SUSY gauge theories, cohomology, and Bethe ansatz

M.D.–Nekrasov '21
Bullimore–Zhang '21
Okounkov et. al. '12
— " — '16
M.D. to appear
M.D.–Nekrasov, to appear

① Vacuum geometry.

Trivially gapped QFT — single gapped vacuum (invertible in the IR). What is its response to slowly varying parameters (without closing the gap...)?
Parameters vary in spacetime, the response is encoded in the effective action: $\Gamma_{eff}[\phi]$, $\phi: \mathbb{R}^n \to \mathcal{P}$ are parameters.

Invertible F.T. $\left(Z[M] = e^{-\Gamma_{eff}[\phi]}\right)$.

Slowly varying parameters $\Leftrightarrow$ adiabatic limit $\Rightarrow$ top. terms in Γ_{eff}.
This is really the QFT version of the Berry's problem.

In D-dim QFT, one gets local connection D-form (on (D−1)-gerbe).

More traditional Berry problem $\rightsquigarrow$ adiabatic thm:
change parameters slowly with time, $\Rightarrow$ local connection one-form.

IR-divergent on non-compact space (Kap. & Spod. ...)

$\Rightarrow$ Study $\underbrace{\Sigma}_{\text{space}} \times \mathbb{R}_t$

Low energy effective action:

$$\Gamma[\phi] = \int dt \left\{ \underbrace{\alpha^{(0)}(\phi)}_{\substack{\text{vacuum energy (0 if SUSY)}\\ \text{dynamical phase}}} + \underbrace{\alpha^{(1)}_i(\phi)\dot\phi^i}_{\substack{\text{geometric phase}\\ \text{(Berry phase)}}} + \overbrace{\alpha^{(2)}_{ij}(\phi)\dot\phi^i\dot\phi^j + \ldots}^{\text{vanish in the adiabatic limit}} \right\}$$

In general, UV divergences can make $\alpha^{(1)}$ ill-defined (ambig.)
This can happen for general enough backgrounds...
Making "old" Berry problem ill-defined in QFT.
One can analyze this by classifying local counterterms.

Problem: Study terms $\mathcal{L}^{(1)}_i(\phi)\dot{\phi}^i$ in the 1d effective action modulo local D-dimensional UV counterterms on $\Sigma\times\mathbb{R}$.

[M.D., to appear]

E.g. tt^* geometry — fortunately, it's well-defined.

We are interested in gapped 3d QFT on $T^2\times\mathbb{R}_t$ coupled to:

$\overset{\shortparallel}{E_\tau}$

$\mathcal{P}=\left(\text{flat connections for } U(1)^r \text{ on } E_\tau\right)$ (plus real $U(1)^r$-masses)

The relevant effective action is

$$\Gamma=\frac{k_{ij}}{4\pi}\int A^i\wedge dA^j \longrightarrow \text{reduce on } E_\tau \text{ w/ } \quad a_1^i=\frac{1}{2\pi}\int_A A^i, \quad a_2^i=\frac{1}{2\pi}\int_B A^i$$

$$a_1^i\sim a_1^i+1,\quad a_2^i\sim a_2^i+1$$

$$\Gamma=\pi k_{ij}\int dt\left(a_1^i\dot{a}_2^j-a_2^i\dot{a}_1^j\right)\Rightarrow \mathcal{B}=\pi k_{ij}\left(a_1^i\,da_2^j-a_2^i\,da_1^j\right).$$

$x^i=a_1^i+\tau a_2^i \quad\Rightarrow\quad \mathcal{B}$ defines holo connection ($d\mathcal{B}$ is $\Omega^{(1,1)}$).

$\Rightarrow$ holomorphic line bundle $\mathcal{L}$ spanned by gapped vac. $\mathbb{C}|0\rangle$ over $\mathcal{E}_T$ = flat T-connections on E_τ = $E_\tau\otimes\mathfrak{t}=\dfrac{\mathfrak{t}\otimes\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$

Each gapped vacuum $p\longrightarrow$ holo line bundle $\mathcal{L}_p$ over $\mathcal{E}_T$.

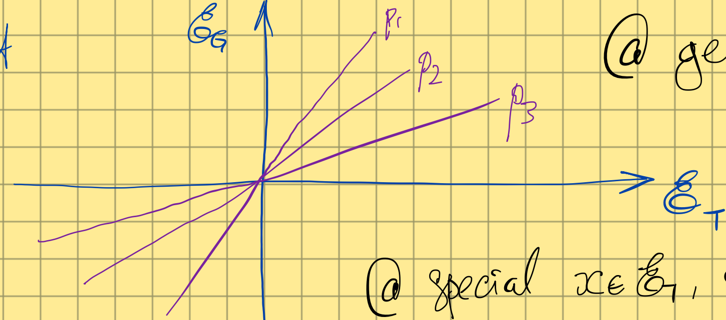
In 2d & 1d trivial. In 3d nontrivial, $c_1(\mathcal{L}_p)=k_{ij}$.
(degen. limit, $\mathfrak{t}\otimes\mathbb{C}^\times$ and $\mathfrak{t}\otimes\mathbb{C}$).

$\mathrm{Ell}_T(pt)=\mathcal{E}_T$; $\mathcal{L}_p$ – bundle over $\mathrm{Ell}_T(pt)$; its sections = "ell coho classes".

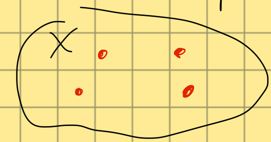
Many points that are gapped $\forall\, x\in\mathcal{E}_T \Rightarrow \mathrm{Ell}_T(pt_1\sqcup pt_2\sqcup\ldots\sqcup pt_n)$
$=\mathcal{E}_T\times\ldots\mathcal{E}_T$
with $\mathcal{L}=\mathcal{L}_1\sqcup\mathcal{L}_2\sqcup\ldots\sqcup\mathcal{L}_n$.

Suppose pt's are not gapped at $x=0$ and X = moduli space of SUSY vacua (Higgs branch) @ $x=0$.

⇒ We get

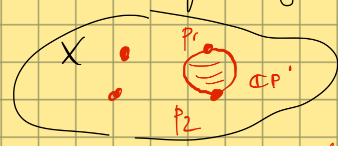


@ generic $x \in \mathcal{E}_T$, pt's isolated



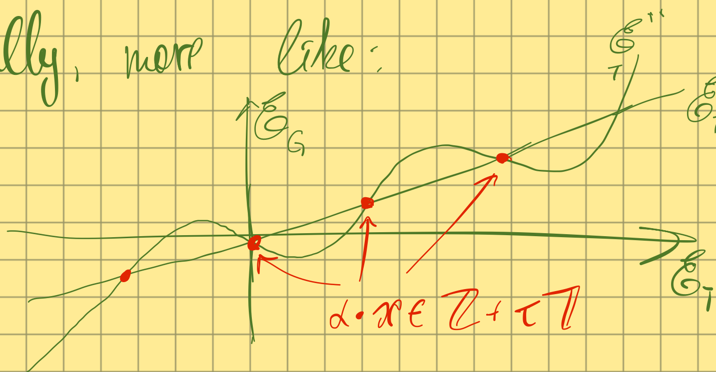
@ special $x \in \mathcal{E}_T$, s.t. $\alpha \cdot x = n + m\tau \in \mathbb{Z} + \tau\mathbb{Z}$
(α - certain weights for global symm T),

part of X regenerates:



α: T-weight for which T_α orbit connects p_1 to p_2.

Really, more like:



Let "~" denote gluing $\mathcal{E}'_T$ to $\mathcal{E}''_T$ along such special loci.

Then $\mathrm{Ell}_T(X) = \dfrac{\mathcal{E}_T \cup \mathcal{E}_T \cup \ldots \cup \mathcal{E}_T}{\sim}$

$\mathrm{Ell}_T(X)$ - elliptic coho variety, scheme over $\mathcal{E}_T$.

Not affine (not Spec R). In gauge theory $s^a = f(x)$.

In 3d, also have A' = torus of top symm.

Eff. C.S. in general has to levels for $T \times A'$ ⇒

⇒ must include $\mathcal{E}_{A'}$: flat A'-conn. on $\mathbb{E}_\tau$.

$E_T(X) = \mathrm{Ell}_T(X) \times \mathcal{E}_{A'}$ - extended ell. coho = scheme over $\mathcal{E}_T \times \mathcal{E}_{A'}$.

In the 2d limit, $\mathrm{Ell}_T(X)$ becomes affine $\mathrm{Spec}\, K_T(X)$

In the 1d limit, — " — $\mathrm{Spec}\, H_T(X)$.

⇒ $\mathrm{Ell}_T(X)$ & $E_T(X)$ are natural elliptic generalization.

⇒ "Spectral data describing classical vacua".

Each irreducible component $\mathcal{E}_T \times \mathcal{E}_{A'}$ of $E_T(X)$ carries a line bundle $\mathcal{L}_p$. They glue together (C.S. levels agree) to $\mathcal{L}$ - a line bundle over $E_T(X)$.

The problem of extension. Let a flavor torus $A \circlearrowright X$.

$$\Gamma(X^A, \mathcal{L}') \xrightarrow{Stab} \Gamma(X, \mathcal{L}) \qquad \text{Need a natural map.}$$

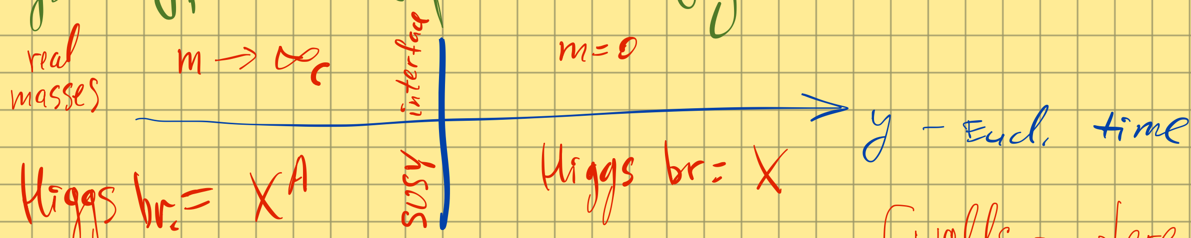
$$\Gamma(X^A, \mathcal{L}') = \bigsqcup_p \mathcal{L}'_p \qquad \Gamma(X, \mathcal{L}) = \bigsqcup_p \mathcal{L}_p/\sim$$

(In K-theory, $K_T(X^A) \to K_T(X)$, coho $H_T(X^A) \to H(X^A)$).

Physically: $\left(\begin{matrix}\text{vacua of} \\ \text{theory with} \\ \text{Higgs br. } X^A\end{matrix}\right) \xrightarrow{Stab} \left(\begin{matrix}\text{vacua of} \\ \text{theory with} \\ \text{Higgs br. } X\end{matrix}\right)$. ?

There is a certain math. definition of Stab based on the gradient flows $\simeq A_{\mathbb{C}}$ - trajectories on X.

Physically, it is reproduced by:



C = chamber in $\mathfrak{a} = Lie(A)$. [walls = where Higgs branch jumps up in size]

(Construction of the interface/Stab depends on C).

Such an interface takes a vacuum localized @ the isolated $p \in X^A$ (represented by an equiv. form Ψ_p) and extends it to Ω_p localized @ $\overline{Attr(p)}$. (grad flow $\leftarrow$ BPS eqn's)

So such interfaces provide maps between spaces of SUSY vacua that realize Stab.

In general X^A can be non-trivial (non-isolated).

e.g. $H_T(X^A) \xrightarrow{Stab} H_T(X)$

$H_T(X^A)$ = $\mathbb{C}$ vec. space spanned by isolated T-fixed points; $H_T(X)$ = same vector space

This allows to obtain a new conceptual understanding of the Bethe / Gauge:

$\Rightarrow R_{C_1,C_2} = \mathrm{Stab}_{C_2}^{-1} \circ \mathrm{Stab}_{C_1}$

$R_{C_1C_2}: \begin{pmatrix}\text{vacua}\\ @\ m\to\infty_{C_1}\end{pmatrix} \to \begin{pmatrix}\text{vacua}\\ @\ m\to\infty_{C_2}\end{pmatrix}$ = square matrix

For us X = Nakaj. quiver var. $X = T^*R /\!/\!/ G$

G_f – flavor group, $A \subset G_f$ – max torus. x = flat conn. (equiv. param)

$U(1)_\hbar$ – special symm. (1) scales ω with wt $= 2$

(2) part of 3d $\mathcal{N}=4$ R-symm., but flavor symm. for $\mathcal{N}=2$.

$\hbar$ = flat conn. (equiv. param).

$T = A \times U(1)_\hbar$ – $\mathcal{N}=2$ flavor; $\mathcal{E}_T$ – torus of flavor (equiv.) param.

A' – top. symm. $z \in \mathcal{E}_{A'}$ – flat conn. for A' (Kähler param).

$R_{C_1,C_2}(x, \hbar, z)$ – elliptic R-matrix.

x = spectral parameter; $\hbar$ – quantum param; z – dynamical param.

(trigon./K-th limit has no z, but has p.c.-dependence on slope

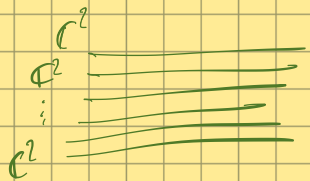
rat/H limit only depends on $x, \hbar$).

(it's "generic q" for quantum group)

- One needs generic $\hbar$ to have interesting basic story.
- $\hbar = 0 \Rightarrow$ trivial.
- Non-generic $\hbar \neq 0$ — something interesting?

Ex. XYZ $\quad T(L) = \bigoplus_{N=0}^{L} \begin{array}{c}\boxed{L}\\ |\\ (N)\end{array}$ — family of QFT's

SUSY vacua $\mathcal{V}(L) \simeq \bigoplus_{N=0}^{L} \mathbb{C}^{\binom{L}{N}} \simeq \mathbb{C}^{2^L} =$ Hilb. space of XYZ



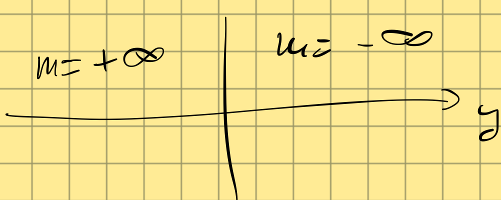
$\Big\}$ L of them, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 = \mathbb{C}^{2^L}$

We want to build the R-matrix, L-op's, more generally, consider:

$R : \mathbb{C}^{2^{L_1}} \otimes \mathbb{C}^{2^{L_2}} \to \mathbb{C}^{2^{L_1}} \otimes \mathbb{C}^{2^{L_2}}$

$$T(L_1+L_2)^m \simeq T(L_1) \otimes T(L_2)$$

$$\text{vacua}\left(T(L_1+L_2)^m\right) = \mathcal{V}(L_1) \otimes \mathcal{V}(L_2)$$



$\Rightarrow R_{+-} : \mathcal{V}(L_1) \otimes \mathcal{V}(L_2) \longrightarrow \mathcal{V}(L_2) \otimes \mathcal{V}(L_1)$

$$R_{+-}(x_1, x_2, \ldots, x_L ; \hbar ; z)$$

$$x_1 x_2 \ldots x_L = 1.$$

The knowledge of such R_{+-} provides representation of the quantum group (Yangian in 1d, q.loop in 2d, q.ellipt. in 3d) on $\mathcal{V}(L)$.

Action of the quantum group is generated by interfaces.

E.g. $\left(\begin{array}{c}\boxed{L+1}\\ |\\ (N+1)\end{array}\right)^m = \begin{array}{c}\boxed{L}\\ |\\ (N+1)\end{array} \oplus \begin{array}{c}\boxed{L}\\ |\\ (N)\end{array} \otimes \begin{array}{c}\boxed{1}\\ |\\ (1)\end{array}$

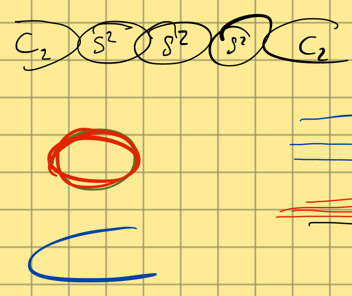
$m \to -m$ interface leads to a natural interface between $\begin{array}{c}\boxed{L}\\ |\\ (N)\end{array}$ and $\begin{array}{c}\boxed{L}\\ |\\ (N+1)\end{array}$ $\rightsquigarrow$ raising op. of the q. group

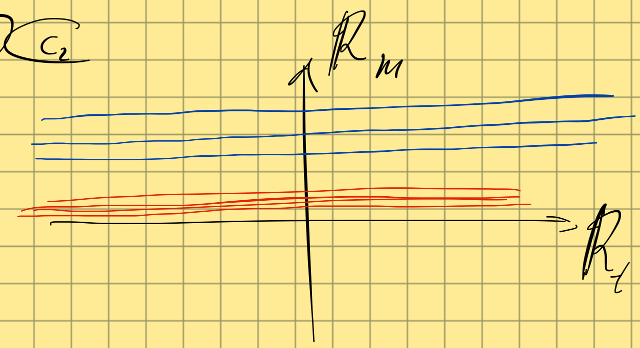
(e.g. Varagnolo, Nakajima via Lagr. corresp.)

we describe q. group in the RTT form.

Such approach works for ∀ quivers, also affine.

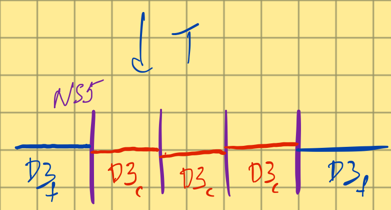
$$\mathbb{E}_\tau \times \mathbb{R}_t \times \mathbb{R}_m \times \mathbb{R}^2_\hbar \times \mathbb{C}^2/\Gamma$$





	$\mathbb{E}_\tau$	$\mathbb{R}_t$	$\mathbb{R}_m$	$\mathbb{R}^2_\hbar$
$D4_c$	X	X	·	·
$D4_f$	X	X	·	·

$$T^*(\mathbb{E}_\tau \times \mathbb{R}_t) = \underbrace{\mathbb{E}_\tau \times \mathbb{R}^2_\hbar}_{T^*\mathbb{E}_\tau} \times \underbrace{\mathbb{R}_t \times \mathbb{R}_m}_{T^*\mathbb{R}_t}$$



Here — carries a repr. of the ADE g.
—, =, ≡ ... are vectors in such a repr.