The Geometry of Magnificent Four

with J. Rennemo

I saw the sign and it opened up my eyes
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Life is demanding without understanding
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The Sign, Ace of Base

Gromov-Witten, Donaldson-Thomas (...) theory: $[M]^{\mathrm{vir}}$, $\mathcal{O}_M^{\mathrm{vir}}$

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$\rightsquigarrow \iota_*[M]^{\mathrm{vir}} = e(V) \in H_{2\mathrm{vd}}(A)$

where $\mathrm{vd} := \mathrm{rk}(T_M^{\mathrm{vir}}) = \dim(A) - \mathrm{rk}(V)$

Koszul dga: $(\Lambda^\bullet V^*, \lrcorner s, \wedge)$ $\quad h^0(\Lambda^\bullet V^*) = \mathcal{O}_M$

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If $\det(\Omega_M^{\mathrm{vir}})^{\frac{1}{2}}$ exists

Nekrasov-Okounkov twist $\quad \widehat{\mathcal{O}}_M^{\mathrm{vir}} := \mathcal{O}_M^{\mathrm{vir}} \otimes \det(\Omega_M^{\mathrm{vir}})^{\frac{1}{2}}$

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$$\int_{[M]^{\mathrm{vir}}} \alpha, \quad \alpha \in H^*(M, \mathbb{Q}), \quad \chi(M, \mathcal{O}_M^{\mathrm{vir}} \otimes \beta), \quad \beta \in K_0(M)$$

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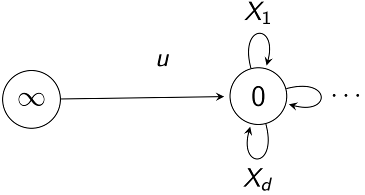
Goal: Apply to Hilbert schemes

$$\mathrm{Hilb}^n(\mathbb{C}^d) = \left\{ Z \subset \mathbb{C}^d : \dim H^0(Z, \mathcal{O}_Z) = n \right\}$$

For any $d > 0$, consider



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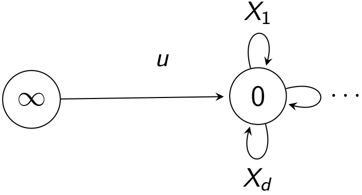

Reps with dimension $(1, n)$

$R := \mathbb{C}^n \oplus \mathrm{End}(\mathbb{C}^n)^{\oplus d}, \quad \mathrm{GL}_n \curvearrowright R$

$(u, X_1, \cdots, X_d) \mapsto (gu, gX_1g^{-1}, \ldots, gX_dg^{-1})$

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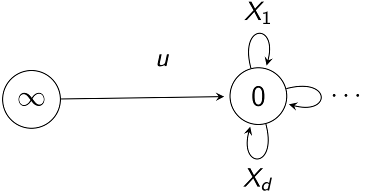

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Open subset $U \subset R$: reps st. $\mathbb{C}\langle X_1, \ldots, X_d \rangle \cdot \langle u \rangle = \mathbb{C}^n$

$\mathrm{GL}_n \curvearrowright U$ is free

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$V|_P = \mathrm{End}(\mathbb{C}^n)^{\binom{d}{2}}$, $s_P = \{[X_i, X_j]\}_{i<j}$, $P = [(u, X_1, \ldots, X_d)]$

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$\boldsymbol{d = 2}$: $\mathrm{vd} = \mathrm{rk}(T_M^{\mathrm{vir}}) = n \neq 2n$

Arbesfeld-Johnson-Lim-Oprea-Pandharipande (...)

$d = 3$

Superpotential: $\Phi : A \to \mathbb{C}$, $\Phi(u, X_1, X_2, X_3) = \mathrm{tr}(X_1[X_2, X_3])$

$\Rightarrow s = d\Phi \in H^0(A, \Omega_A), \quad M = Z(s) \subset A$

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"Local model DT theory *compact* Calabi-Yau 3-folds"

Brav-Bussi-Joyce, Pantev-Toën-Vaquié-Vezzosi

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Question How about $d > 3$?

3-term obstruction theory

Back to general setting... Given data:

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Self-dual: $\Omega_M^{\mathrm{vir}} \cong T_M^{\mathrm{vir}}[2]$

Virtual dim: $\mathrm{vd} = \mathrm{rk}(T_M^{\mathrm{vir}}) = 2\dim(A) - \mathrm{rk}(V) > \dim(A) - \mathrm{rk}(V)$

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BRAV-BUSSI-JOYCE, PANTEV-TOËN-VAQUIÉ-VEZZOSI

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Orientation: $o : \det \Omega_M^{\mathrm{vir}} \xrightarrow{\cong} \mathcal{O}_M$ st. $o \otimes o = D$

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Suppose $\dim(V)$ even, and $\exists\ \Lambda \subset V$ maximal isotropic subbundle

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Note: $e(\Lambda)^2 = (-1)^{\frac{\mathrm{rk}(V)}{2}} e(V)$

$d = 4$

Recall $A := \mathrm{ncHilb}^n(\mathbb{C}^4)$, $\quad M := Z(s) = \mathrm{Hilb}^n(\mathbb{C}^4)$

at $P = [(u, X_1, X_2, X_3, X_4)]$:

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Define quadratic form on V

$q(\{Y_{ij}\}_{i<j}) = \mathrm{tr}(Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23})$

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Take $\Lambda|_P := \bigoplus_{1 \le i < j = 4} \mathrm{End}(\mathbb{C}^n) \rightsquigarrow$ orientation

Standard torus action $(\mathbb{C}^*)^4 \curvearrowright \mathbb{C}^4, \quad T = \{t_1 t_2 t_3 t_4 = 1\} \leq (\mathbb{C}^*)^4$

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$$\mathrm{Hilb}^n(\mathbb{C}^4)^T = \Big\{ I \subset \mathbb{C}[x_1, x_2, x_3, x_4] : \text{monomial ideal, colength } n \Big\}$$

$$= \Big\{ \pi \subset \mathbb{Z}^4_{\geq 0} \text{ 4D partition } : |\pi| = n \Big\}$$

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$[M]^{\mathrm{vir}} \in H^T_{2n}(M, \mathbb{Z}[\tfrac{1}{2}])$, $\quad \widehat{\mathcal{O}}^{\mathrm{vir}}_M \in K^T_0(M, \mathbb{Z}[\tfrac{1}{2}])$

Tautological bundle: $\mathcal{O}^{[n]}|_Z = H^0(Z, \mathcal{O}_Z)$

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Interested in

$$\sum_n \chi(\mathrm{Hilb}^n(\mathbb{C}^3), \widehat{\mathcal{O}}^{\mathrm{vir}})(-q)^n$$

$$\sum_n \chi(\mathrm{Hilb}^n(\mathbb{C}^4), \widehat{\mathcal{O}}^{\mathrm{vir}} \otimes \widehat{\Lambda}_{-1}(\mathcal{O}^{[n]})^* \otimes y)q^n$$

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Method 1. Compute by localization

$d = 3$ Graber-Pandharipande localization

$$T = (\mathbb{C}^*)^3 \curvearrowright M, \qquad T_M^{\mathrm{vir}}|_{M^T} = \underbrace{(T_M^{\mathrm{vir}}|_{M^T})^f}_{=0!} + (T_M^{\mathrm{vir}}|_{M^T})^m$$

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$$\chi(M, \widehat{\mathcal{O}}_M^{\mathrm{vir}}) = \chi\Big(M^T, \frac{\mathcal{O}_{M^T}^{\mathrm{vir}} \otimes \det(\Omega_M^{\mathrm{vir}})^{\frac{1}{2}}|_{M^T}}{\Lambda_{-1}\Omega_M^{\mathrm{vir}}|_{M^T}^m}\Big)$$

$$= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}\Omega_M^{\mathrm{vir}}|_Z)}$$

$$= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}(\Omega_A - T_A)|_Z)}$$

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$$\begin{aligned}\chi(M, \widehat{\mathcal{O}}^{\mathrm{vir}}_M) &= \chi\Big(M^T, \frac{\mathcal{O}^{\mathrm{vir}}_{M^T} \otimes \det(\Omega^{\mathrm{vir}}_M)^{\frac{1}{2}}|_{M^T}}{\Lambda_{-1}\Omega^{\mathrm{vir}}_M|^m_{M^T}}\Big) \\ &= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}\Omega^{\mathrm{vir}}_M|_Z)} \\ &= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}(\Omega_A - T_A)|_Z)}\end{aligned}$$

$Z \leftrightarrow$ 3D partition π

$$\begin{aligned}T^{\mathrm{vir}}_M|_Z &= T_A|_Z - \Omega_A|_Z \\ &= \mathrm{Ext}^1(I_Z, I_Z) - \mathrm{Ext}^2(I_Z, I_Z) =: \mathsf{V}^{\mathrm{DT3}}_\pi\end{aligned}$$

$d = 3$ Graber-Pandharipande localization

$$T = (\mathbb{C}^*)^3 \curvearrowright M, \qquad T_M^{\mathrm{vir}}|_{M^T} = \underbrace{(T_M^{\mathrm{vir}}|_{M^T})^f}_{=0!} + (T_M^{\mathrm{vir}}|_{M^T})^m$$

$$\chi(M, \widehat{\mathcal{O}}_M^{\mathrm{vir}}) = \chi\Big(M^T, \frac{\mathcal{O}_{M^T}^{\mathrm{vir}} \otimes \det(\Omega_M^{\mathrm{vir}})^{\frac{1}{2}}|_{M^T}}{\Lambda_{-1}\Omega_M^{\mathrm{vir}}|_{M^T}^m}\Big)$$

$$= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}\Omega_M^{\mathrm{vir}}|_Z)}$$

$$= \sum_{Z \in M^T} \frac{1}{\mathrm{ch}(\widehat{\Lambda}_{-1}(\Omega_A - T_A)|_Z)}$$

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Notation: $[E] := \mathrm{ch}(\widehat{\Lambda}_{-1}E^*), \quad E \in K_0^T(\mathrm{pt})$

$d = 4$ Oh-Thomas localization

$$\chi(M, \widehat{\mathcal{O}}_M^{\mathrm{vir}} \otimes \widehat{\Lambda}_{-1}(\mathcal{O}^{[n]})^* \otimes y) = \sum_{Z \in M^T} \frac{\mathrm{ch}(\widehat{\Lambda}_{-1} H^0(\mathcal{O}_Z)^* \otimes y)}{\sqrt{(-1)^n \mathrm{ch}(\widehat{\Lambda}_{-1} \Omega_M^{\mathrm{vir}}|_Z)}}$$

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$$T_M^{\mathrm{vir}}|_Z = (T_A - \Lambda^*)|_Z + (\Omega_A - \Lambda)|_Z$$

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$Z \leftrightarrow$ 4D partition π

$$(T_A - \Lambda^*)|_Z - H^0(\mathcal{O}_Z)^* \otimes y =: \mathsf{V}_\pi^{\mathrm{DT4}}$$

For 3D partition π, define $Z_\pi := \sum_{(i,j,k)\in\pi} t_1^i t_2^j t_3^k$

$$\mathsf{V}_\pi^{\mathrm{DT3}} = Z_\pi - \frac{Z_\pi^*}{t_1 t_2 t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} Z_\pi^* Z_\pi$$

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Note: for $y = t_4$

$\pi \not\subset \{x_4 = 0\} \Rightarrow [-\mathsf{V}_\pi^{\mathrm{DT4}}] = 0$

$\pi \subset \{x_4 = 0\} \Rightarrow [-\mathsf{V}_\pi^{\mathrm{DT4}}] = [-\mathsf{V}_\pi^{\mathrm{DT3}}]$ RHS: MNOP, Okounkov

New feature:

$$(-1)^{\tau_\pi} = (-1)^{\dim\ \mathrm{coker}\left(T_A|_Z \xrightarrow{ds} V|_Z \to \Lambda^*|_Z\right)^f} = (-1)^{|\pi| + |\{(i,i,i,j)\in\pi\,:\, i<j\}|}$$

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Measure $(-1)^{\tau_\pi}[-\mathsf{V}_\pi^{\mathrm{DT4}}]$ discovered by physicists
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Conclusion:

$$\sum_n \chi(\mathrm{Hilb}^n(\mathbb{C}^3), \widehat{\mathcal{O}}^{\mathrm{vir}})(-q)^n = \sum_\pi [-\mathsf{V}_\pi^{\mathrm{DT3}}](-q)^{|\pi|}$$

MNOP, OKOUNKOV

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$$\sum_n \chi(\mathrm{Hilb}^n(\mathbb{C}^4), \widehat{\mathcal{O}}^{\mathrm{vir}} \otimes \widehat{\Lambda}_{-1}(\mathcal{O}^{[n]})^* \otimes y)q^n = \sum_\pi (-1)^{\tau_\pi}[-\mathsf{V}_\pi^{\mathrm{DT4}}]q^{|\pi|}$$

K-RENNEMO

Second specializes to first for $y = t_4$

Method 2. Compute by pushing down $\mathrm{Hilb}^n(\mathbb{C}^d) \xrightarrow{\nu_n} \mathrm{Sym}^n(\mathbb{C}^d)$

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Okounkov's factorizability $\rightsquigarrow \exists$ K-theory classes A_n, B_n

$$\sum_n \chi(\mathrm{Hilb}^n(\mathbb{C}^3), \widehat{\mathcal{O}}^{\mathrm{vir}})(-q)^n = \mathrm{Exp}\Big(\sum_{n=1}^{\infty} \chi(\mathbb{C}^3, A_n) q^n\Big)$$

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Plethystic exponential: for $F(z_1, \ldots, z_k) \in \mathbb{Q}[[z_1, \ldots, z_k]]$

$$\mathrm{Exp}(F(z_1, \ldots, z_k)) := \exp\left(\textstyle\sum_{n=1}^{\infty} \frac{1}{n} F(z_1^n, \ldots, z_k^n)\right)$$

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Determine Exponent:

- $d = 3$ hard combinatorics Okounkov
- $d = 4$ determined by $y = t_4$ K-Rennemo

Proves conjectures Nekrasov, Nekrasov-Piazzalunga

Theorem (Okounkov)

$$\sum_{\pi}[-\mathsf{V}_{\pi}^{\mathrm{DT3}}](-q)^{|\pi|} = \mathrm{Exp}\left(\frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]} \frac{1}{[(t_1 t_2 t_3)^{\frac{1}{2}} q][(t_1 t_2 t_3)^{\frac{1}{2}} q^{-1}]} \right)$$

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Theorem (K-Rennemo)

$$\sum_\pi (-1)^{\tau_\pi} [-\mathsf{V}_\pi^{\mathrm{DT4}}] q^{|\pi|} = \mathrm{Exp}\left(\frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3][t_4]} \frac{[y]}{[y^{\frac{1}{2}} q][y^{\frac{1}{2}} q^{-1}]} \right)$$

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Theorem (K-Rennemo)

$$\sum_\pi (-1)^{\tau_\pi} [-\mathsf{V}_\pi^{\mathrm{DT4}}] q^{|\pi|} = \mathrm{Exp}\left(\frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3][t_4]} \frac{[y]}{[y^{\frac{1}{2}} q][y^{\frac{1}{2}} q^{-1}]} \right)$$

Another direction *Magnificent Four with Colour*

... i.e. *upgrade to Quot scheme*

$$\mathrm{Quot}_r^n(\mathbb{C}^4) = \left\{ [\mathcal{O}_{\mathbb{C}^4}^{\oplus r} \twoheadrightarrow Q] \, : \, \dim Q = n \right\}$$

The sign

$P \in M^T$ reduced $\Rightarrow ds : T_A|_P^f \hookrightarrow V|_P^f$ maximal isotropic

compare to $\Lambda|_P^f \subset V|_P^f$ maximal isotropic

$$(-1)^{\tau_Z} = \begin{cases} 1 & \text{if } ds(T_A|_P^f) \text{ positive} \\ -1 & \text{otherwise} \end{cases}$$

classical fact:

$$ds(T_A|_P^f), \Lambda|_P^f \text{ same ori} \Leftrightarrow \dim(ds(T_A|_P^f) \cap \Lambda|_P^f) = \frac{\dim(V|_P^f)}{2} \mod 2$$

$$\rightsquigarrow (-1)^{\tau_\pi} = (-1)^{\dim\ \operatorname{coker}\left(T_A|_P \overset{ds}{\to} V|_P \to \Lambda^*|_P\right)^f}$$

Factorizability

Given: Y smooth T-variety and $\{F_n \in \mathrm{Coh}^T(\mathrm{Sym}^n(Y))\}_{n=1}^{\infty}$

$\forall \lambda = (1^{m_1} 2^{m_2} \cdots) \vdash n$, consider $\prod_k \mathrm{Sym}^{m_k}\mathrm{Sym}^k(Y) \xrightarrow{f_\lambda} \mathrm{Sym}^n(Y)$

define $U \subset \prod_k \mathrm{Sym}^{m_k}\mathrm{Sym}^k(Y)$:

open subset with $x_i \neq x_j$ for x_i, x_j in different groups

factorizability: $f_\lambda^* F_n|_U \cong \boxtimes_k \mathrm{Sym}^{m_k} F_k|_U$ (compatible with subdivision)

Okounkov constructs $\{G_n \in K_0^T(Y)\}_{n=1}^{\infty}$ such that

$$1 + \sum_{n=1}^{\infty} \chi(\mathrm{Sym}^n(Y), F_n)\, q^n = \mathrm{Exp}\Big(\sum_{n=1}^{\infty} \chi(Y, G_n)\, q^n \Big)$$

$F_n := R\nu_{n*}\mathcal{O}^{\mathrm{vir}} \in K_0^T(\mathrm{Sym}^n(\mathbb{C}^2))$: take $n = k_1 + k_2$

consider $\mathrm{Sym}^{k_1}(\mathbb{C}^2) \times \mathrm{Sym}^{k_2}(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$

goal: construct $f_\lambda^* F_n|_U \cong F_{k_1} \boxtimes F_{k_2}|_U$, consider

$A_n := \mathrm{ncHilb}^n(\mathbb{C}^2) \supset \mathrm{Hilb}^n(\mathbb{C}^2) = Z(s_n) =: M_n$, $s_n \in \Gamma(A_n, V_n)$

consider

$$\begin{array}{ccc}
A_{k_1} \times A_{k_2}|_U & \xrightarrow{g_\lambda} & A_n \\
\uparrow & & \uparrow \\
M_{k_1} \times M_{k_2}|_U & \longrightarrow & M_n \\
\downarrow & & \downarrow \\
U & \xrightarrow[\text{étale}]{f_\lambda} & \mathrm{Sym}^n(\mathbb{C}^2)
\end{array}$$

Over $P = [(u_1, X_1, Y_1)] \times [(u_2, X_2, Y_2)] \in M_{k_1} \times M_{k_2}|_U$:

$$(V_{k_1} \boxplus V_{k_2})|_P \subset g_\lambda^* V_n|_P, \begin{pmatrix} \mathrm{End}(\mathbb{C}^{k_1}) & 0 \\ 0 & \mathrm{End}(\mathbb{C}^{k_2}) \end{pmatrix} \subset \mathrm{End}(\mathbb{C}^n)$$

$$s_P = \left[\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \right] = \begin{pmatrix} [X_1, Y_1] & 0 \\ 0 & [X_2, Y_2] \end{pmatrix}$$

$$ds|_P : N_{A_{k_1} \times A_{k_2}/A_n}|_P \xrightarrow{\cong} g_\lambda^*(V_n/(V_{k_1} \boxplus V_{k_2}))|_P$$

$$\Rightarrow g_\lambda^* \Lambda^\bullet V_n|_U \cong \Lambda^\bullet V_{k_1} \boxtimes \Lambda^\bullet V_{k_2} \boxtimes \Lambda^\bullet N_{A_{k_1} \times A_{k_2}/A_n}|_U$$

for $E_n := h^{\mathrm{even}}(\Lambda^\bullet V_n^*) \oplus h^{\mathrm{odd}}(\Lambda^\bullet V_n^*)$ with class $\mathcal{O}_{M_n}^{\mathrm{vir}}$

$$\Rightarrow g_\lambda^* E_n|_U \cong E_{k_1} \boxtimes E_{k_2}|_U, \quad \Rightarrow f_\lambda^* F_n|_U \cong F_{k_1} \boxtimes F_{k_2}|_U$$