Diagrammatic Expansion of Non-perturbative Little String Free Energies

Stefan Hohenegger

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New Trends in Non-Perturbative Gauge/ String Theory and Integrability

(27/June/2022)

Based on: 2011.06323, 1911.08172
2009.00797 (with Amer Iqbal)
1811.03387 (with Brice Bastian)
ongoing work with Baptiste Filoche

Little String Theories

Over the last decades string theory has provided insights into strongly coupled quantum systems

Specifically: prediction of existence of new interacting conformal field theories in dimensions $D > 4$

e.g.: [Seiberg 1996]

String theory
- extended objects
- gravitation

$\xrightarrow[\text{gravity}]{\text{suitable decoupling of}}$

quantum field theory
- point-like degrees of freedom
- well defined energy momentum tensor

String theory also predicts the existence of new 'non-local theories', e.g. little string theories (LSTs)

String theory
- extended objects
- gravitation
- compactification to $D > 4$

$\xrightarrow[\text{gravity}]{\text{suitable decoupling of}}$

little string theory
- intrinsic string scale M_{string} remains

$\ll M_{\text{string}}$: effective QFT (point-like dofs)

$> M_{\text{string}}$: UV-comp. contains stringy dofs

- well defined energy momentum tensor

different approaches: [Witten 1995]
[Aspinwall, Morrison 1997]
[Seiberg 1997]
[Intriligator 1997]
[Hanany, Zaffaroni 1997]
[Brunner, Karch 1997]

Classification of LSTs (ADE type for theories with $\mathcal{N} = (2,0)$ supersymmetry)

[Bhardwaj, Del Zotto, Heckman, Morrison, Rudelius, Vafa 2016]

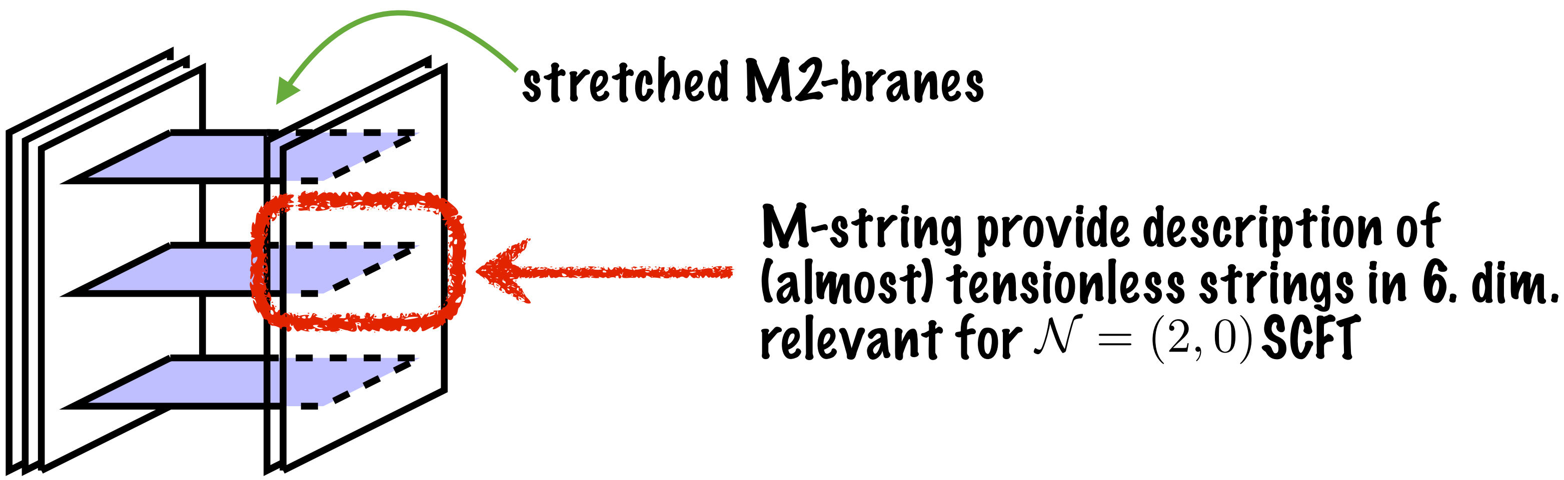
Rich class of examples realised M-theory through branewebs

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozçaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]
[Haghighat 2015]
[SH, Iqbal, Rey 2015]
[Haghighat, Murthy, Vafa, Vandoren 2015]

Parallel M5-branes

brane webs: M-branes arranged in a fashion to preserve (some amount of) supersymmetry

* String-like objects arise at the intersection of M5- and M2-branes



* many dual realisations allowing to explicitly compute quantities (e.g. partition function)

notably: F-theory compactification on toric, non-compact Calabi-Yau threefolds

[Morrison, Vafa 1996]
[Heckman, Morrison, Vafa 2013]
[Del Zotto, Heckman, Tomasiello, Vafa 2014]
[Heckman 2014]
[Haghighat, Klemm, Lockhart, Vafa 2014]
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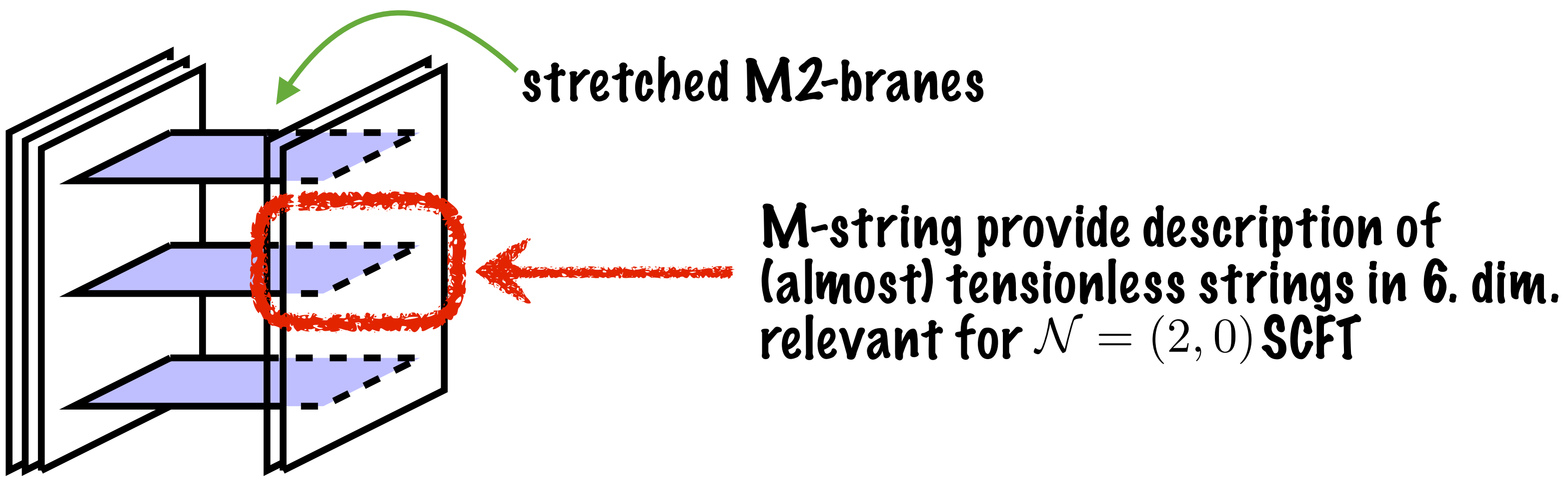

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* depending on the details of the brane configuration, a large class of different Little Strings (or their duals) can be realised and studied very explicitly

* low energy limit associated with non-abelian supersymmetric field theories (mass deformed $\mathcal{N}=2^*$ theories upon compactification to 4 dimensions)

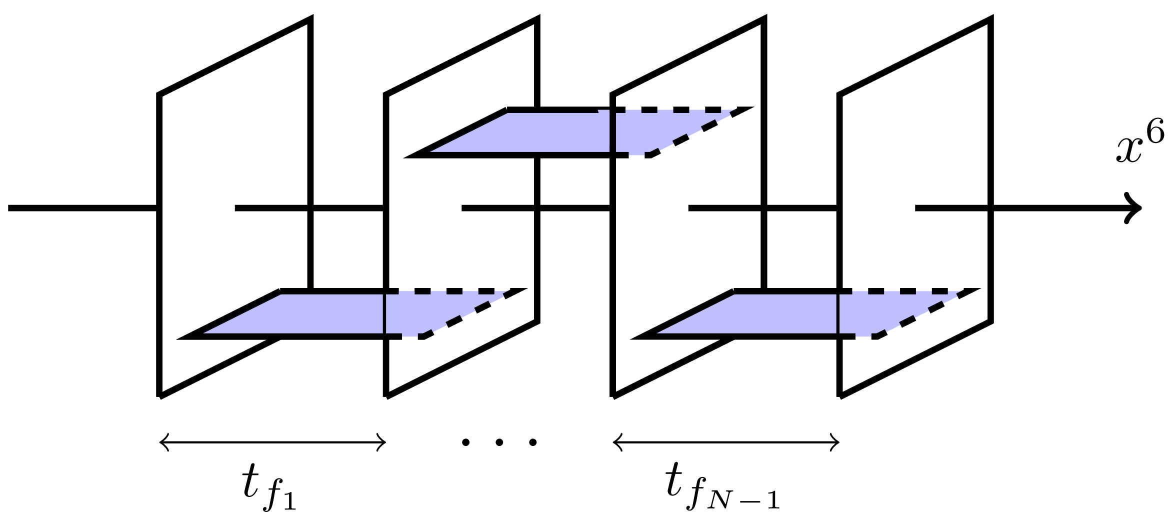
Class of theories exhibits interesting (and non-expected) dualities (trialities)! [Bastian, SH, Iqbal, Rey 2016, 2017, 2018] [Bastian, SH 2018]

Details of the Brane Configurations

	0	1	2	3	4	5	6	7	8	9	10
M5-branes	●	●	●	●	●	●					
M2-branes	●	●					●				

$\underbrace{2 \quad 3 \quad 4 \quad 5}_{\mathbb{R}^4_{||}}$ $\quad$ $\underbrace{7 \quad 8 \quad 9 \quad 10}_{\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M}$

non-compact case: $\mathbb{R}$



leads to CFT on M5-brane world-volume

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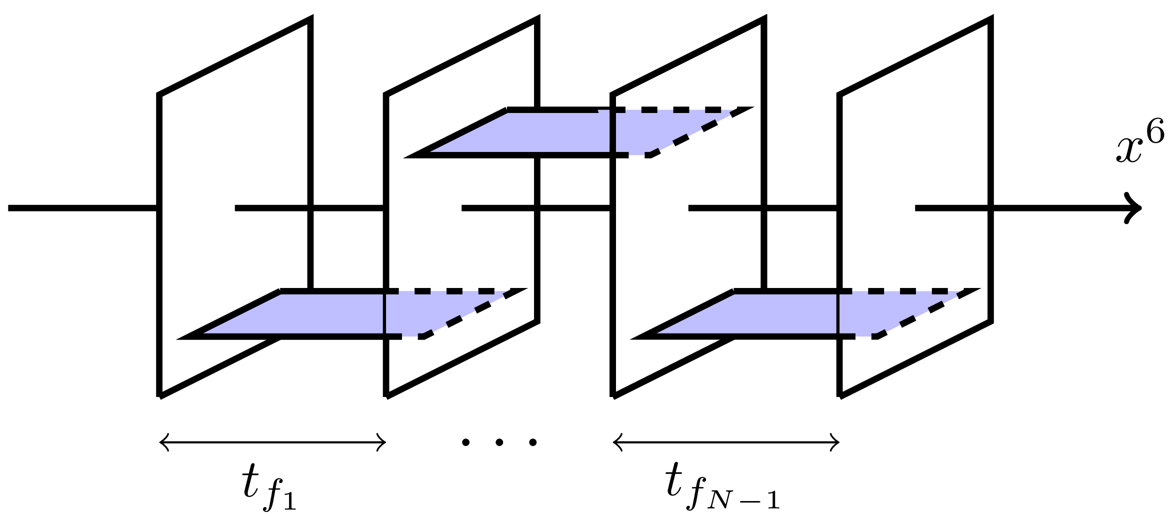
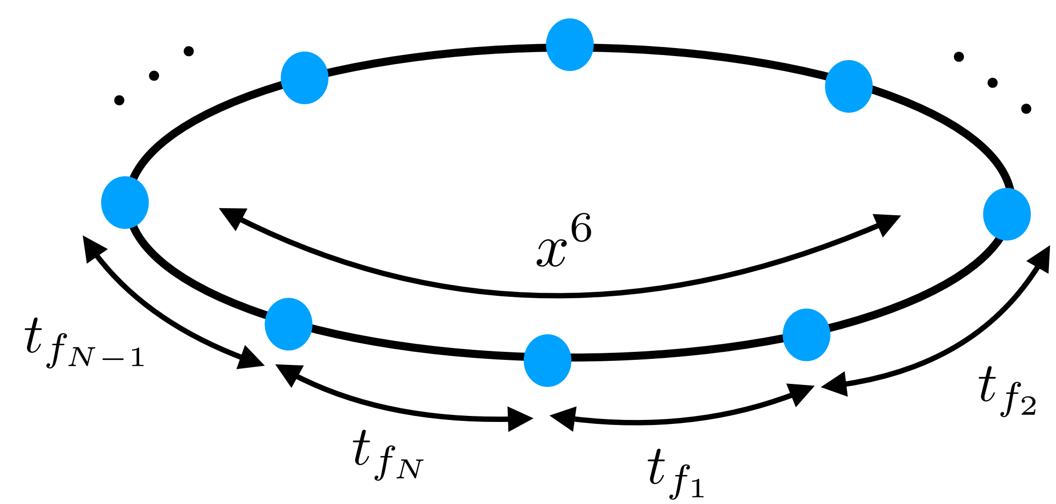
non-compact case: $\mathbb{R}$

compact case: S^1 with radius $R_6 = \dfrac{\rho}{2\pi i}$





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leads to a LST on M5-brane world-volume

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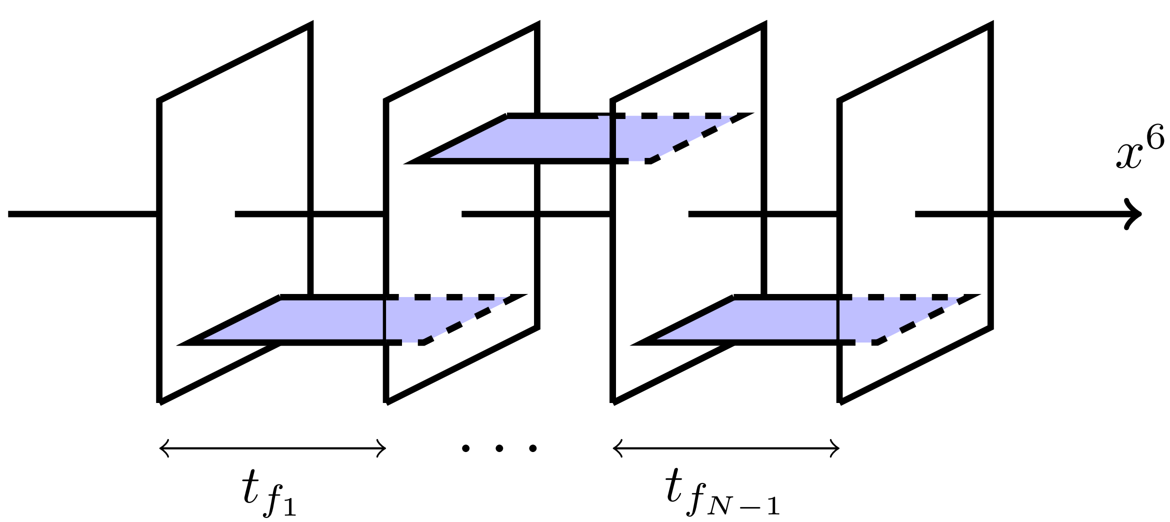
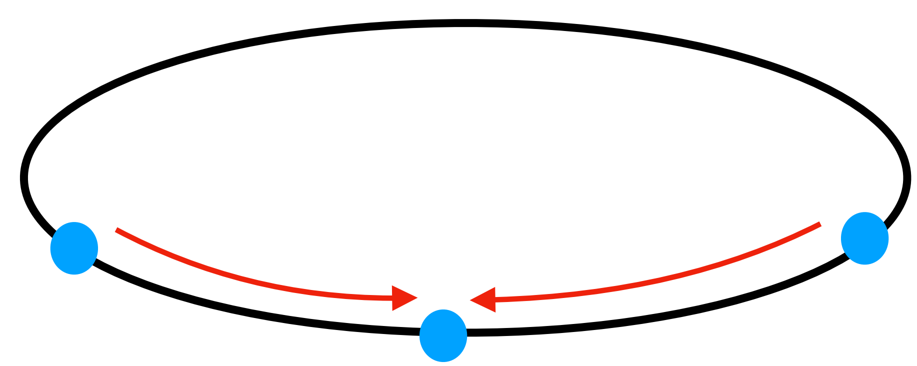


move all M5-branes to one location



still leaves winding modes around R_6

leads to CFT on M5-brane world-volume

leads to a LST on M5-brane world-volume

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M5-branes	●	●	●	●	●	●					
M2-branes	●	●					●				
ϵ_1			○	○				○	○	○	○
ϵ_2					○	○		○	○	○	○

Compactification: Compactify (0,1) to $T^2 \sim S^1 \times S^1$ with radii R_0 and $R_1 =: \frac{\tau}{2\pi i}$

Deformations: there are two types of deformations with respect to the compactified (0,1)-directions introducing complex coordinates $(z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)$ and $(w_1, w_2) = (x_7 + ix_8, x_9 + ix_{10})$

(0)-direct.: $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2}$: $(z_1, z_2) \to (e^{2\pi i\epsilon_1} z_1, e^{2\pi i\epsilon_2} z_2)$ and $(w_1, w_2) \to (e^{-i\pi(\epsilon_1+\epsilon_2)} w_1, e^{-i\pi(\epsilon_1+\epsilon_2)} w_2)$

(1)-direct.: $U(1)_m$: $(w_1, w_2) \to (e^{2\pi i m} w_1, e^{-2\pi i m} w_2)$

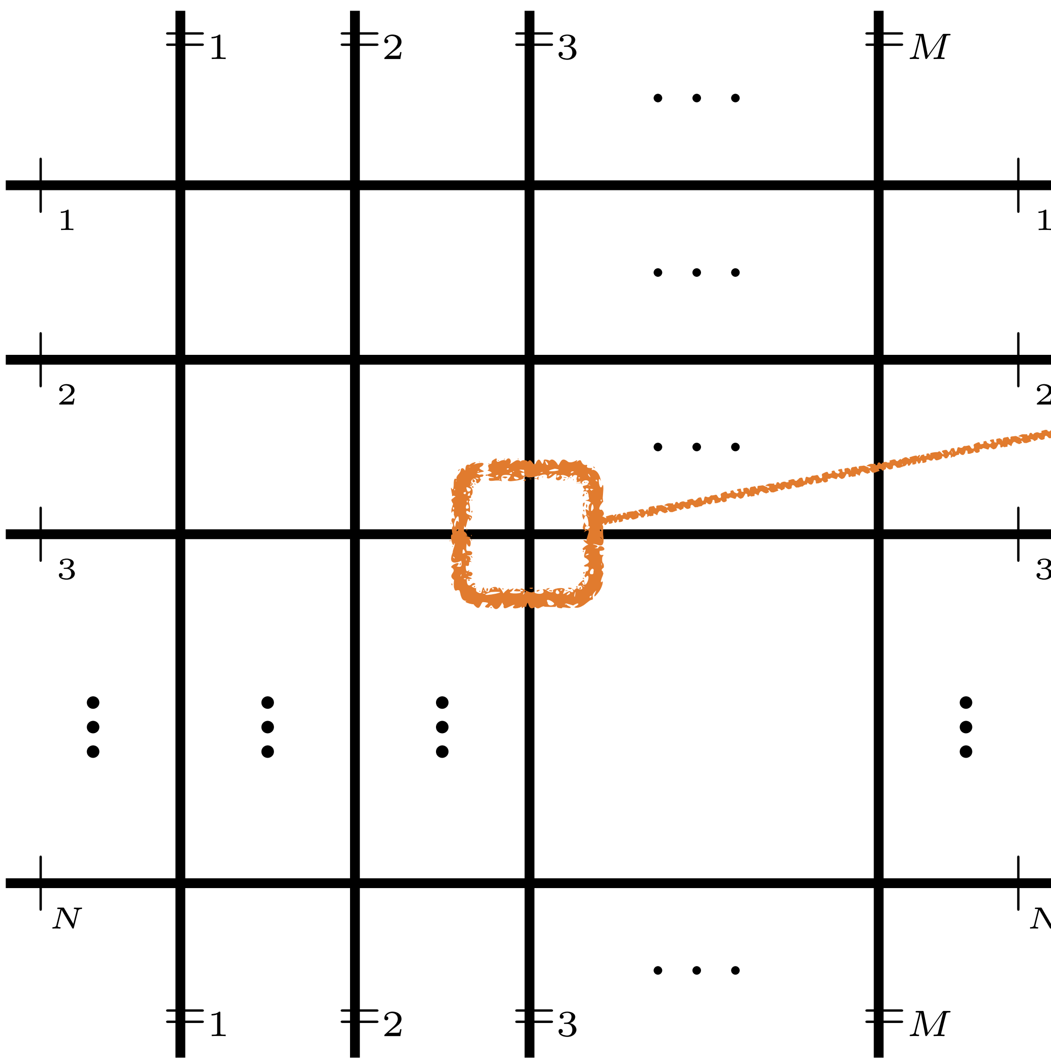
gauge theory: Omega-background [Nekrasov 2012]

mass-deformation

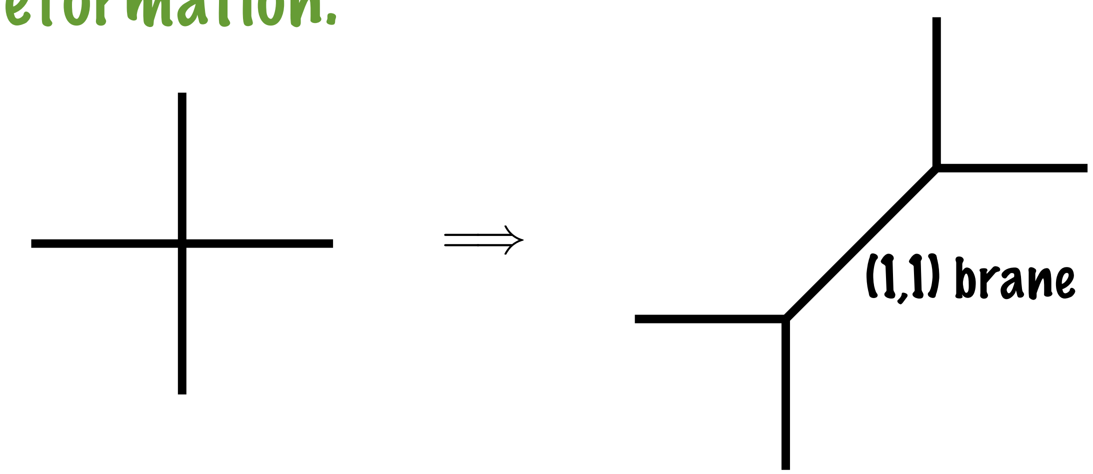
Dual Type II Setup

For vanishing mass deformation ($m=0$) the M-brane configuration is dual to D5-NS5-branes in IIB

	0	1	2	3	4	5	6	7	8	9
D5 branes	•	•	•	•	•	•	–			
NS5 branes	•	•	•	•	•	–	•			
	gauge theory					(p,q)-plane		transverse $\mathbb{R}^3$		



Deformation:



uplift the deformed type II configuration to M-th.
on an elliptically fibered Calabi-Yau threefold $X_{N,M}$

[Leung, Vafa 1997]

topic diagram of $X_{N,M}$ same as deformed brane web

Dual Calabi-Yau 3-fold Description

2-parameter series of toric, double elliptically fibered Calabi-Yau threefolds $X_{N,M}$

Toric Web Diagram:

* (N,M) web on a torus
* double elliptic fibration structure with parameters (ρ, τ)
* $3NM$ different parameters representing the area of various curves C of the CY3

$$d = \int_C \omega$$

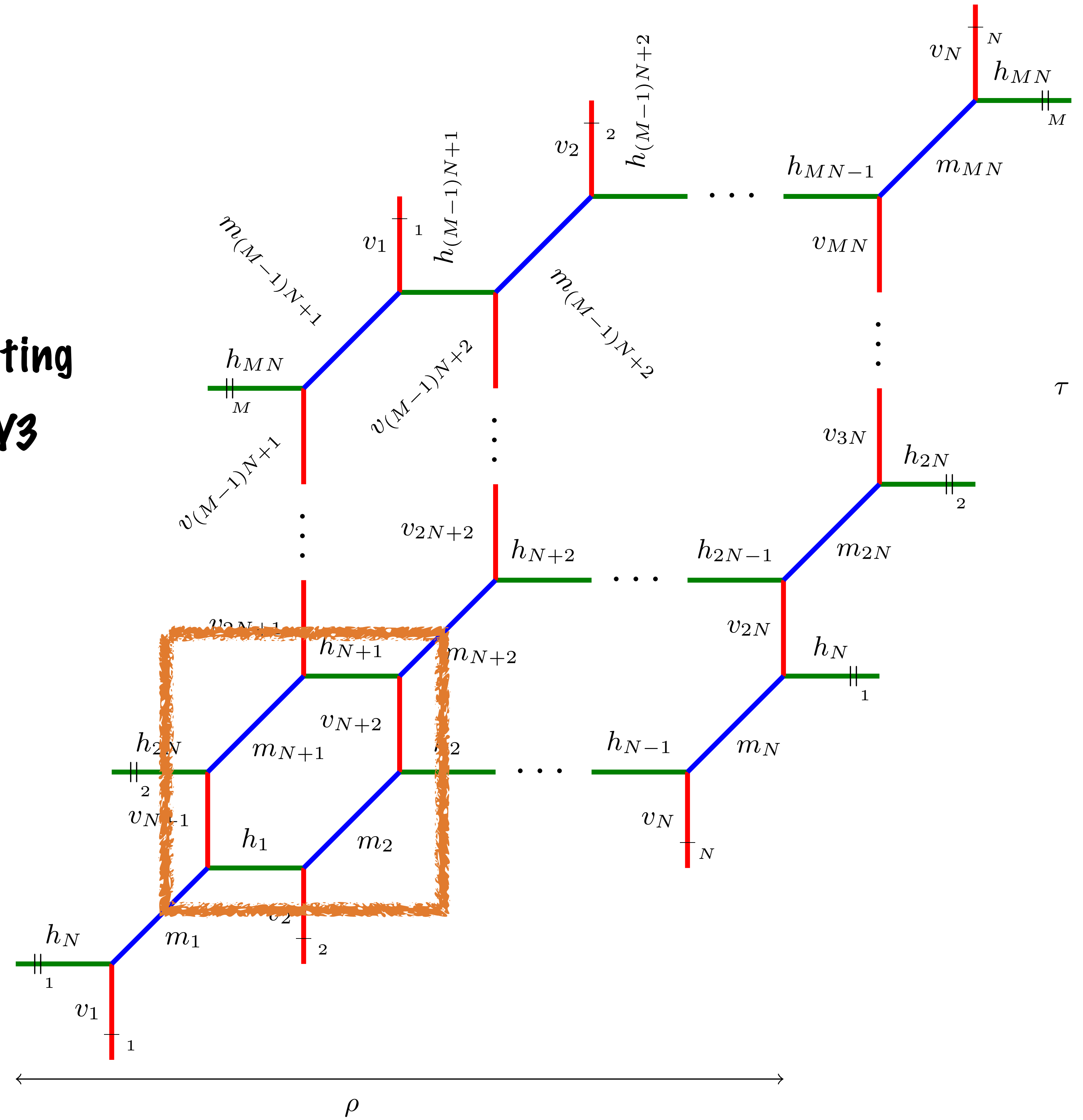
Kähler form

-) NM horizontal lines $h_{1,\dots,NM}$

-) NM vertical lines $v_{1,\dots,NM}$

-) NM diagonal lines $m_{1,\dots,NM}$

* only $NM+2$ independent parameters due to consistency conditions



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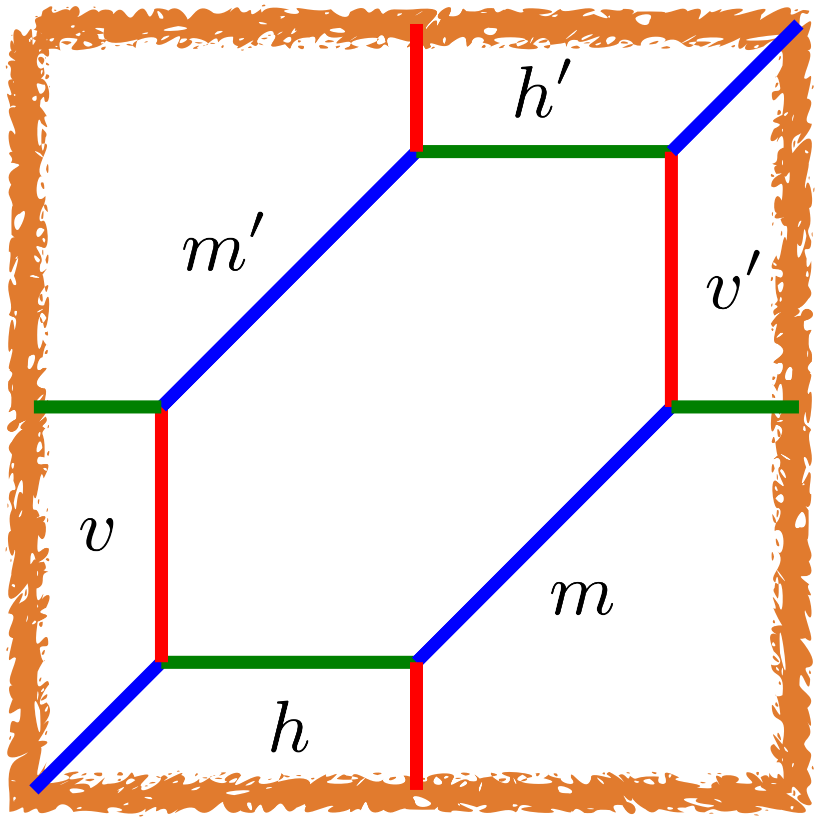


$$h + m = h' + m'$$
$$v + m' = m + v'$$

different possible choices for set of independent parameters

BPS Partition Function

Free Energy: Counts number of BPS configurations, i.e. M2-branes wrapping holomorphic curves on the CY3 $X_{N,M}$. Captured by topological free energy $F_{N,M} = \ln \mathcal{Z}_{N,M}$ of $X_{N,M}$

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozcaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]

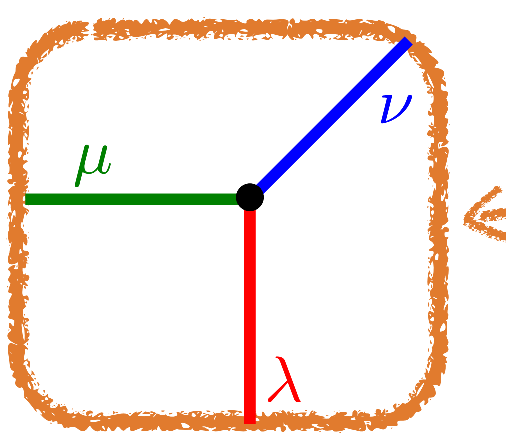
Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the refined topological vertex

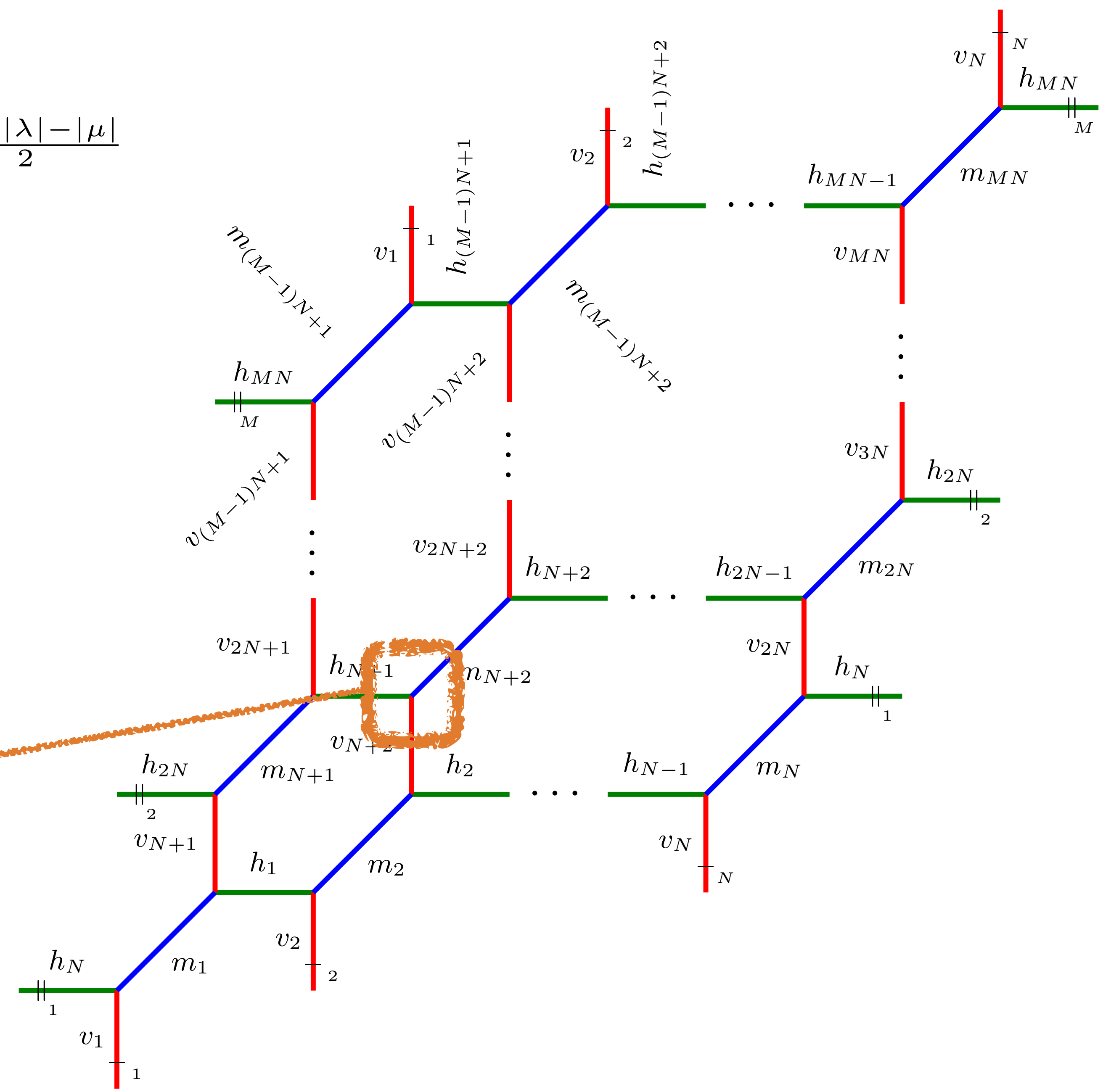
-) assign trivalent vertex to each intersection

$$C_{\lambda\mu\nu} = q^{\frac{||\mu||^2}{2}}\, t^{-\frac{||\mu^t||^2}{2}}\, q^{\frac{||\nu||^2}{2}}\, \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}}$$

$$\times\; s_{\lambda^t/\eta}(t^{-\rho}q^{-\nu})\, s_{\mu/\eta}(q^{-\rho}t^{-\nu^t})$$

$$\tilde{Z}_\nu(t,q) = \prod_{(i,j)\in\nu} \left(1 - t^{\nu_j^t - i + 1} q^{\nu_i - j}\right)^{-1},$$





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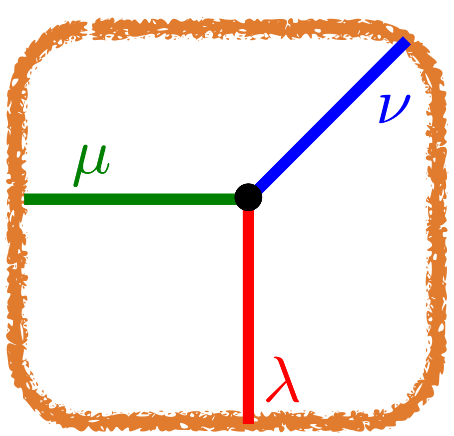
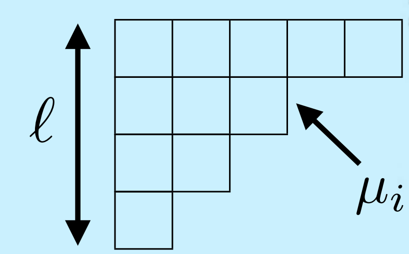


Notation:

$q = e^{2\pi i \epsilon_1}$ and $t = e^{-2\pi i \epsilon_2}$

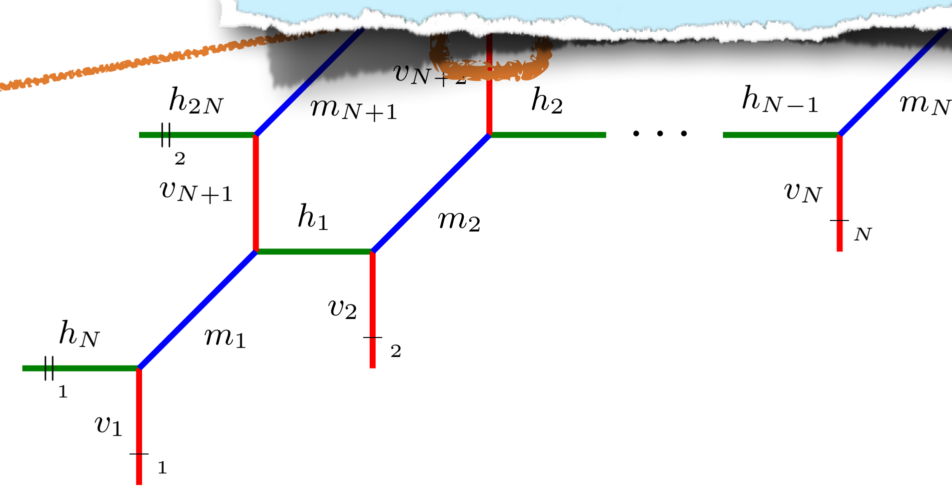
$\mu\,,\nu\,,\lambda$ integer partitions

$$|\mu| = \sum_{i=1}^{\ell} \mu_i$$



$$||\mu||^2 = \sum_{i=1}^{\ell} \mu_i^2$$

$s_{\mu/\eta}$ skew Schur function



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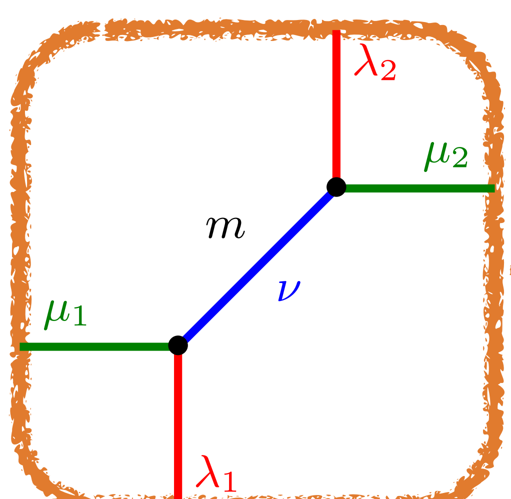
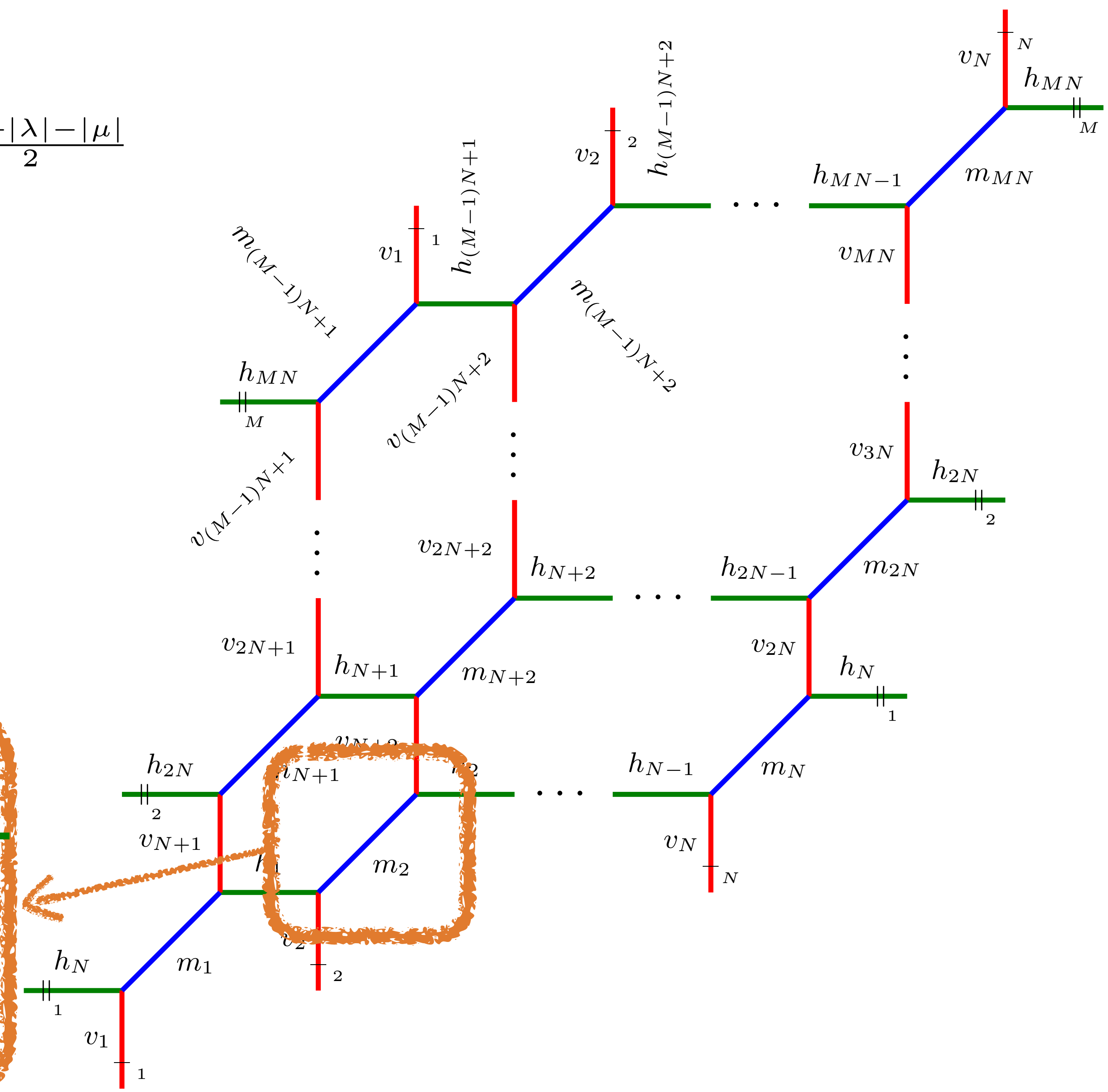
-) glue vertices according to web diagram

$$\sum_\nu (-e^{2\pi i m})^{|\nu|} C_{\mu_1\lambda_1\nu} C_{\mu_2^t\lambda_2^t\nu^t}$$





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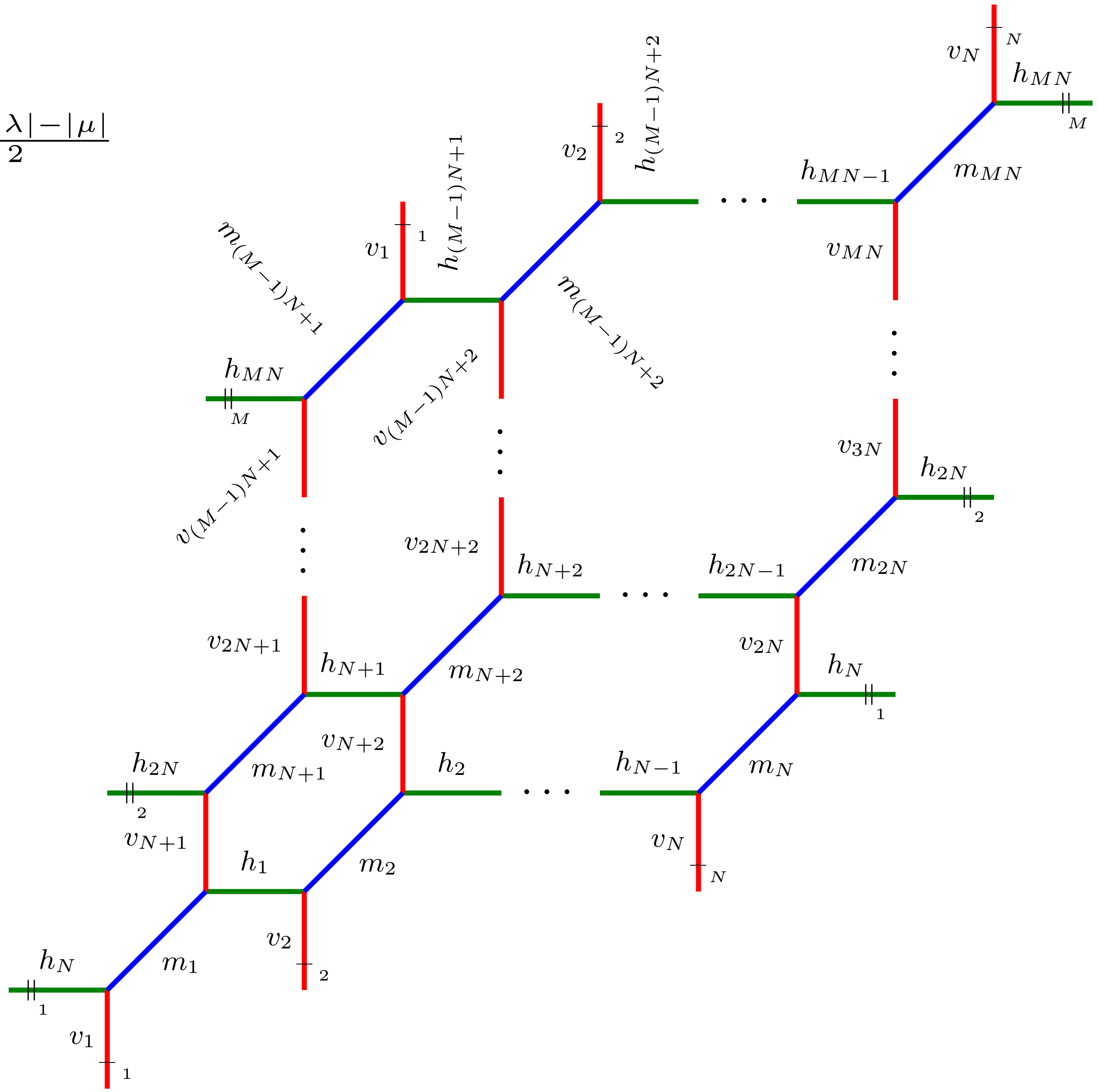
-) choose preferred direction



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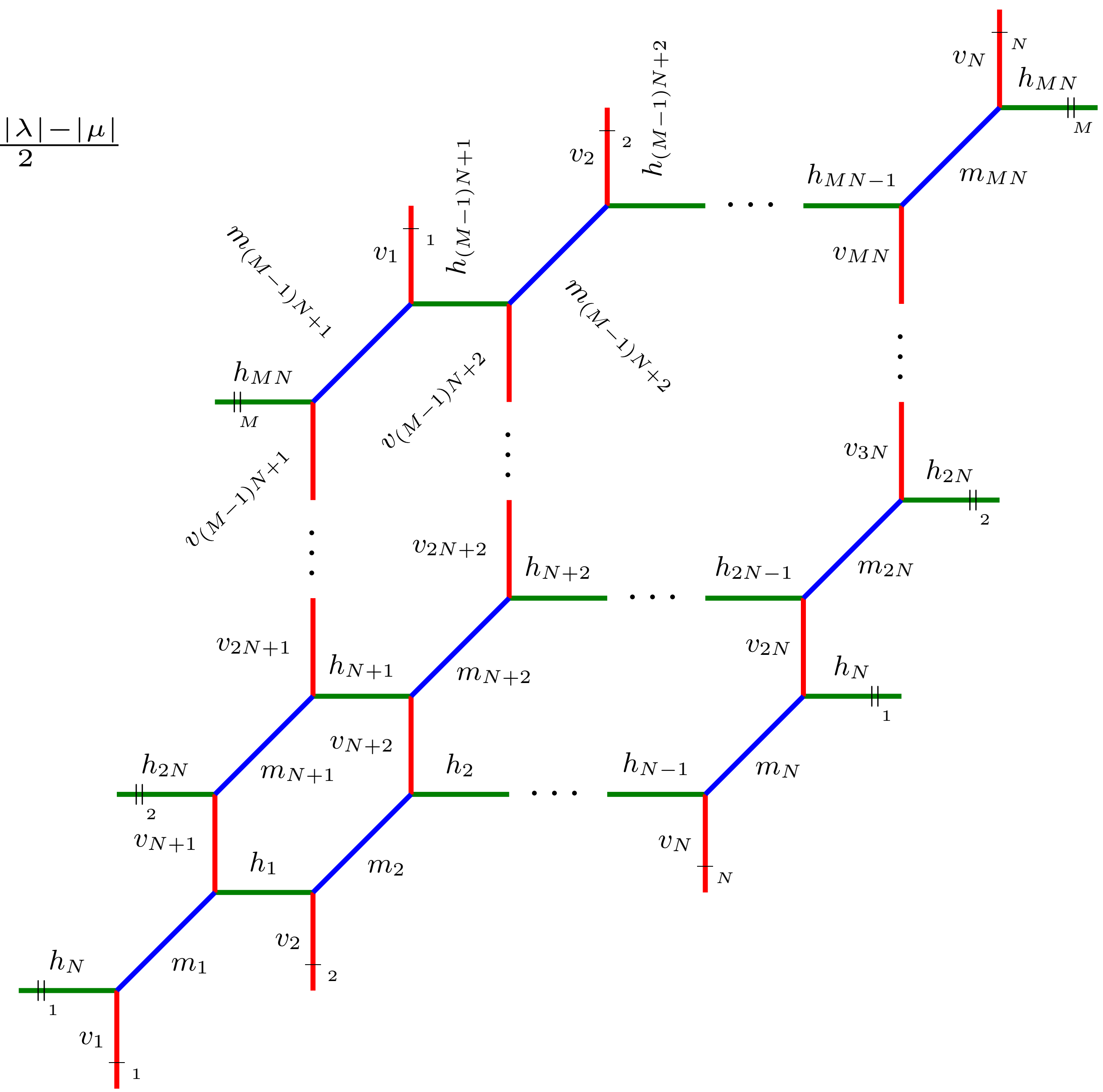
-) choose preferred direction

must be common to all vertices of diagram



Instanton Partition Functions

Different choices of preferred direction afford different (but equivalent) expansions:

$$\mathcal{Z}_{N,M}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = Z_p(\{v\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{h}}\, Z_{\vec{k}}(\{v\},\{m\}) = Z^{(N,M)}_{\text{hor}}$$

$$= Z_p(\{h\},\{m\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{v}}\, Z_{\vec{k}}(\{h\},\{m\}) = Z^{(N,M)}_{\text{vert}}$$

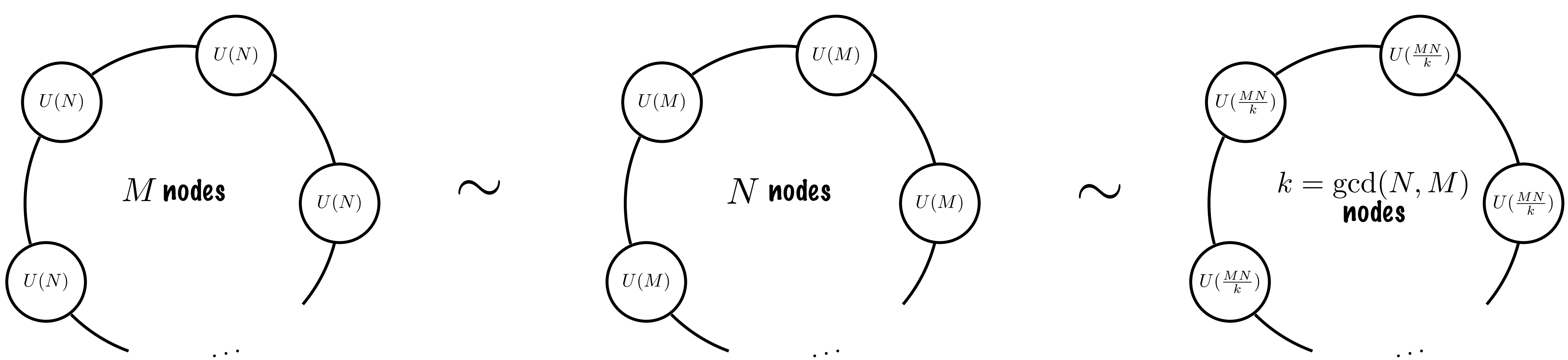
$$= Z_p(\{h\},\{v\}) \sum_{\vec{k}} e^{-\vec{k}\cdot\mathbf{m}}\, Z_{\vec{k}}(\{h\},\{v\}) = Z^{(N,M)}_{\text{diag}}$$

common normalisation factor
(perturbative partition function)

Compare different series expansions with instanton partition functions of quiver gauge theories.

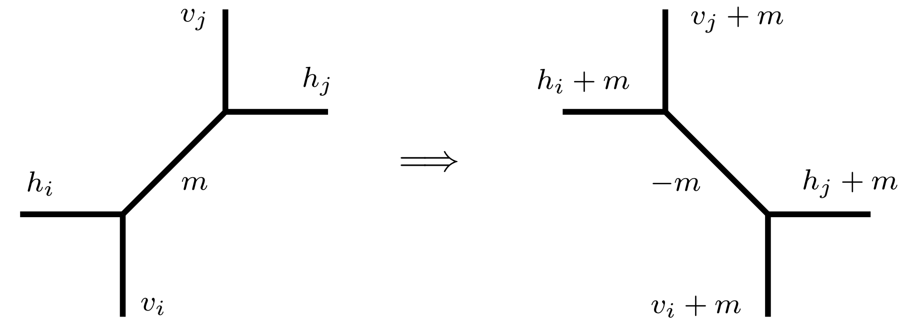
Equalities of partition functions implies dualities among different gauge theories: Triality

[Bastian, SH, Iqbal, Rey 2017]

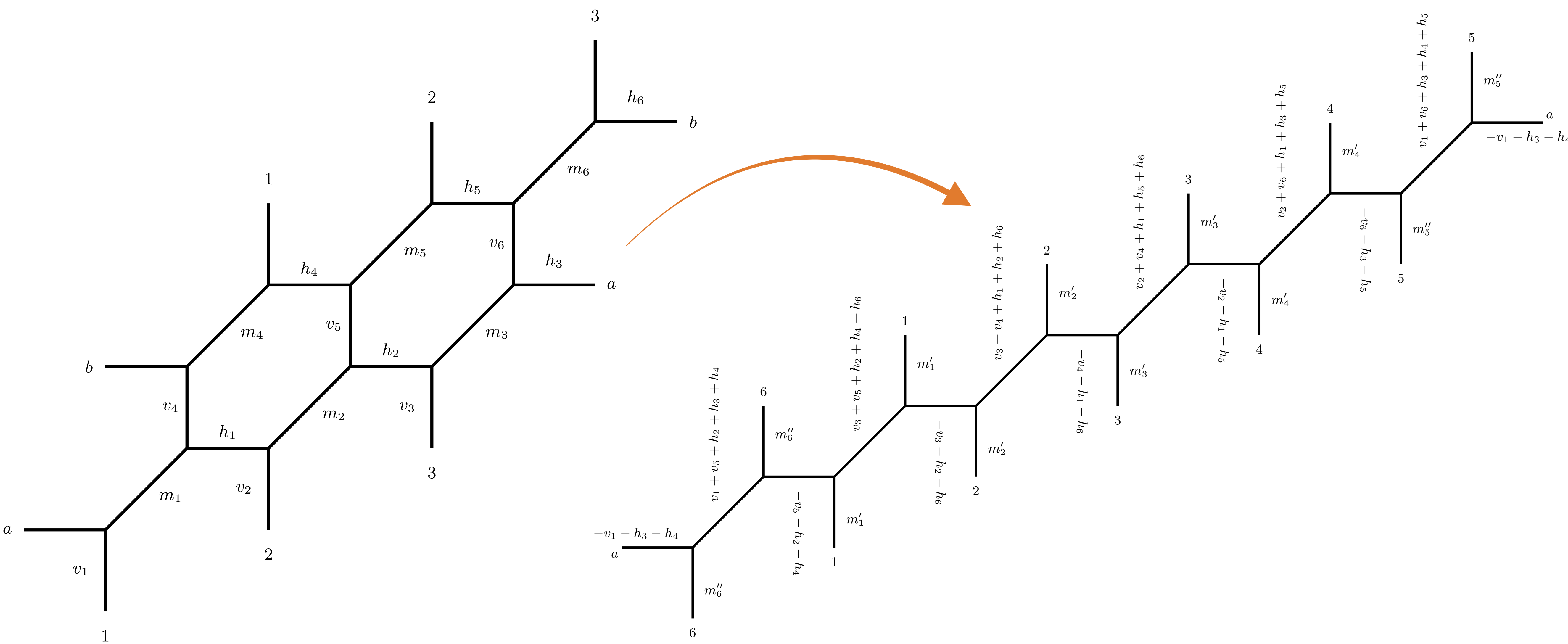


Further Dualities from $X_{N,M}$

Flop transition for any two curves in the diagram:



Example: Series of flop and SL(2,Z) transformations for $X_{3,2} \sim X_{6,1}$ [SH, Iqbal, Rey 2016]



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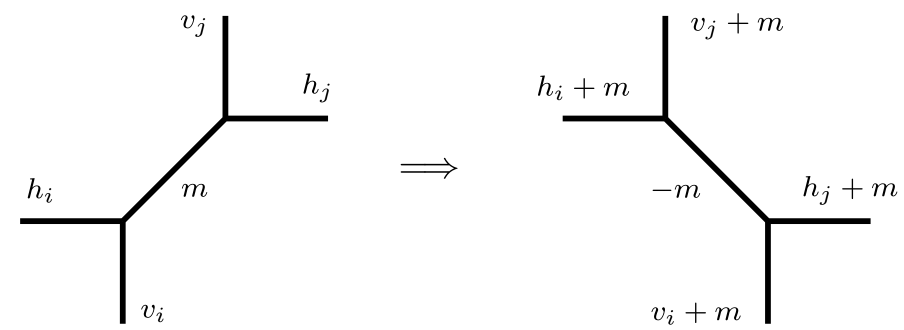

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Duality leaves partition function invariant

$$\mathcal{Z}_{3,2}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = \mathcal{Z}_{6,1}(\{h'\},\{v'\},\{m'\},\epsilon_{1,2})$$

[Bastian, SH, Iqbal, Rey 2017]

Kähler parameters implied by duality transformation

Vertical expansion of $\mathcal{Z}_{6,1}$ gives rise to a gauge theory with gauge group $U(6)$ and part. fct. $\mathcal{Z}^{(6,1)}_{\text{vert}}$

Further dualities among larger classes of gauge symmetries

Network of Dual Theories

Extended moduli space of $X_{N,M}$:

$$X_{N,M} \sim X_{N',M'} \qquad \textbf{for} \qquad \begin{aligned} NM &= N'M' \\ \gcd(N,M) &= \gcd(N',M') \end{aligned}$$

Partition function invariant

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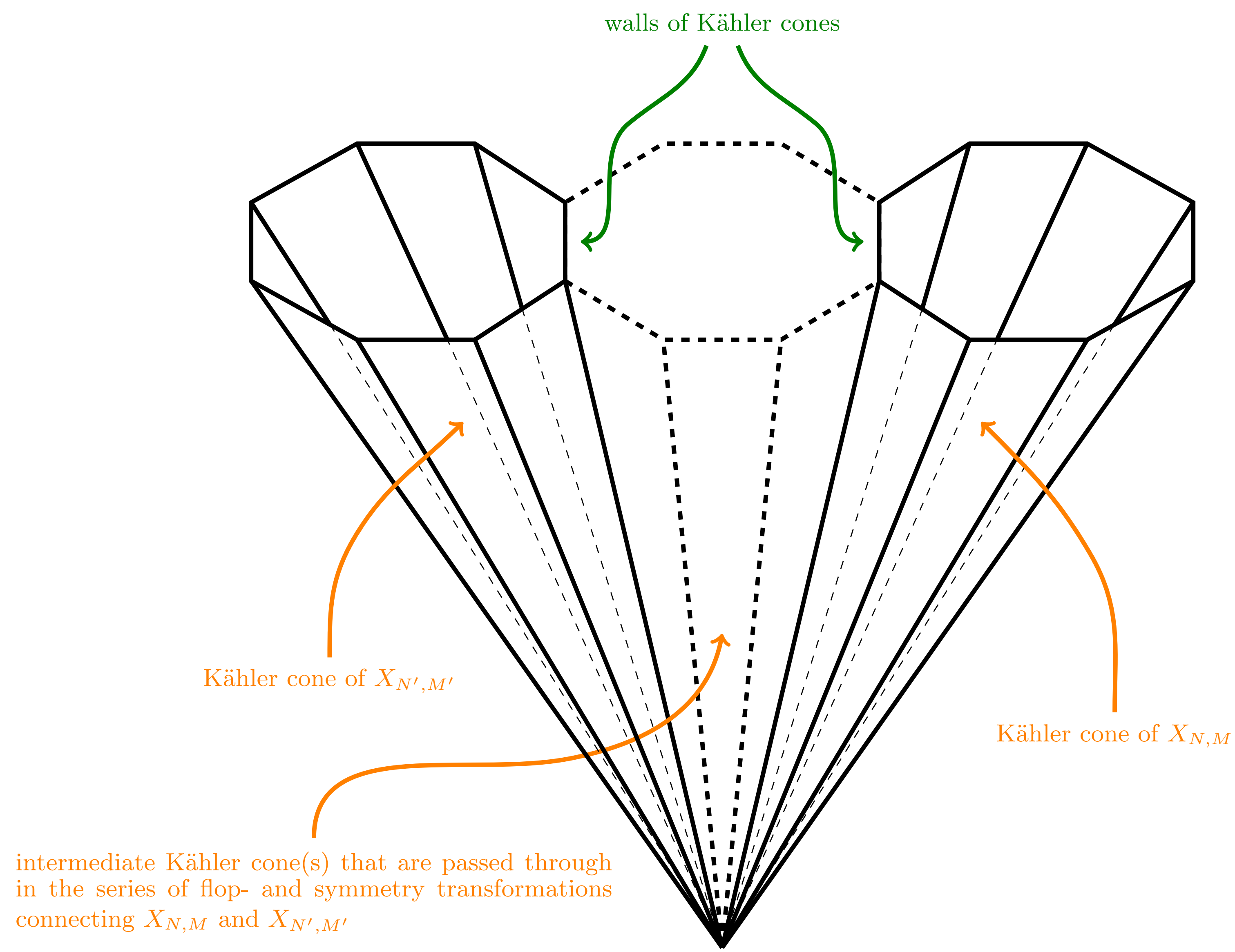
(partial) proves: [Bastian, SH, Iqbal, Rey 2017]
[Haghighat, Sun 2018]



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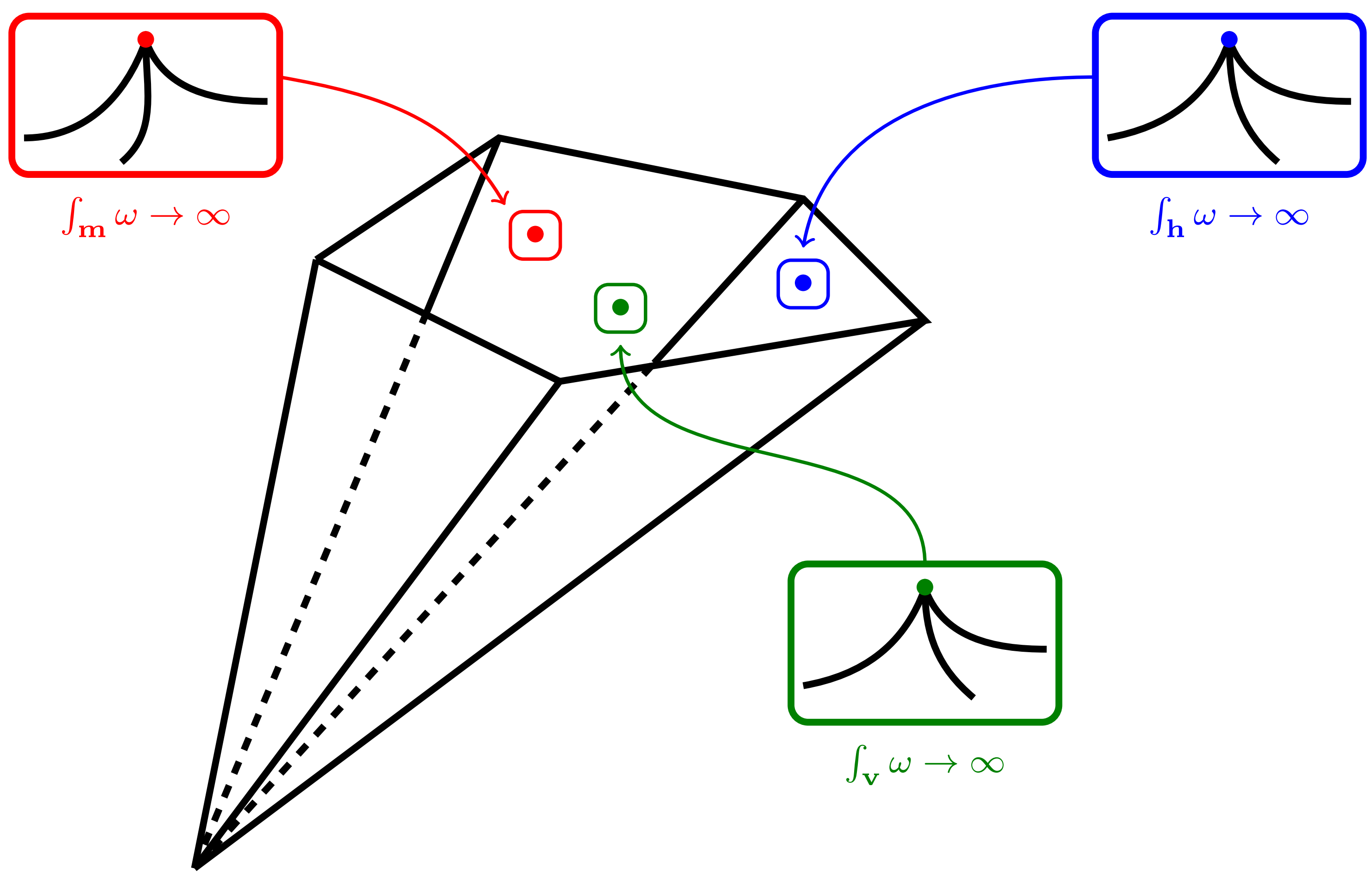
[Haghighat, Sun 2018]

Weak coupling regions **within each Kähler cone:**



Network of Dual Theories

Extended moduli space of $X_{N,M}$:

$$X_{N,M} \sim X_{N',M'} \qquad \text{for} \qquad \begin{array}{c} NM = N'M' \\ \gcd(N,M) = \gcd(N',M') \end{array}$$

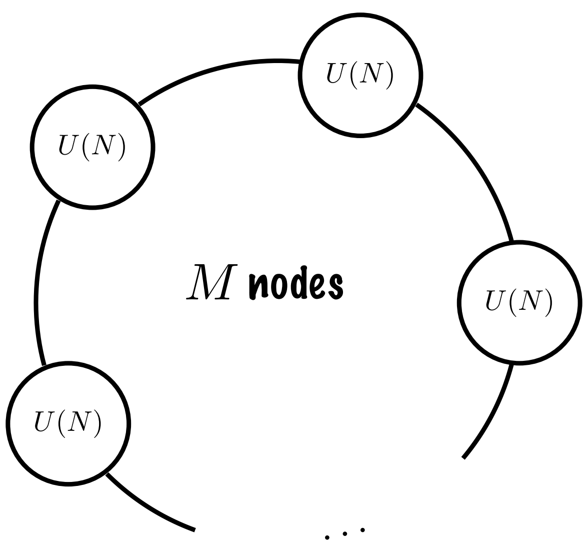
Partition function invariant

[SH, Iqbal, Rey 2016]

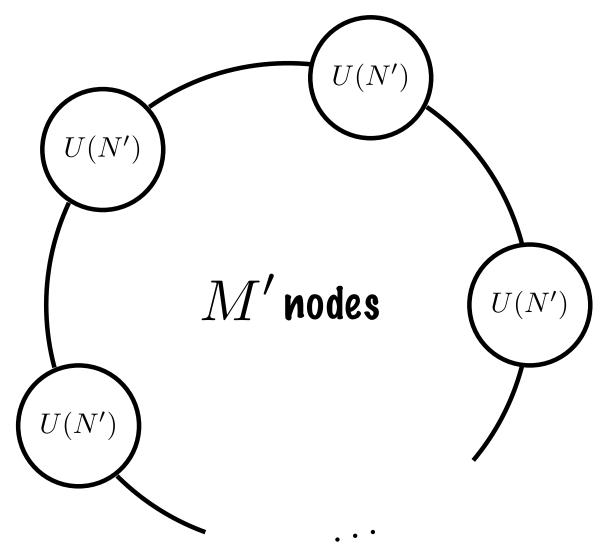
$$\mathcal{Z}_{N,M}(\{h\},\{v\},\{m\},\epsilon_{1,2}) = \mathcal{Z}_{N',M'}(\{h'\},\{v'\},\{m'\},\epsilon_{1,2})$$

(partial) proves: [Bastian, SH, Iqbal, Rey 2017]
[Haghighat, Sun 2018]

Network of dual theories: [Bastian, SH, Iqbal, Rey 2017]



$\sim$



for any (N',M') with

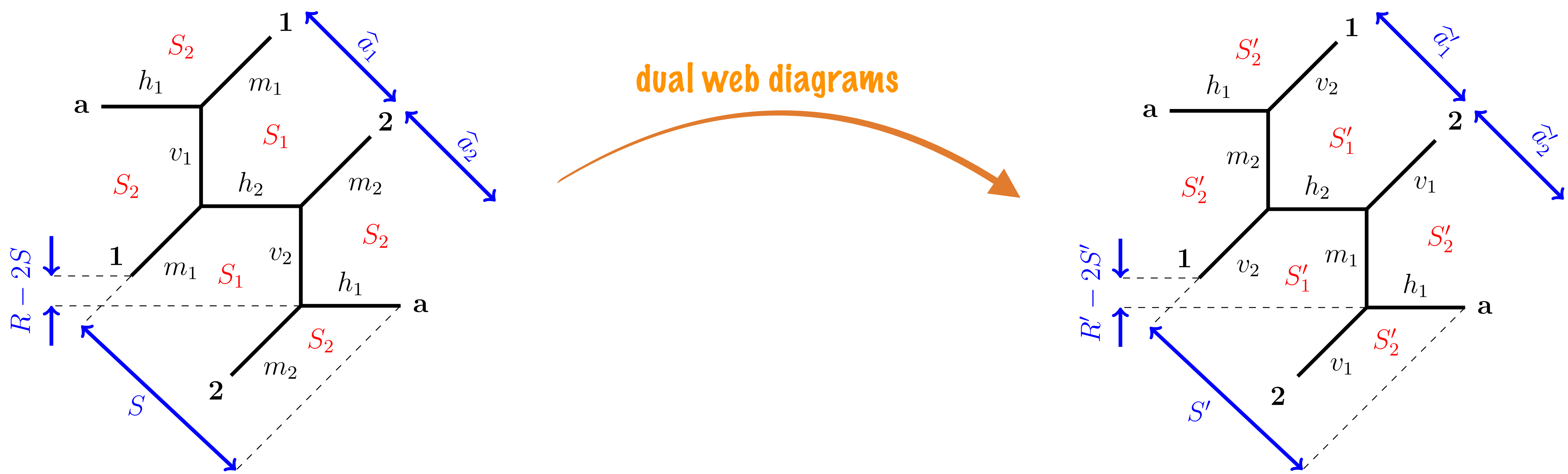
$$NM = N'M'$$
$$\gcd(N,M) = \gcd(N',M')$$

Dihedral Symmetries of Configurations (N,1)

Web of dualities among different theories can be turned into symmetries for individual theories

[SH, Bastian 2018]

Example (N,M)=(2,1):



$$\widehat{a}_1 = v_1 + h_2\,, \qquad \widehat{a}_2 = v_2 + h_1\,,$$
$$S = h_2 + v_2 + h_1\,, \qquad R - 2S = m_1 - v_2\,.$$

$$\widehat{a}'_1 = m_1 + h_1\,, \qquad \widehat{a}'_2 = m_2 + h_2\,,$$
$$S' = h_2 + m_1 + h_1\,, \qquad R' - 2S' = v_2 - m_1\,.$$

Implies the following symmetry of the partition function:

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \widehat{a}'_1 \\ \widehat{a}'_2 \\ S' \\ R' \end{pmatrix}$$

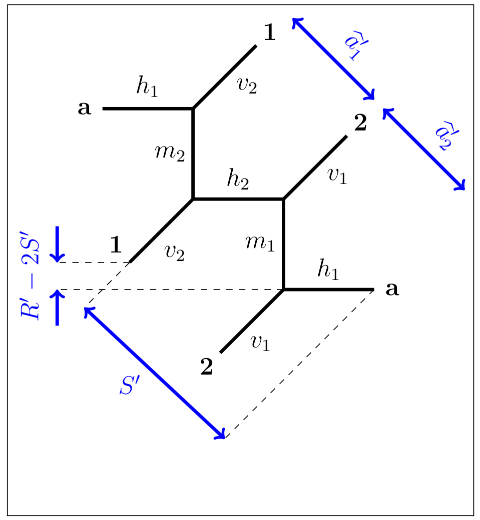
where $G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **with**

$$\det G_1 = 1$$
$$G_1 \cdot G_1 = \mathbb{1}_{4\times 4}$$

Generalising to include other duality transformations:

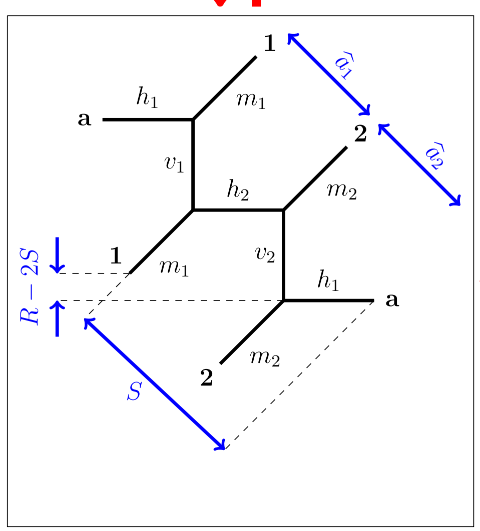
	$\mathbb{1}_{4\times4}$	G_1	G_2	G_3
$\mathbb{1}_{4\times4}$	$\mathbb{1}_{4\times4}$	G_1	G_2	G_3
G_1	G_1	$\mathbb{1}_{4\times4}$	G_3	G_2
G_2	G_2	G_3	$\mathbb{1}_{4\times4}$	G_1
G_3	G_3	G_2	G_1	$\mathbb{1}_{4\times4}$

$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



G_1 G_1

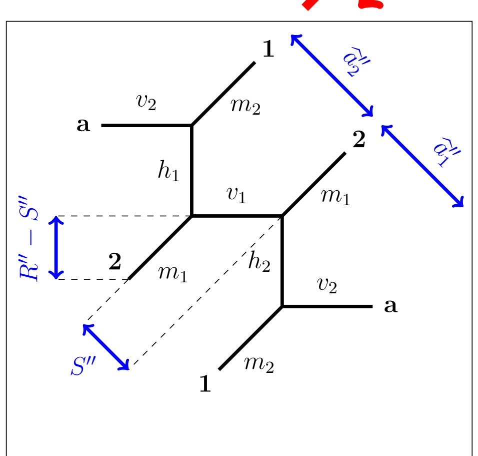
$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & 2 & -4 & 1 \end{pmatrix}$$

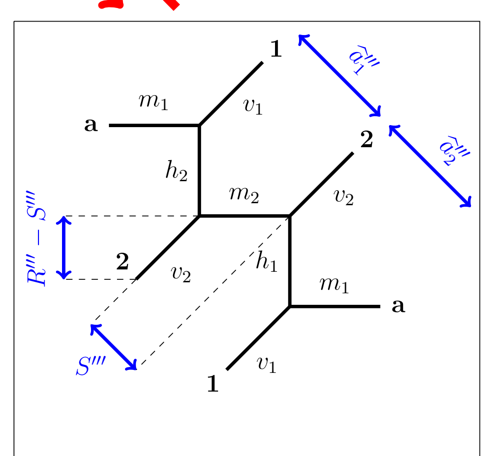


G_2 G_2

G_3 G_3

$$G_3 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 \\ 2 & 2 & -4 & 1 \end{pmatrix}$$





Group Structure:

$$\{\mathbb{1}_{4\times4}, G_1, G_2, G_3\} \cong \mathrm{Dih}_2$$

Generalisation to (N,1): Symmetry group

$$\mathbb{G}(N) \times \mathrm{Dih}_N$$

'shuffling' of roots

where

$$\mathbb{G}(N) \cong \begin{cases} \mathrm{Dih}_3 & \text{if} \quad N = 1\,, \\ \mathrm{Dih}_2 & \text{if} \quad N = 2\,, \\ \mathrm{Dih}_3 & \text{if} \quad N = 3\,, \\ \mathrm{Dih}_\infty & \text{if} \quad N \geq 4\,. \end{cases}$$

Explicitly

$$\mathbb{G}(N) \cong \left\langle \{\mathcal{G}_2(N), \mathcal{G}_2'(N) \big| (\mathcal{G}_2(N))^2 = (\mathcal{G}_2'(N))^2 = (\mathcal{G}_2(N)\cdot\mathcal{G}_2'(N))^n = \mathbb{1}\} \right\rangle$$

$$n = \begin{cases} 3 & \text{for} \quad N = 1,3 \\ 2 & \text{for} \quad N = 2 \\ \infty & \text{for} \quad N \geq 4 \end{cases}$$

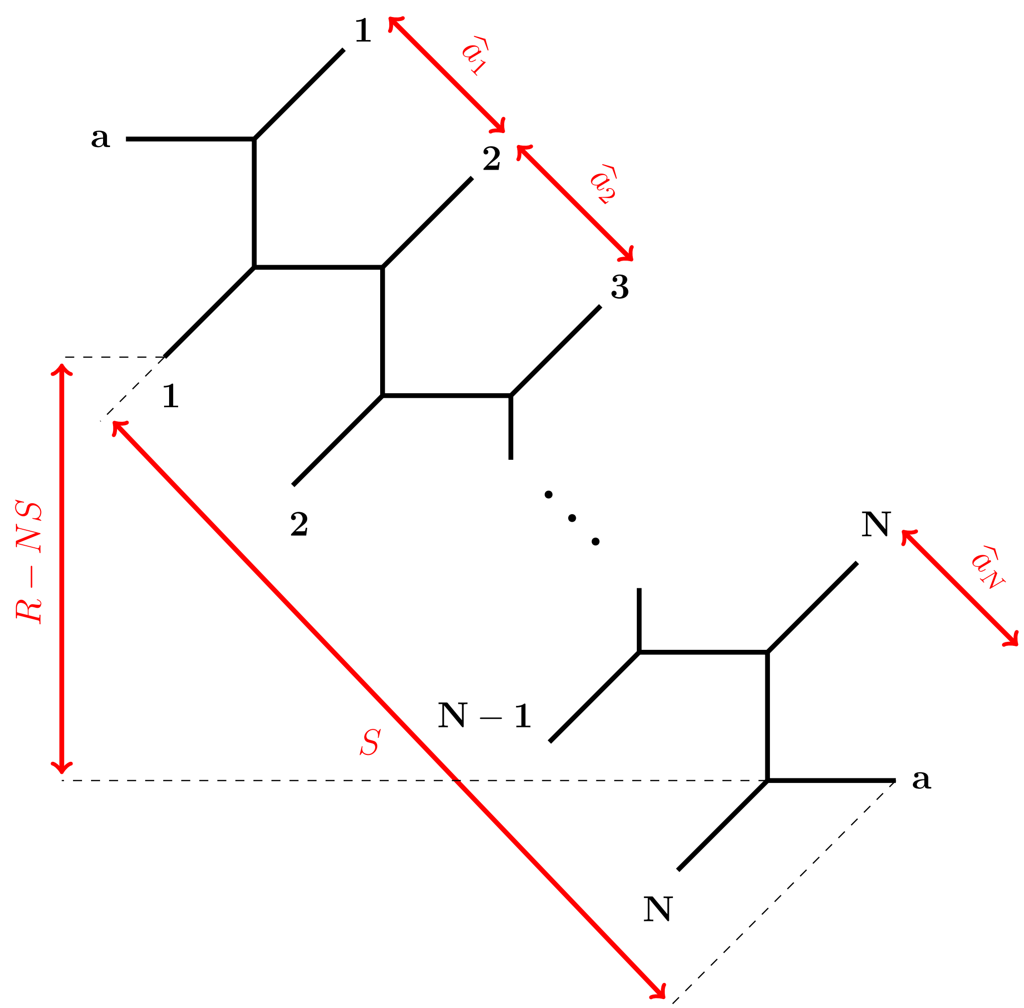
with the $(N+2)\times(N+2)$ **matrices**

$$\mathcal{G}_2(N) = \begin{pmatrix} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{pmatrix} \quad \textbf{and} \quad \mathcal{G}_2'(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Action on the Free Energy: $(N,1)$

Fourier Expansion of the Free Energy:

$$F_{N,1}(\widehat{a}_i,S,R;\epsilon_{1,2}) = \ln\, \mathcal{Z}_{N,1}(\widehat{a}_i,S,R;\epsilon_{1,2}) = \sum_{s_1,s_2=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i_1,\ldots,i_N}^{\infty}\sum_{k\in\mathbb{Z}} \epsilon_1^{s_1-1}\epsilon_2^{s_2-1} f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,r}\, Q_{\widehat{a}_1}^{i_1}\ldots Q_{\widehat{a}_N}^{i_N}\, Q_S^k\, Q_R^r$$



Action of $\mathbb{G}(N)\times \mathrm{Dih}_N$ on Fourier coefficients

$$f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,n} = f^{(s_1,s_2)}_{i'_1,\ldots,i'_N,k',n'}$$

for

$$(i'_1,\ldots,i'_N,k',n')^T = G^T\cdot(i_1,\ldots,i_N,k,n)^T$$

$$G\in\mathbb{G}(N)\times\mathrm{Dih}_N$$

checked explicitly in numerous examples

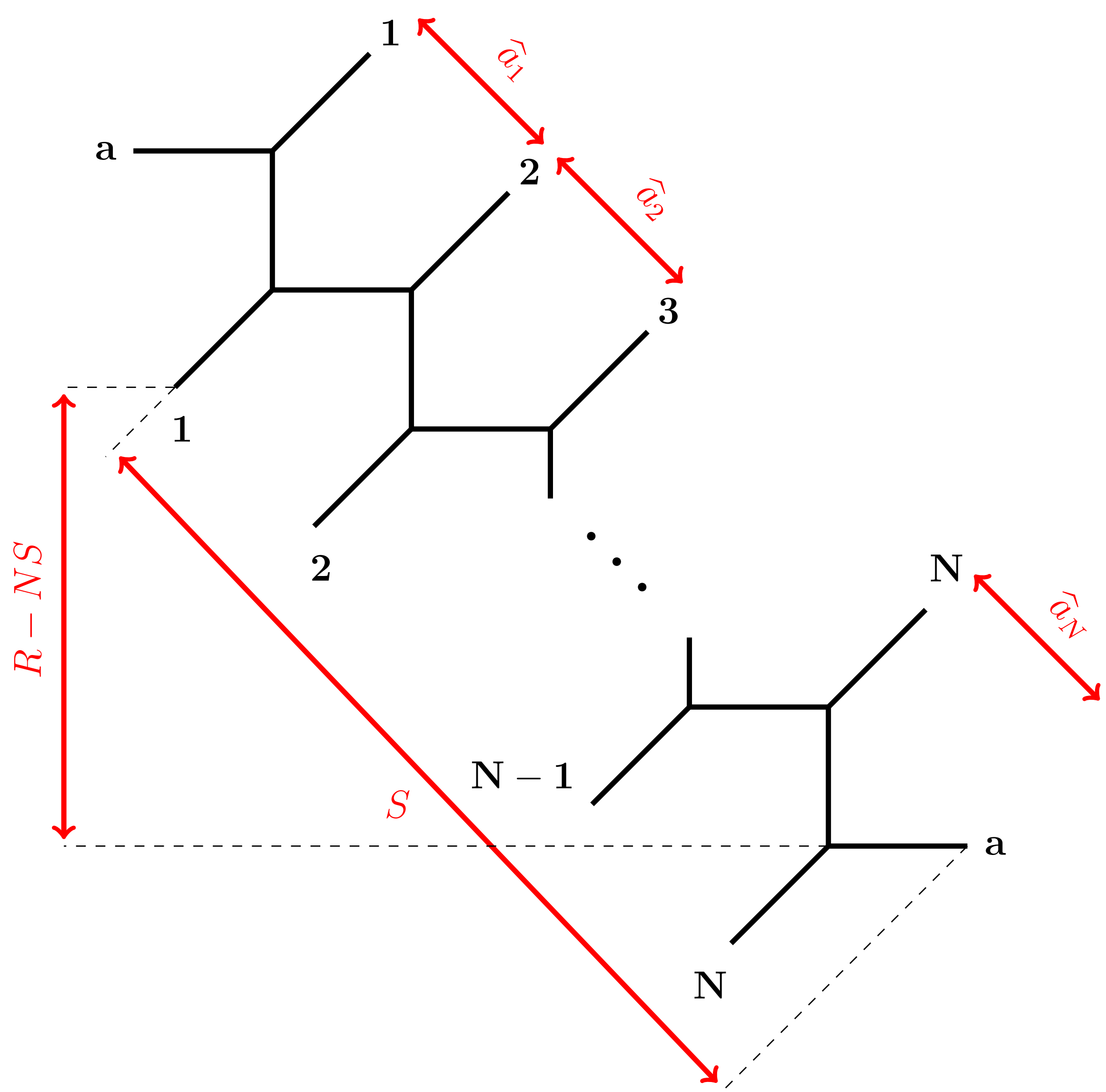
Notation:

$$Q_{\widehat{a}_i} = e^{2\pi i\widehat{a}_i}$$
$$Q_S = e^{2\pi iS}$$
$$Q_R = e^{2\pi iR}$$
$$\rho = \sum_{i=1}^{N}\widehat{a}_i$$

Symmetry constrains form of the coefficients of the free energy

Re-organise expansion in different objects

Implications for the Free Energy: $(N,1)$



Free Energy:

$$\mathcal{F}_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2}) = \ln \mathcal{Z}_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2})$$

Unrefined Limit: $\epsilon_1 = -\epsilon_2 = \epsilon$

Instanton Expansion:

$Q_R = e^{2\pi i R}$

$$\mathcal{F}_{N,1}(\widehat{a}_i, S, R; \epsilon, -\epsilon) = \sum_r Q_R^r \, P_N^{(r)}(\widehat{a}_i, S, \epsilon)$$

Consider ϵ- expansion for individual orders in r

$$P_N^{(r)}(\widehat{a}_i, S, \epsilon) = \sum_{s=0}^{\infty} \epsilon^{2s-2} \, P_{N,(s)}^{(r)}(\widehat{a}_i, S)$$

Fourier expansion in gauge parameters

$$P_{N,(s)}^{(r)}(\widehat{a}_i, S) = \sum_{n_1,\ldots,n_N} Q_{\widehat{a}_1}^{n_1} \ldots Q_{\widehat{a}_N}^{n_N} \, P_{N,(s)}^{(r),\{n_1,\ldots,n_N\}}(S) \quad \textbf{with} \quad Q_{\widehat{a}_j} = e^{2\pi i \widehat{a}_j}$$

Leading Instanton Level $r = 1$ for $(N = 2, 1)$

Following expansion compatible with the symmetry group $\mathbb{G}(N) \times \mathrm{Dih}_N$ [SH 2019]

(Form verified by calculating the Fourier expansion up to very high orders)

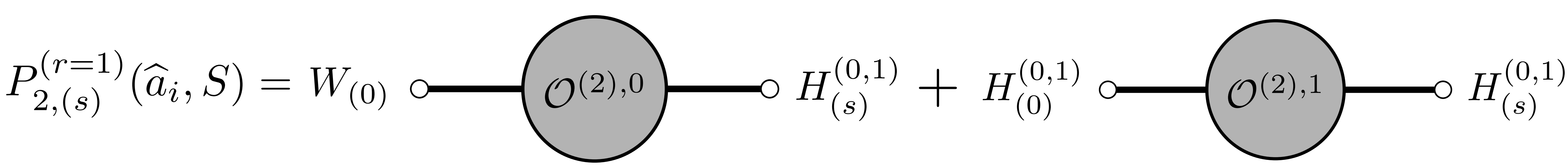
$$P^{(r=1)}_{2,(s)}(\widehat{a}_1, \widehat{a}_2, S) = \boxed{H^{(1),\{0\}}_{(s)} \, W^{(1)}_{(0)}} \boxed{\mathcal{O}^{(2),0}} + \boxed{H^{(1),\{0\}}_{(s)} \, H^{(1),\{0\}}_{(0)}} \boxed{\mathcal{O}^{(2),1}(\widehat{a}_1, \rho)}$$

External states:

- only depend on S and $\rho = \widehat{a}_1 + \widehat{a}_2$
- universal definition for any N

Couplings:

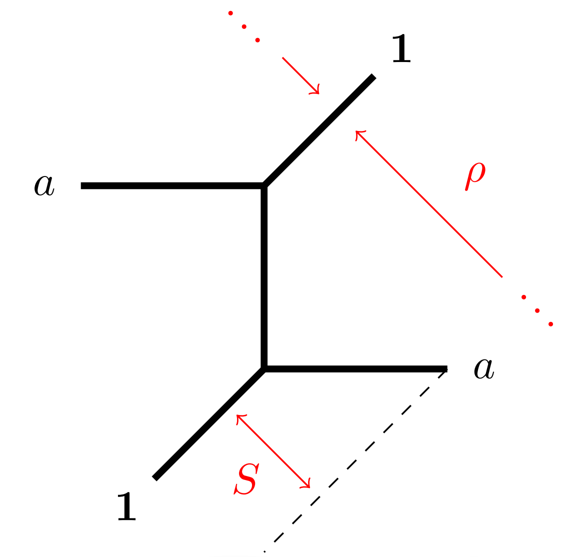
- only depend on $\widehat{a}_1$ and ρ
- independent of S
- encodes the gauge structure of the spectrum of the gauge theory
- definition specific for $N = 2$



External states:

Free energy for $N=1$

$$P^{(r)}_{N=1,(s)}(\rho,S)=H^{(r),\{0\}}_{(s)}(\rho,S)$$



Jacobi Form of weight $w=2s-2$ **and index** $m=r$

Jacobi Form of weight w **and index** m**:**

$\phi:\ \mathbb{H}\times\mathbb{C}\to\mathbb{C}$

With the following transformation properties

$$\phi\left(\frac{a\rho+b}{c\rho+d},\frac{z}{c\rho+d}\right)=(c\rho+d)^w\,e^{\frac{2\pi i m c\,z^2}{c\rho+d}}\,\phi(\rho,z)\,,\quad\forall\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\,,$$

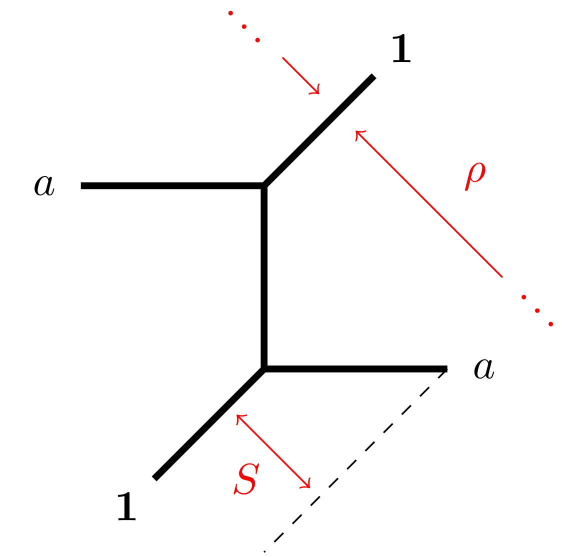
$$\phi(\rho,z+k_1\,\rho+k_2)=e^{-2\pi i m(k_1^2\rho+2k_1 z)}\,\phi(\rho,z)\,,\qquad\forall k_{1,2}\in\mathbb{N}\,,$$

and the Fourier series

$$\phi(z,\rho)=\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}c(n,k)\,Q_\rho^n\,e^{2\pi i z k}\,,\qquad\text{with}\qquad c(n,k)=(-1)^w\,c(n,-k)$$

External states:

Free energy for $N = 1$



$$P^{(r)}_{N=1,(s)}(\rho, S) = H^{(r),\{0\}}_{(s)}(\rho, S)$$

Jacobi Form of weight $w = 2s - 2$ **and index** $m = r$

Explicitly for $r = 1$

$$H^{(1),\{0\}}_{(s)}(\rho, S) = \begin{cases} -\phi_{-2,1}(\rho, S) & \text{if} \quad s = 0\,, \\ \frac{\phi_{0,1}(\rho,S)}{24} & \text{if} \quad s = 1\,, \\ \frac{(-1)^s B_{2s}\, E_{2s}(\rho)}{(2s-3)!!(2s)!!}\, \phi_{-2,1}(\rho, S) & \text{if} \quad s > 1\,, \end{cases}$$

Standard Jacobi Forms:

$$\phi_{0,1}(\rho, z) = 8 \sum_{a=2}^{4} \frac{\theta_a^2(z;\rho)}{\theta_a^2(0,\rho)}$$

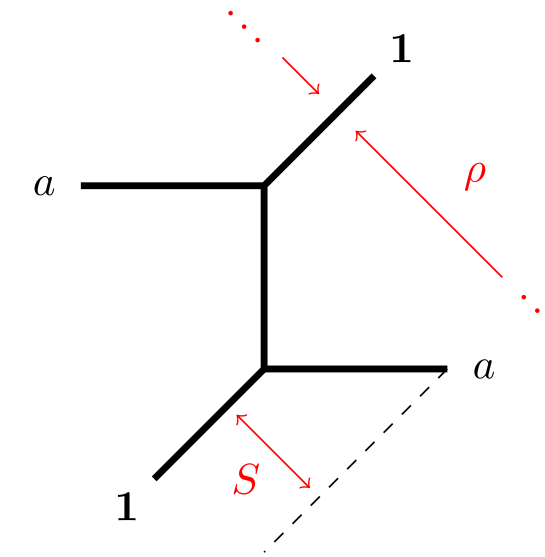
$$\phi_{-2,1}(\rho, z) = \frac{\theta_1^2(z;\rho)}{\eta^6(\rho)}$$

Eisenstein series:

$$E_{2k}(\rho) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, Q_\rho^n$$

External states:

Free energy for $N=1$



$$P^{(r)}_{N=1,(s)}(\rho,S)=H^{(r),\{0\}}_{(s)}(\rho,S)$$

Jacobi Form of weight $w=2s-2$ **and index** $m=r$

Explicitly for $r=1$

$$H^{(1),\{0\}}_{(s)}(\rho,S)=\begin{cases}-\phi_{-2,1}(\rho,S) & \text{if} \quad s=0\,,\\ \frac{\phi_{0,1}(\rho,S)}{24} & \text{if} \quad s=1\,,\\ \frac{(-1)^s B_{2s}\,E_{2s}(\rho)}{(2s-3)!!(2s)!!}\,\phi_{-2,1}(\rho,S) & \text{if} \quad s>1\,,\end{cases}$$

Different instanton levels related by action of a Hecke operator

$$H^{(r),\{0\}}_{(s)}(\rho,S)=\mathcal{H}_r\left(H^{(1),\{0\}}_{(s)}(\rho,S)\right)$$

Hecke Operator:

$\mathcal{J}_{w,m}$ space of Jacobi forms of weight w and index m

$$\mathcal{H}_n:\ \mathcal{J}_{w,m}\longrightarrow\mathcal{J}_{w,nm}$$

$$\phi(\rho,z)\longmapsto\mathcal{H}_n(f)=n^{w-1}\sum_{\substack{d|n\\ b \bmod d}} d^{-w} f\left(\frac{n\rho+bd}{d^2},\frac{nz}{d}\right)$$

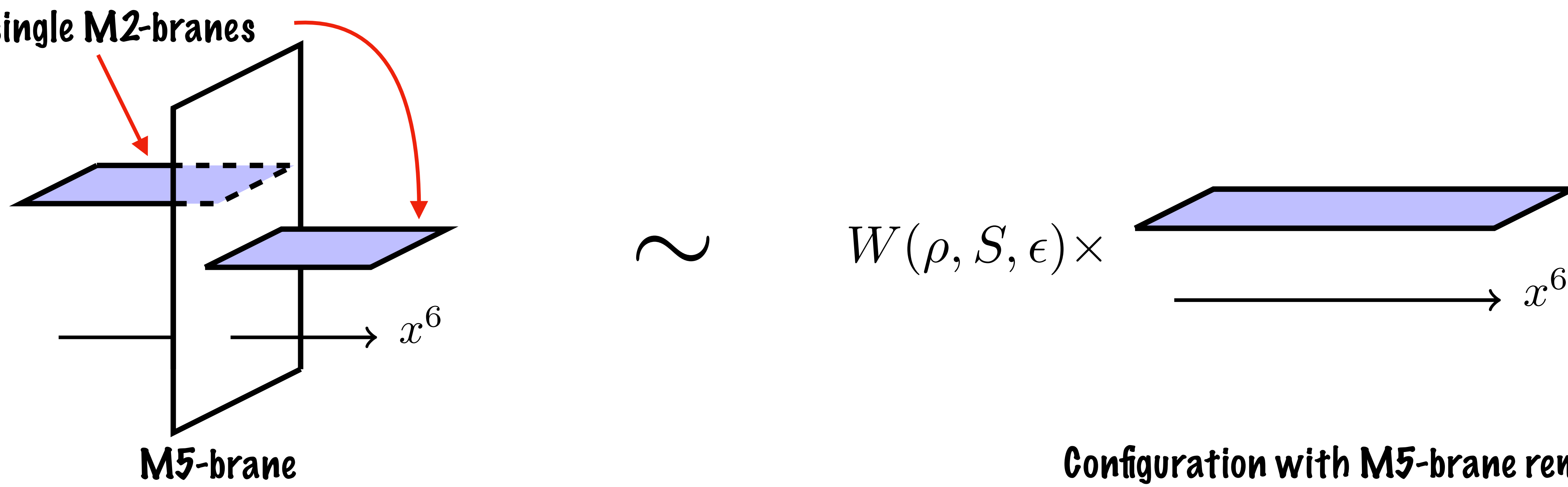
(Quasi) Jacobi Form

[SH, Iqbal, Rey 2015]
[Ahmed, SH, Iqbal, Rey 2017]

$$W(\rho, S, \epsilon) = \frac{\theta_1^2(\rho, S) - \theta_1(\rho, S+\epsilon)\theta_1(\rho, S-\epsilon)}{\theta_1^2(\rho, \epsilon)} = \sum_{s=0}^{\infty} \epsilon^{2s}\, W_{(s)}^{(1)}(\rho, S)$$

W governs the counting of (single particle) BPS states of single M2-branes ending on M5-branes on both sides

[SH, Iqbal, Rey 2015]



$W_{(s)}^{(1)}$ are Jacobi forms of weight $w = 2s$ and index $m = 1$, explicitly

$$W_{(0)}^{(1)} = \frac{1}{24}(\phi_{0,1} + 2E_2\phi_{-2,1}), \quad W_{(1)}^{(1)} = \frac{E_2^2 - E_4}{288}\,\phi_{-2,1}, \quad W_{(2)}^{(1)} = \frac{5E_2^3 + 3E_2E_4 - 8E_6}{51840}\,\phi_{-2,1}$$

Higher instanton levels through action of Hecke operators

$$W_{(s)}^{(r)}(\rho, S) = \mathcal{H}_r\left(W_{(s)}^{(1)}(\rho, S)\right)$$

Couplings:

$$\mathcal{O}^{(2),0}(\widehat{a}_1,\rho) = 2$$

$$\mathcal{O}^{(2),1}(\widehat{a}_1,\rho) = -\sum_{n=1}^{\infty}\frac{2n}{1-Q_\rho^n}\left(Q_{\widehat{a}_1}^n + \frac{Q_\rho^n}{Q_{\widehat{a}_1}^n}\right)$$

[SH 2019]

Intriguing re-writing of $\mathcal{O}^{(2),1}$

$$\mathcal{O}^{(2),1}(\widehat{a}_1,\rho) = -\frac{2}{(2\pi i)^2}\left[\frac{\pi^2}{3}E_2(\rho) + \wp(\widehat{a}_1;\rho)\right] = -\frac{2}{(2\pi i)^2}\left[\mathbb{G}''(\widehat{a}_1;\rho) + \frac{2\pi i}{\rho-\bar{\rho}}\right]$$

Weierstrass elliptic Function: $\wp(z;\rho) = \frac{1}{z^2} + 2\sum_{k=1}^{\infty}(2k+1)\,\zeta(2k+2)\,E_{2k+2}(\rho)\,z^{2k}$

2-Point Function of a free scalar field ϕ on a torus:

$$\mathbb{G}(z-w;\rho) = \langle\phi(z)\,\phi(w)\rangle = -\ln\left|\frac{\theta_1(z-w;\rho)}{\theta_1'(0,\rho)}\right|^2 - \frac{\pi}{2\mathrm{Im}(\rho)}(z-w-(\bar{z}-\bar{w}))^2$$

satisfies: $\Delta_z\,\mathbb{G}(z;\rho) = 4\pi\,\delta^{(2)}(z) - \frac{2\pi}{\mathrm{Im}(\rho)}$

$\boxed{\mathcal{O}^{(2),1} \text{ given by the propagator of a free scalar field!}}$

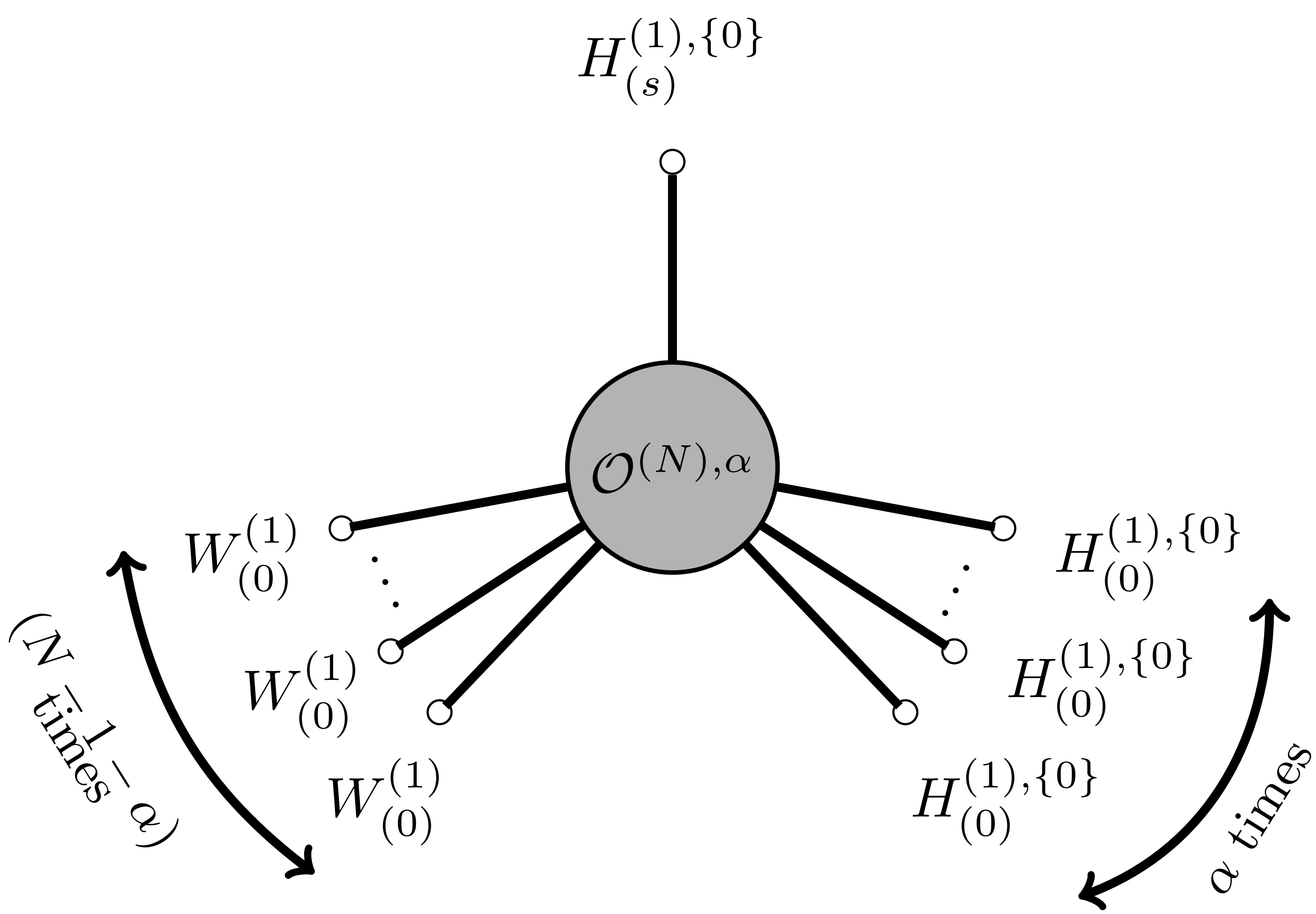
Leading Instanton Level $r = 1$ for $(N, 1)$

[SH 2020]

Conjectured form for general N

$$P^{(r=1)}_{N,(s)}(\widehat{a}_{1,\ldots,N}, S) = H^{(1),\{0\}}_{(s)}(\rho, S) \sum_{\alpha=0}^{N-1} \left(W^{(1)}_{(0)}(\rho, S)\right)^{N-1-\alpha} \left(H^{(1),\{0\}}_{(0)}(\rho, S)\right)^{\alpha} \mathcal{O}^{(N),\alpha}(\widehat{a}_{1,\ldots,N-1}, \rho)$$

Individual terms have the structure of N - point Feynman diagrams:



- same general external states $W^{1}_{(0)}$ and $H^{(1),\{0\}}_{(s)}$
- individual couplings $\mathcal{O}^{(N),\alpha}$ that encode the gauge structure
- explicitly verified through Fourier expansions up to $N = 5$

Couplings are still combinations of propagators of a free scalar field on a torus

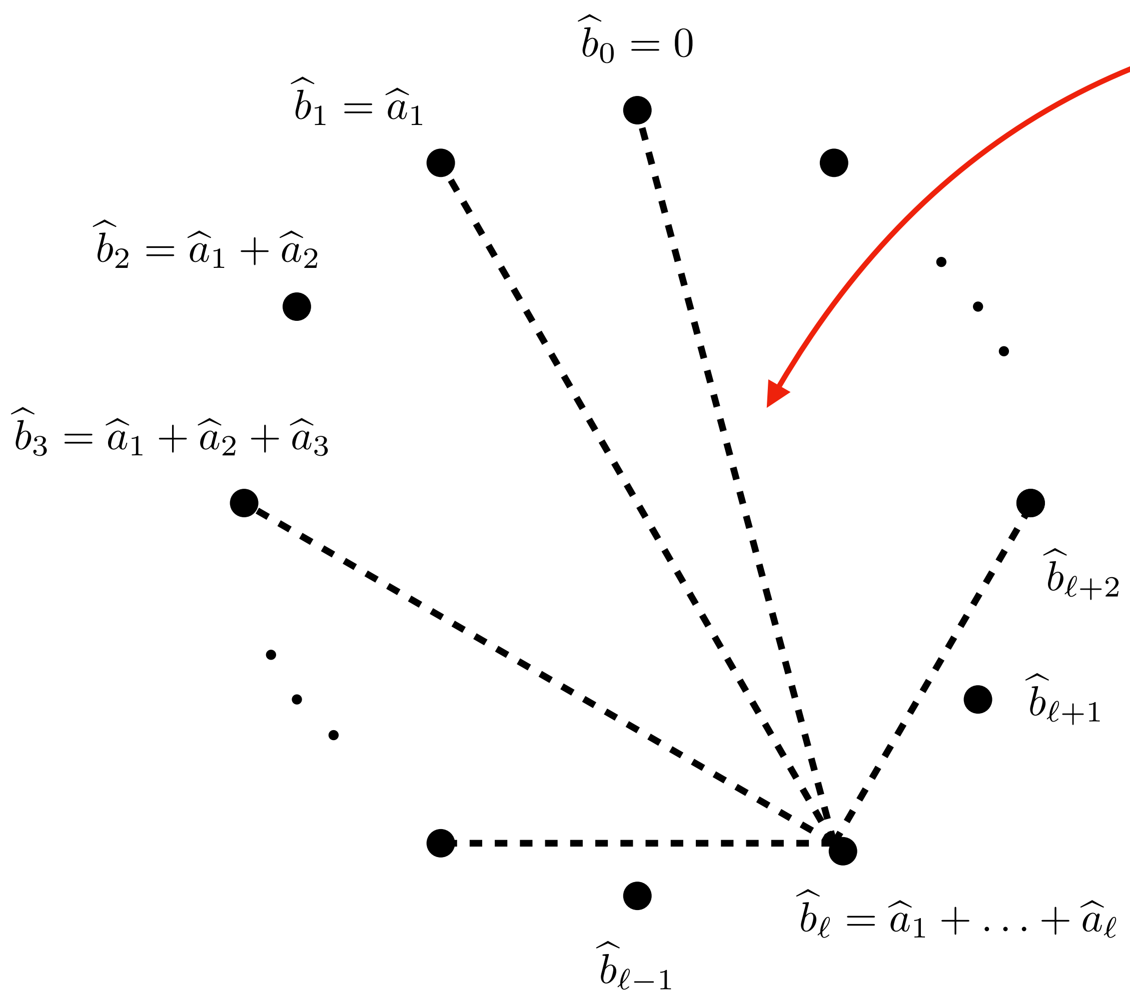
$$\mathcal{O}^{(N),\alpha}(\widehat{a}_{1,\ldots,N-1},\rho) = \frac{1}{(2\pi)^{2\alpha}} \sum_{\ell=0}^{N-1} \sum_{\substack{\mathcal{S}\subset\{0,\ldots,N-1\}\setminus\{\ell\} \\ |\mathcal{S}|=\alpha}} \prod_{j\in\mathcal{S}} \left(\mathbb{G}''(\widehat{b}_\ell - \widehat{b}_j;\rho) + \frac{2\pi i}{\rho-\bar{\rho}} \right)$$

with positions $\widehat{b}_0 = 0$ **and** $\widehat{b}_j = \sum_{n=1}^{j} \widehat{a}_n \,, \qquad \forall j = 1,\ldots,N-1$

Graphical representation:

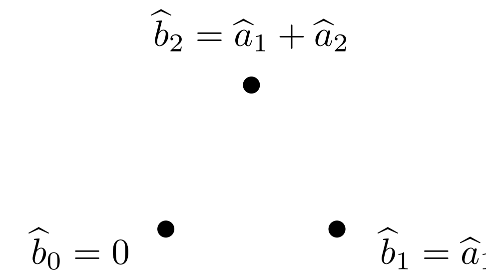
free propagators from $\widehat{b}_\ell$

to α **distinct other positions**

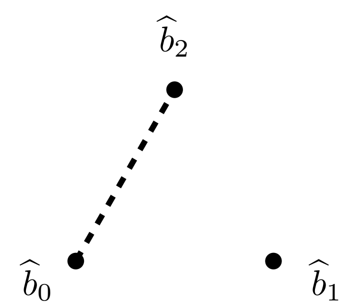


(Graphical) example: $N = 3$

$$\mathcal{O}^{(3),0}(\widehat{a}_1, \widehat{a}_2, \rho) = 3$$



$$\mathcal{O}^{(3),1}(\widehat{a}_1, \widehat{a}_2, \rho) =$$



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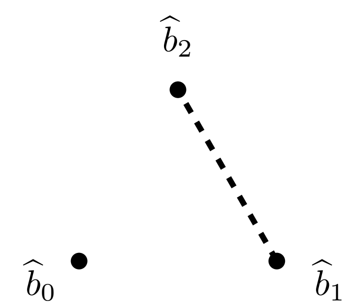

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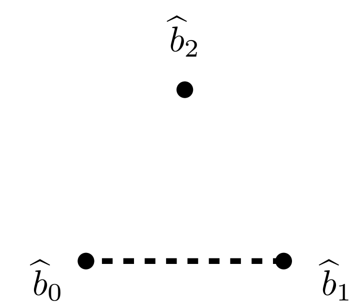
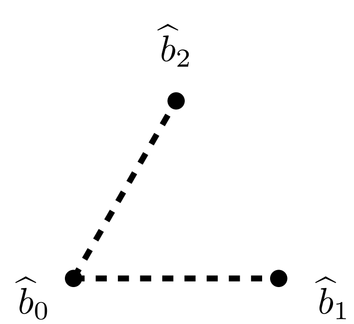


$$\mathcal{O}^{(3),2}(\widehat{a}_1, \widehat{a}_2, \rho) =$$



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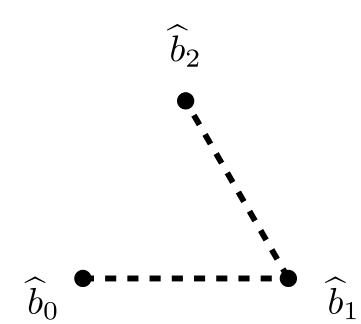

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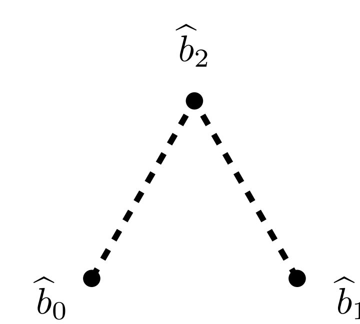


Second Instanton ($r=2$) Level for $(N=2,1)$

Similar decomposition in terms of $H^{(r),\{0\}}_{(s)}$ and $W^{(r)}_{(s)}$, (for $r\in\{0,1\}$), however, not unique!

Still intriguing features and structures.

Unambiguous decomposition at $s=0$ (leading order in ϵ) that is compatible with all symmetries:

$$P^{(2)}_{2,(0)}(\widehat{a}_1,\widehat{a}_2,S)=\boxed{\frac{2}{3}\,H^{(2),\{0\}}_{(0)}\,W^{(2)}_{(0)}\,\mathcal{O}^{(2),0}+\frac{4}{3}\,H^{(2),\{0\}}_{(0)}\,H^{(2),\{0\}}_{(0)}\,\mathcal{O}^{(2),1}}$$

$$+\left(H^{(1),\{0\}}_{(0)}\right)^4\,\mathcal{O}^{(2),\text{4-pt}}_{r=2,s=0}+\left(H^{(1),\{0\}}_{(0)}\right)^2\,H^{(2),\{0\}}_{(0)}\,\mathcal{O}^{(2),\text{3-pt}}_{r=2,s=0}$$

[SH 2020]

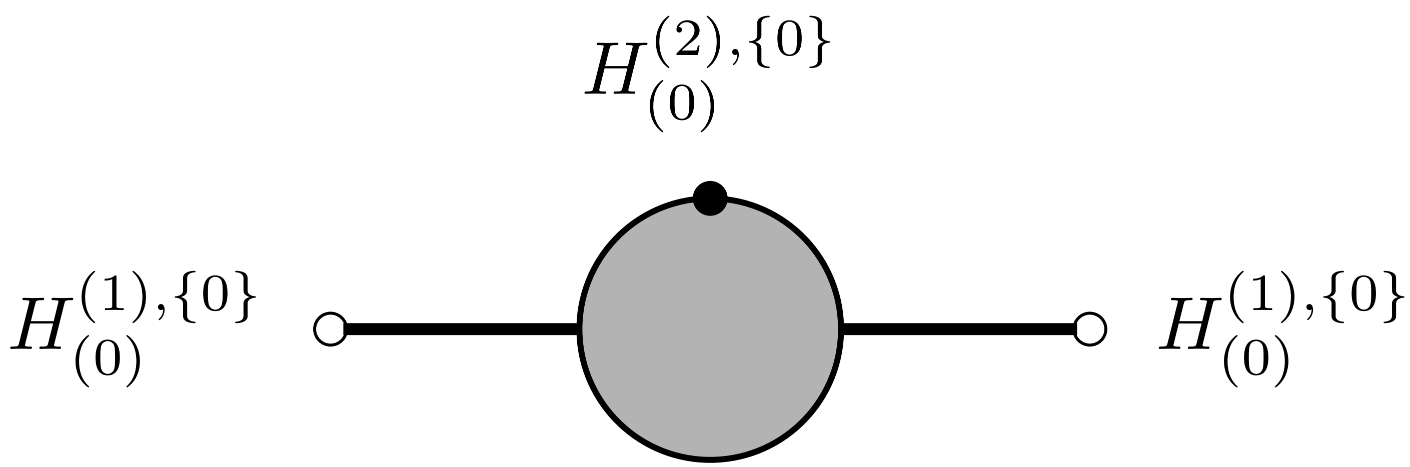
Two-Point Functions:

- same structure as two-point functions as for $r=1$
- same coupling functions $\mathcal{O}^{(2),0}$ and $\mathcal{O}^{(2),1}$ as for $r=1$
- external states obtained through Hecke operators from external states for $r=1$

$$H^{(2),\{0\}}_{(0)}=\mathcal{H}_2\left(H^{(1),\{0\}}_{(0)}\right)\qquad\qquad W^{(2)}_{(0)}=\mathcal{H}_2\left(W^{(1)}_{(0)}\right)$$

Three-Point Functions:

- **Two** external states of the same form as for $r = 1$
- **One** external state obtained through Hecke transform



- **Coupling function:**

$$\mathcal{O}^{(2),\text{3-pt}}_{r=2,s=0}(\widehat{a}_1,\rho) = \frac{4}{3}\mathcal{I}_1(\rho,\widehat{a}_1)$$

Functions $\mathcal{I}_k$ for $k \in \mathbb{N} \cup \{0\}$:

$$\mathcal{I}_k(\rho,\widehat{a}) = \sum_{n=1}^{\infty} \frac{n^{2k+1}}{1-Q_\rho^n}\left(Q_{\widehat{a}}^n + \frac{Q_\rho^n}{Q_{\widehat{a}}^n}\right)$$

Iterative relation:

$$\mathcal{I}_k(\rho,\widehat{a}) = D_{\widehat{a}}^{2k}\,\mathcal{I}_0(\rho,\widehat{a})$$

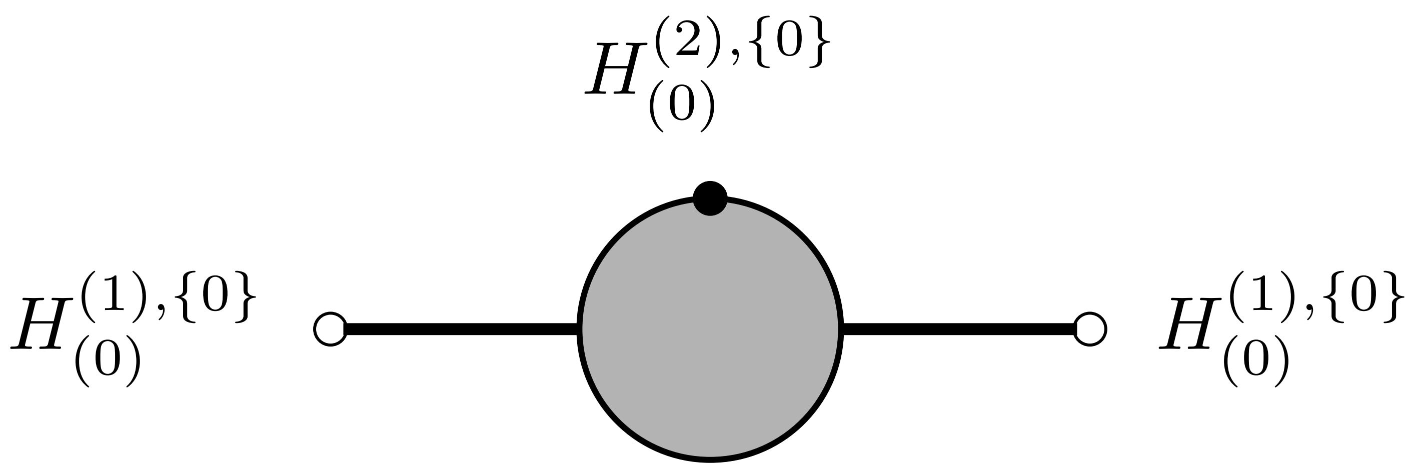
with $D_{\widehat{a}} = \dfrac{1}{2\pi i}\dfrac{\partial}{\partial \widehat{a}} = Q_{\widehat{a}}\dfrac{\partial}{\partial Q_{\widehat{a}}}$

Starting function:

$$\begin{aligned}\mathcal{I}_0(\rho,\widehat{a}) &= -2\,\mathcal{O}^{(2),1}(\widehat{a},\rho)\\ &= \frac{1}{(2\pi i)^2}\left[\mathbb{G}''(\widehat{a};\rho) + \frac{2\pi i}{\rho-\bar{\rho}}\right]\end{aligned}$$

Three-Point Functions:

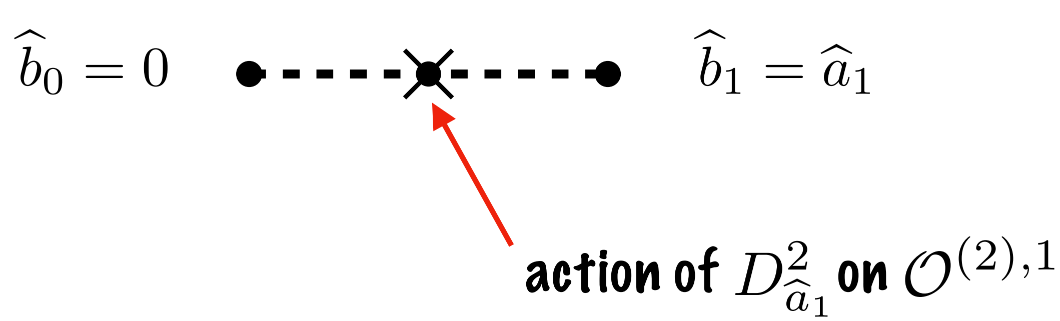
- **Two** external states of the same form as for $r = 1$
- **One** external state obtained through Hecke transform



- **Coupling function:**

$$\mathcal{O}^{(2),\text{3-pt}}_{r=2,s=0}(\widehat{a}_1,\rho) = \frac{4}{3}\mathcal{I}_1(\rho,\widehat{a}_1) = -\frac{8}{3}\,D^2_{\widehat{a}_1}\,\mathcal{O}^{(2),1}(\widehat{a}_1,\rho)$$

graphical representation:



Functions $\mathcal{I}_k$ for $k \in \mathbb{N}\cup\{0\}$:

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Iterative relation:

$$\mathcal{I}_k(\rho,\widehat{a}) = D_{\widehat{a}}^{2k}\,\mathcal{I}_0(\rho,\widehat{a})$$

with $D_{\widehat{a}} = \frac{1}{2\pi i}\frac{\partial}{\partial \widehat{a}} = Q_{\widehat{a}}\frac{\partial}{\partial Q_{\widehat{a}}}$

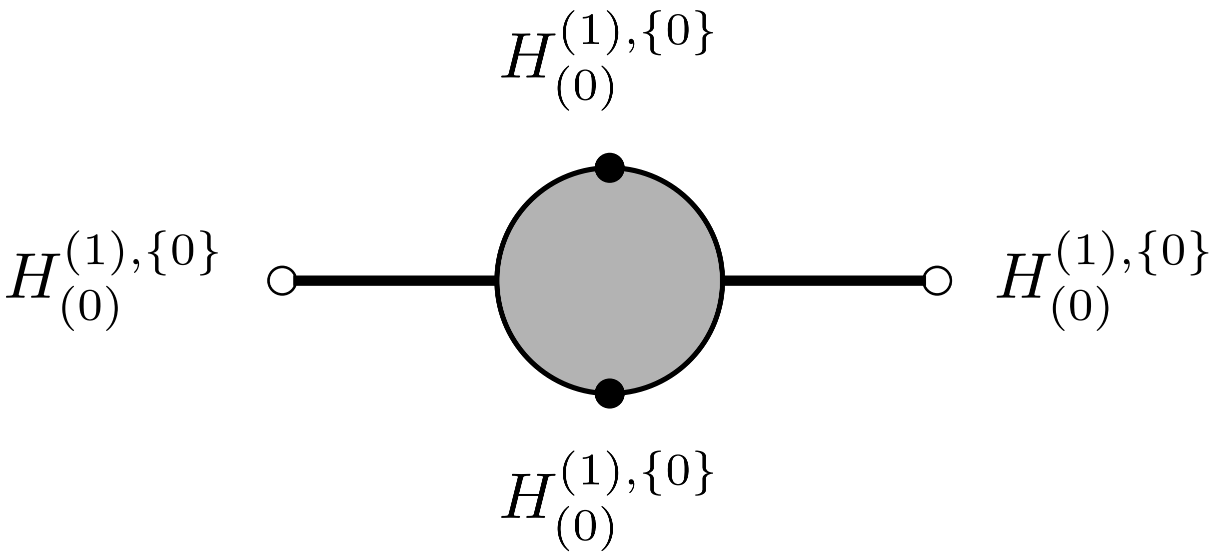
Starting function:

$$\mathcal{I}_0(\rho,\widehat{a}) = -2\,\mathcal{O}^{(2),1}(\widehat{a},\rho) = \frac{1}{(2\pi i)^2}\left[\mathbb{G}''(\widehat{a};\rho) + \frac{2\pi i}{\rho-\bar{\rho}}\right]$$

Four-Point Functions:

$$\left(H^{(1),\{0\}}_{(0)}\right)^4 \; \mathcal{O}^{(2),\text{4-pt}}_{r=2,s=0}$$

- **Four external states of the same form as for $r = 1$**



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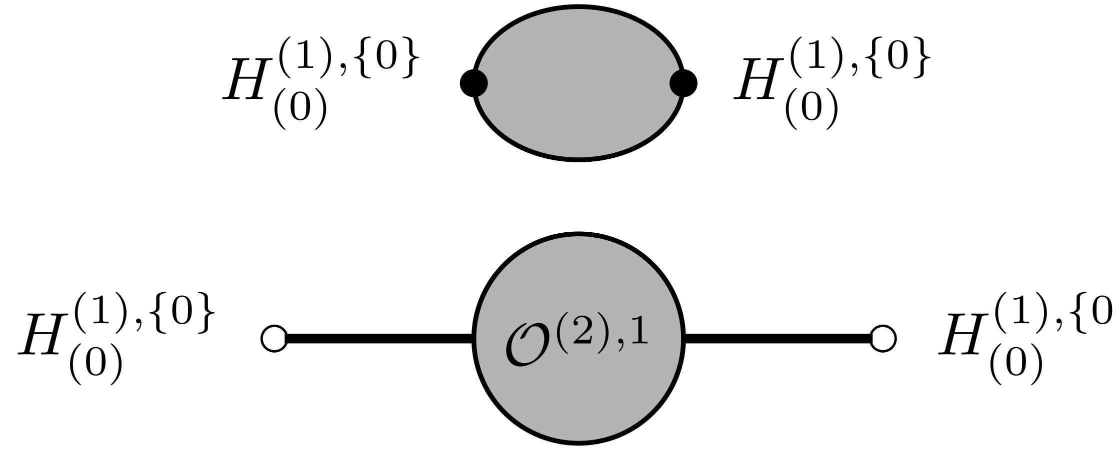


- **coupling function**

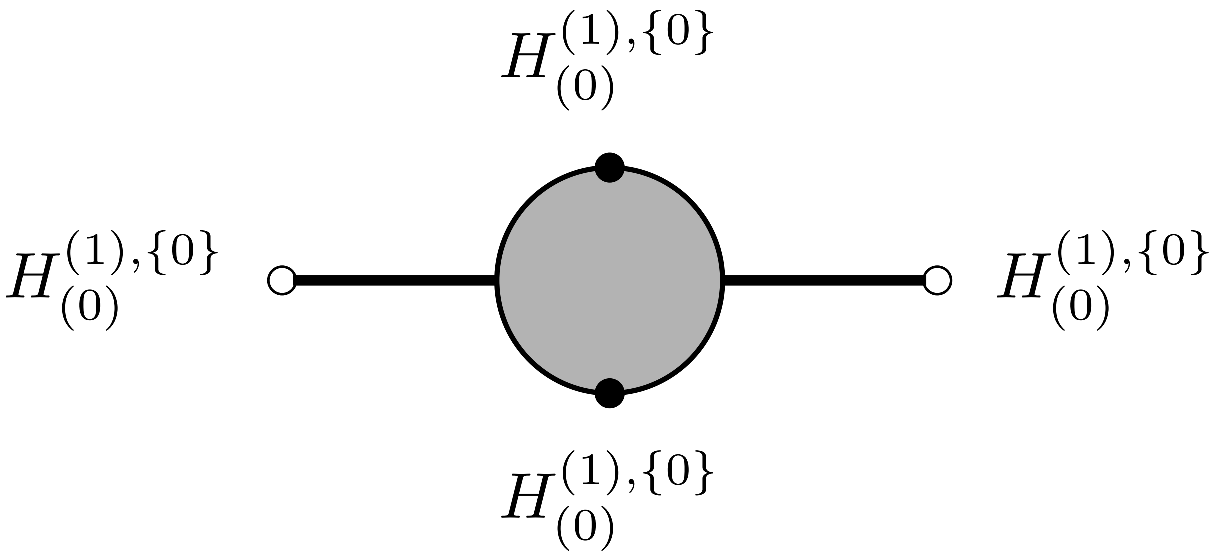
$$\mathcal{O}^{(2),\text{4-pt}}_{r=2,s=0}(\widehat{a}_1,\rho) = -\frac{1}{48}\left[2\mathcal{I}_2(\rho,\widehat{a}_1) + \mathfrak{d}E_4(\rho) + 4E_4(\rho)\,\mathcal{I}_0(\rho,\widehat{a}_1)\right]$$

$$\mathfrak{d} := Q_\rho \frac{d}{dQ_\rho}$$

Four-Point Functions:

$$\left(H^{(1),\{0\}}_{(0)}\right)^4 \ \mathcal{O}^{(2),\text{4-pt}}_{r=2,s=0}$$

- **Four external states of the same form as for $r = 1$**



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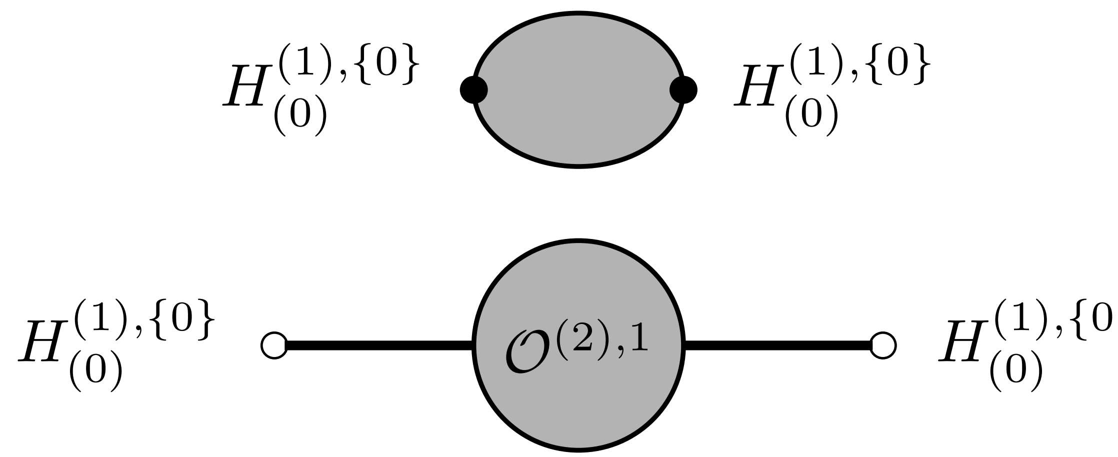


- **coupling function**

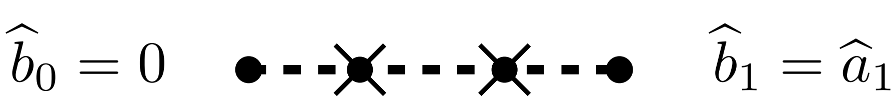
$$\mathcal{O}^{(2),\text{4-pt}}_{r=2,s=0}(\widehat{a}_1,\rho) = -\frac{1}{48}\left[2\mathcal{I}_2(\rho,\widehat{a}_1) + \mathfrak{d}E_4(\rho) + 4E_4(\rho)\,\mathcal{I}_0(\rho,\widehat{a}_1)\right]$$

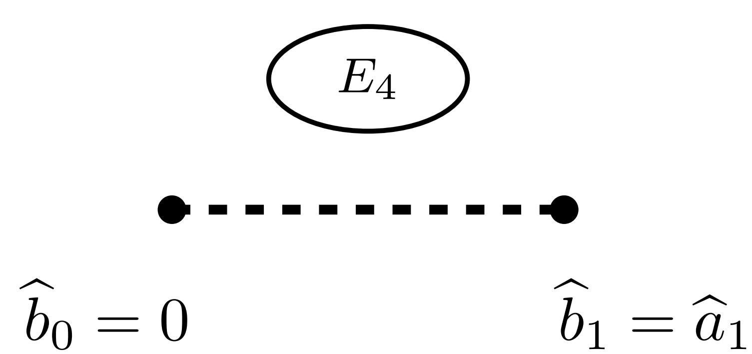
$$= \boxed{\mathcal{O}^{(2),\text{4-pt},1}_{r=2,s=0}} + \boxed{\mathcal{O}^{(2),\text{4-pt},2}_{r=2,s=0}}$$

$$= -\frac{1}{24}\,\mathcal{I}_2(\rho,\widehat{a}_1) = \frac{1}{48}\,D^4_{\widehat{a}_1}\,\mathcal{O}^{(2),1}(\widehat{a}_1,\rho)$$

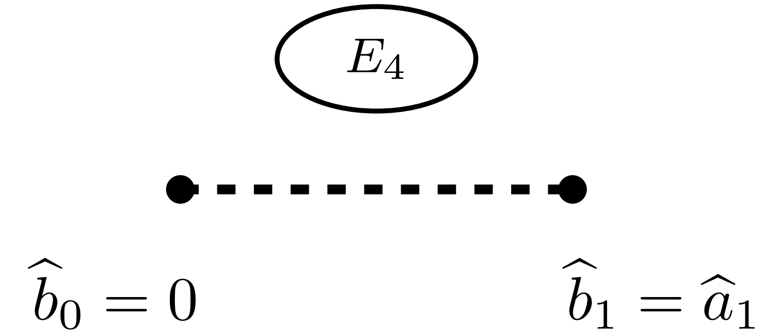
$$= \frac{E_4}{24}\,\mathcal{O}^{(2),1}(\widehat{a}_1,\rho) - \frac{1}{48}\,\mathfrak{d}E_4$$

$$= \frac{1}{24}\left[E_4\,\mathcal{O}^{(2),1}(\widehat{a}_1,\rho) - \text{quasi-holomorphic}\right]$$

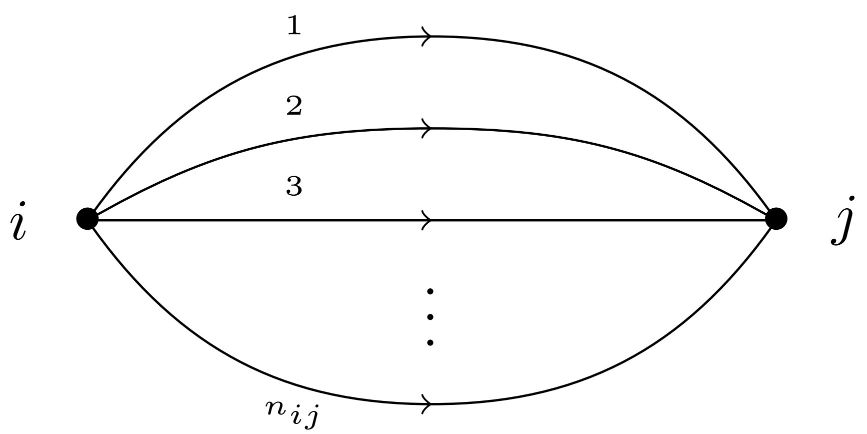




Graph Functions and Graph Forms



Γ graph of N vertices (labelled by i, j) with r_{ij} oriented edges



weight of graph $w = \sum_{1 \leq i < j \leq N} r_{ij}$

[D'Hoker, Green, Gürdogan, Vanhove 2015]
[Broedel, Matthes, Schlotterer 2015]
[D'Hoker, Green 2016]
[D'Hoker, Green, Pioline 2017]
[Zerbini 2018]
[Gerken, Kleinschmidt, Schlotterer 2018-19]
[Mafra, Schlotterer 2019]

Modular graph function associated with Γ

$$\mathcal{C}_\Gamma(\rho) = \prod_{k=1}^{N} \int_{\Sigma_\rho} \frac{d^2 z_k}{\mathrm{Im}\rho} \prod_{1 \leq i < j \leq N} \mathbb{G}(z_i - z_j; \rho)^{r_{ij}}$$

invariant under $SL(2, \mathbb{Z})$

Non-holomorphic modular function

Modular graph forms for $N = 2$: decorate each edge of Γ by a pair of integers (a_r, b_r)

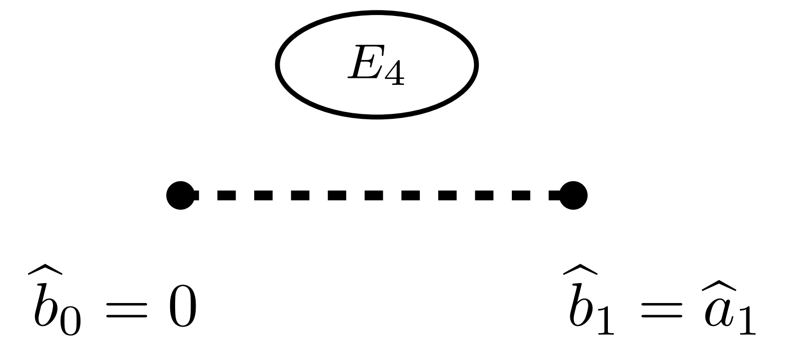
$$\mathcal{C}\left[\begin{smallmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{smallmatrix}\right](\rho) = \sideset{}{'}\sum_{p_1, \ldots, p_n \in \Lambda} \prod_{r=1}^{n} \frac{1}{p_r^{a_r} \bar{p}_r^{b_r}} \, \delta\left(\sum_{s=1}^{N} p_s\right)$$

with $\Lambda = \mathbb{Z} \oplus \rho\, \mathbb{Z}$ [D'Hoker, Green 2016]

Examples: $\mathcal{C}\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right](\rho) = 0 \qquad \forall a_1, b_1 \in \mathbb{N}$

$$\mathcal{C}\left[\begin{smallmatrix} 2k & 0 \\ 0 & 0 \end{smallmatrix}\right](\rho) = \begin{cases} 2\zeta(2)\, \widehat{E}_2(\rho, \bar{\rho}) & \text{if} \quad k = 1\,, \\ 2\zeta(2k)\, E_{2k}(\rho) & \text{if} \quad k > 1\,. \end{cases}$$

For the current case:



$$2\,\zeta(4)\,E_4(\rho) = \mathcal{C}\left[\begin{smallmatrix}4 & 0\\ 0 & 0\end{smallmatrix}\right](\rho)$$

$$= \frac{1}{3!\,(\mathrm{Im}\rho)^4}\,\nabla^2\,(\mathrm{Im}(\rho))^2\,\mathcal{C}\left[\begin{smallmatrix}2 & 0\\ 2 & 0\end{smallmatrix}\right](\rho)$$

$$\nabla = 2i\,(\mathrm{Im}\rho)^2\,\frac{\partial}{\partial\rho}$$

$\mathcal{C}\left[\begin{smallmatrix}2 & 0\\ 2 & 0\end{smallmatrix}\right]$ given as an integral over scalar Greens function

$$\mathcal{C}\left[\begin{smallmatrix}2 & 0\\ 2 & 0\end{smallmatrix}\right](\rho) = \mathcal{C}\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right](\rho) = \int_\Sigma \frac{d^2z_1}{\mathrm{Im}\rho}\int_\Sigma \frac{d^2z_2}{\mathrm{Im}\rho}\,\left(\frac{\pi}{(\mathrm{Im}\rho)}\,\mathbb{G}(z_1 - z_2;\rho)\right)^2$$

suggests corrections to the 2-point function from integrated vertex insertions

Similar contributions appear for $r > 2$

Summary and Conclusions

Studied dualities in a class of Little String Orbifolds:

* partition function $\mathcal{Z}_{N,M}$ compute as topological string partition function on $X_{N,M}$
* weak coupling regions give rise to different (but equivalent) expansions of $\mathcal{Z}_{N,M}$ that can be interpreted as instanton partition functions, dualities:

$$[U(M)]^N \Longleftrightarrow [U(M')]^{N'} \quad \text{for} \quad \begin{array}{l} NM = N'M' \\ \gcd(N,M) = \gcd(N',M') \end{array}$$

* non-perturbative symmetries suggest Feynman graph decomposition of the free energy

  leading instanton level: 'tree level' graphs as combinations of Greens functions of a free scalar field on the torus

  higher instanton levels: integrated corrections similar to loop corrections
  relation to modular graph forms of integrated propagators

Future directions:

* geometric reason for the decomposition of the free energy
* generalisation beyond free energy
* extension to further (phenomenologically realistic) theories