^{-1}\rangle$$

One has $\langle A_s^1 - A_s^{-1}\rangle = -2\pi\partial_L\langle P_s\rangle$, so for zero-momentum states,

$$\partial_\lambda\langle P_s\rangle = 2\pi^2\langle P_u\rangle\partial_L\langle P_s\rangle \quad \text{if } \langle P_1 - P_{-1}\rangle = 0$$

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Starting from a CFT, spectrum is exactly solvable.
Asymptotic density of states $\rho_{\text{CFT}}(E) = \exp(\sim \sqrt{E})$ becomes

$$\rho(E) = \exp(\sim E^{(|u|+1)/2})$$

For $u = 0$ ($J\overline{T}$ deformation) get Cardy growth with a different coefficient
For $u = \pm 1$ ($T\overline{T}$ deformation) get Hagedorn behaviour $\sum e^{-\beta E}$ blows up at β_c
For $|u| > 1$ completely new behaviour, arbitrarily strong

Work in progress: $d > 2$

Continuous q-form symmetries $d \star J^{(q+1)} = 0$ (standard case: $q = 0$)

Gauge theory $U(1)$ on $\mathbb{R}^D$ → "electric" 1-form symmetry ($J = F$)
→ "magnetic" $(D-3)$-form symmetry ($J = \star F$)

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Collision $\star J^{(1)} \wedge \star J^{(2)}$ defined up to derivatives

- Example: $\int d^3x\, \epsilon_{\mu\nu\rho} F^{\mu\nu} J^\rho$ in 3D (with conditions)
- Example: (mixed) theta term $\int F \wedge F$ for 4D $U(1)$ gauge theory
- Analogue of $J\overline{T}$: Lorentz-breaking deformation $\int u_\mu T^{\mu\nu} \partial_\nu \phi$ for some fixed direction u

Factorization works too!

These constructions seem to work in lattice gauge theories too

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Thank you!