# Insights from the quantum modularity of 3-manifold invariants

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**New Trends in Non-Perturbative Gauge/String Theory and Integrability**
**Institut de Mathématiques de Bourgogne**
**28 June 2022**

## Based on…

### Quantum modularity of higher rank homological blocks

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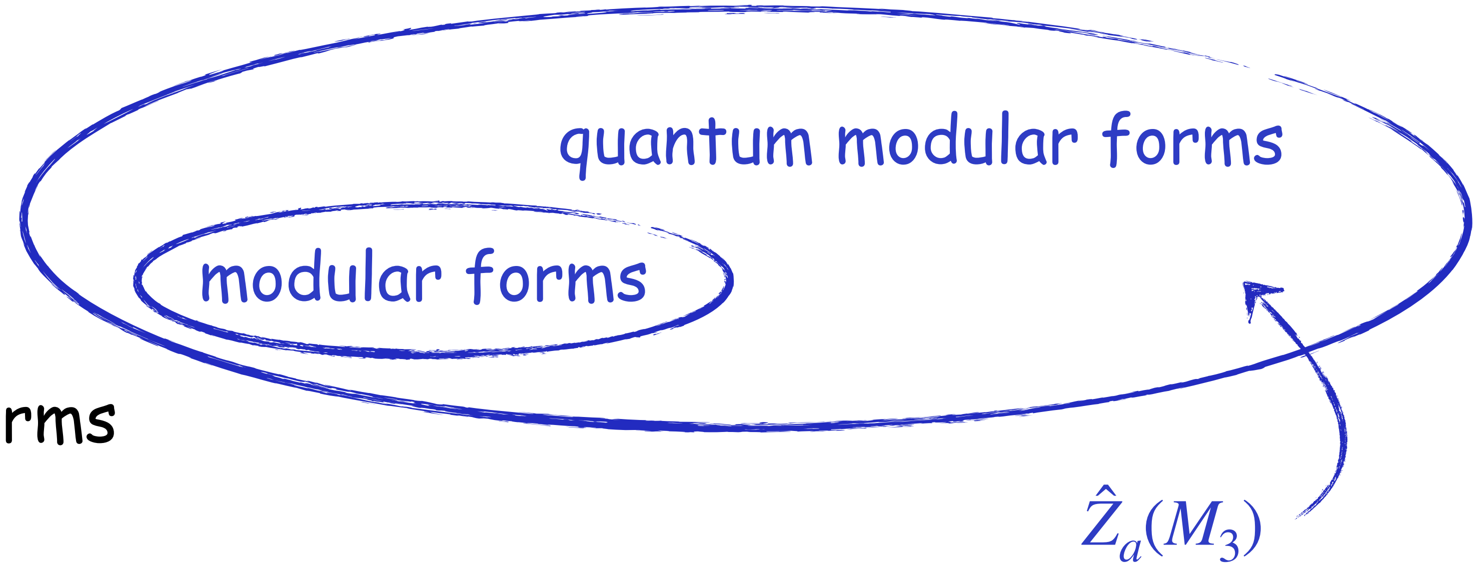
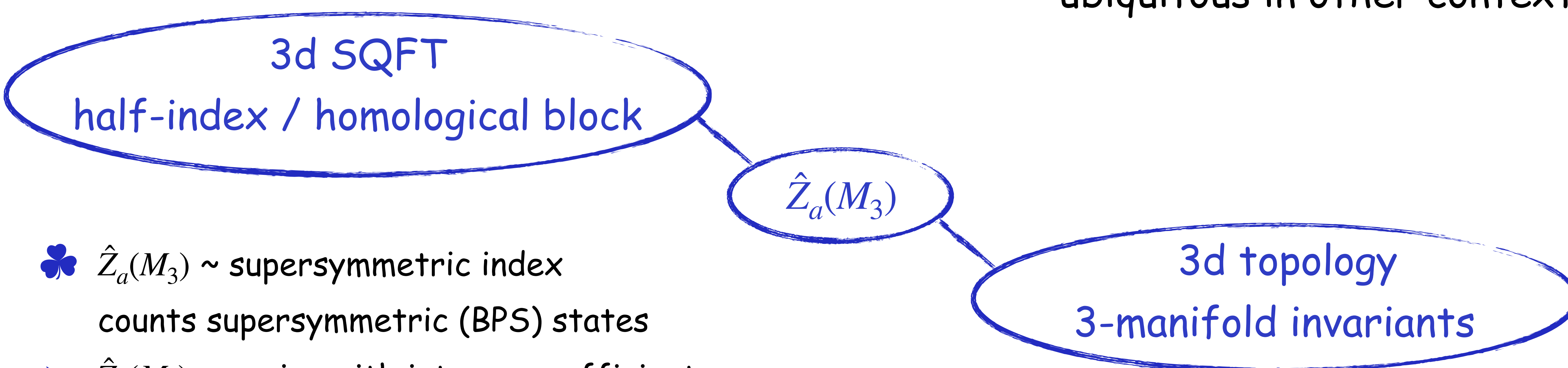
# Context and motivation




Modular forms & quantum modular forms

$\hat{Z}_a(M_3)$ are q-series originally introduced from a physics perspective [Gukov, Putrov, Vafa '16] [Gukov, Pei, Putrov, Vafa '17]

ubiquitous in other contexts




- ♣ $\hat{Z}_a(M_3)$ ~ supersymmetric index
  counts supersymmetric (BPS) states
- → $\hat{Z}_a(M_3)$ q-series with integer coefficients
- ♣ 3d SQFT has an M-theory realisation

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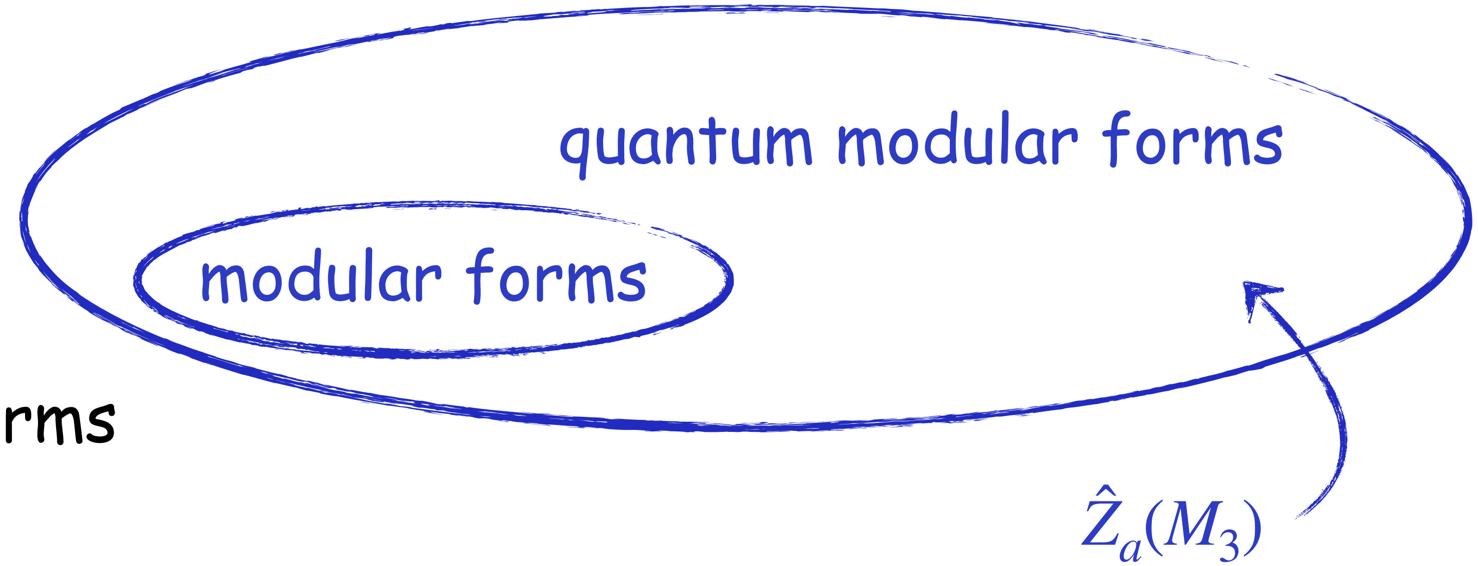


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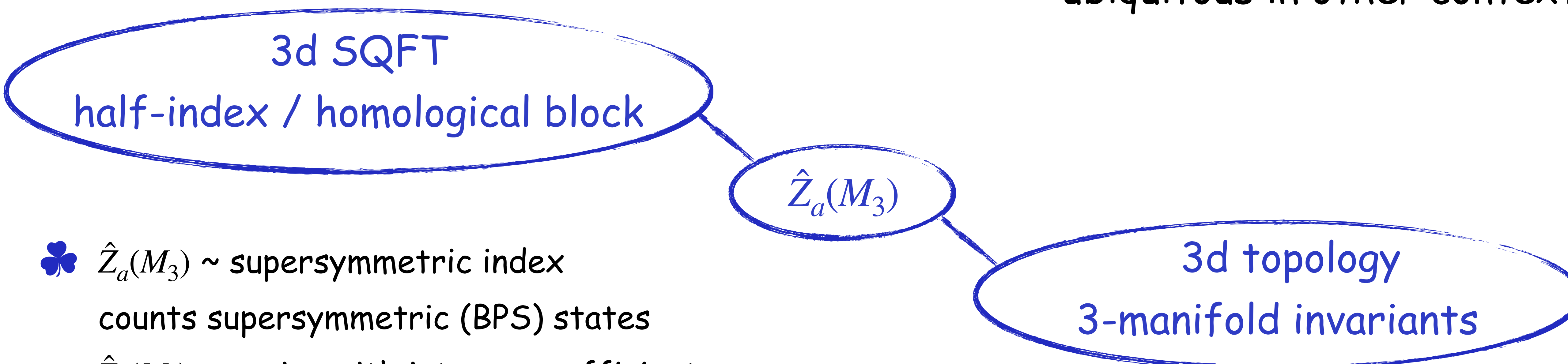



- $\hat{Z}_a(M_3)$ ~ supersymmetric index
  counts supersymmetric (BPS) states

→ $\hat{Z}_a(M_3)$ q-series with integer coefficients

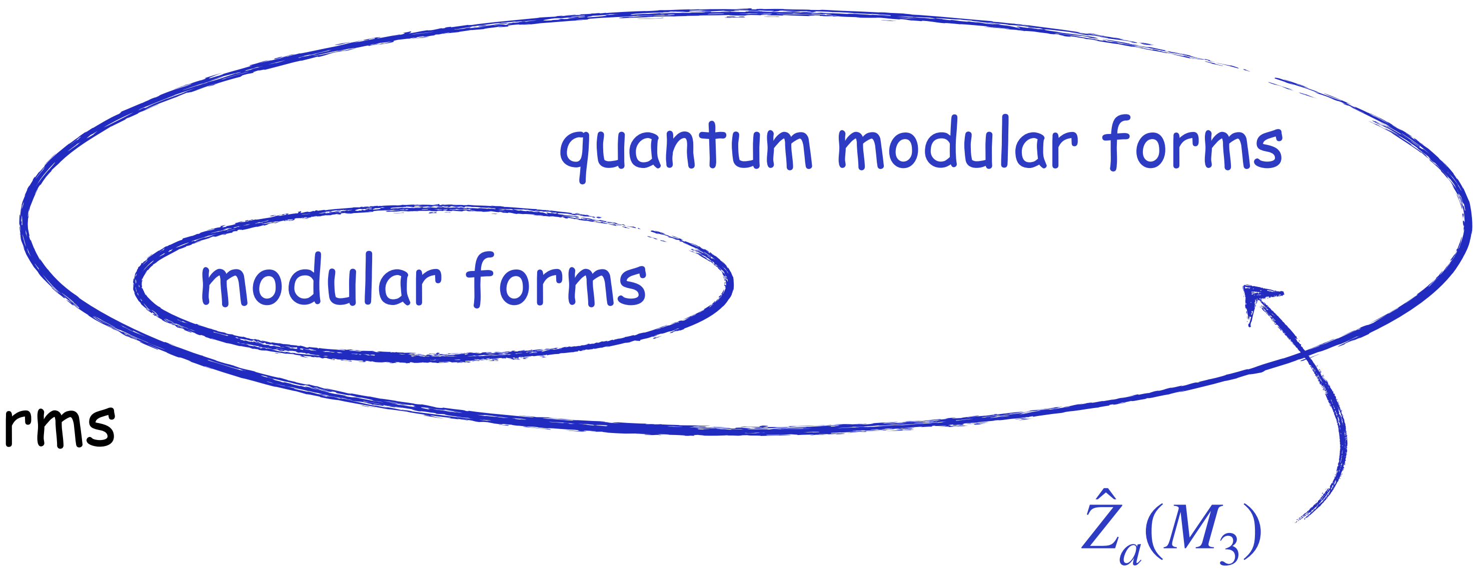
- 3d SQFT has an M-theory realisation

- M-theory and 3d-3d correspondence
  associates manifold $M_3$ to the 3d SQFT

→ $\hat{Z}_a(M_3)$ topological invariants of $M_3$

# Context and motivation

Modular forms & quantum modular forms

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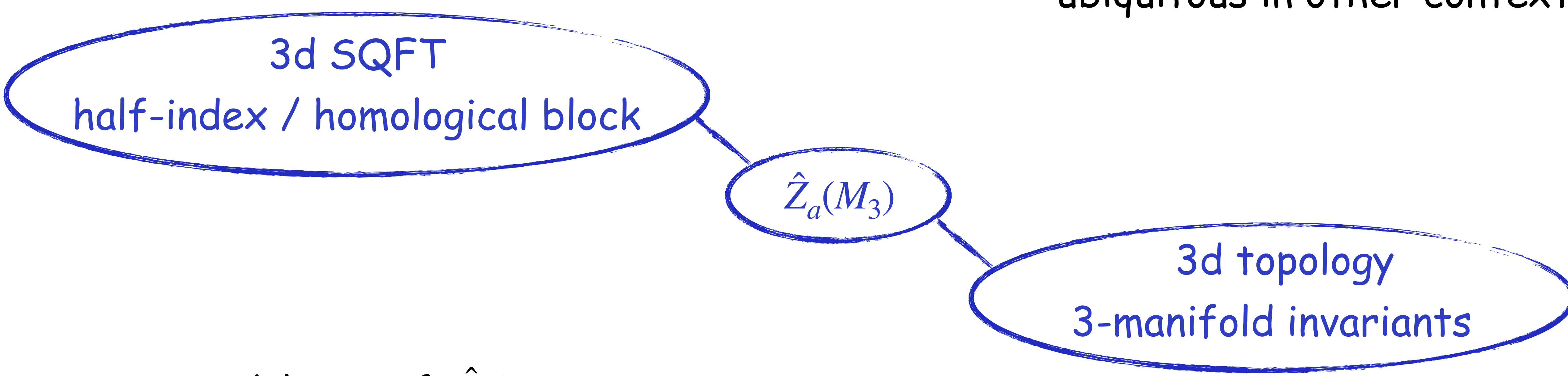


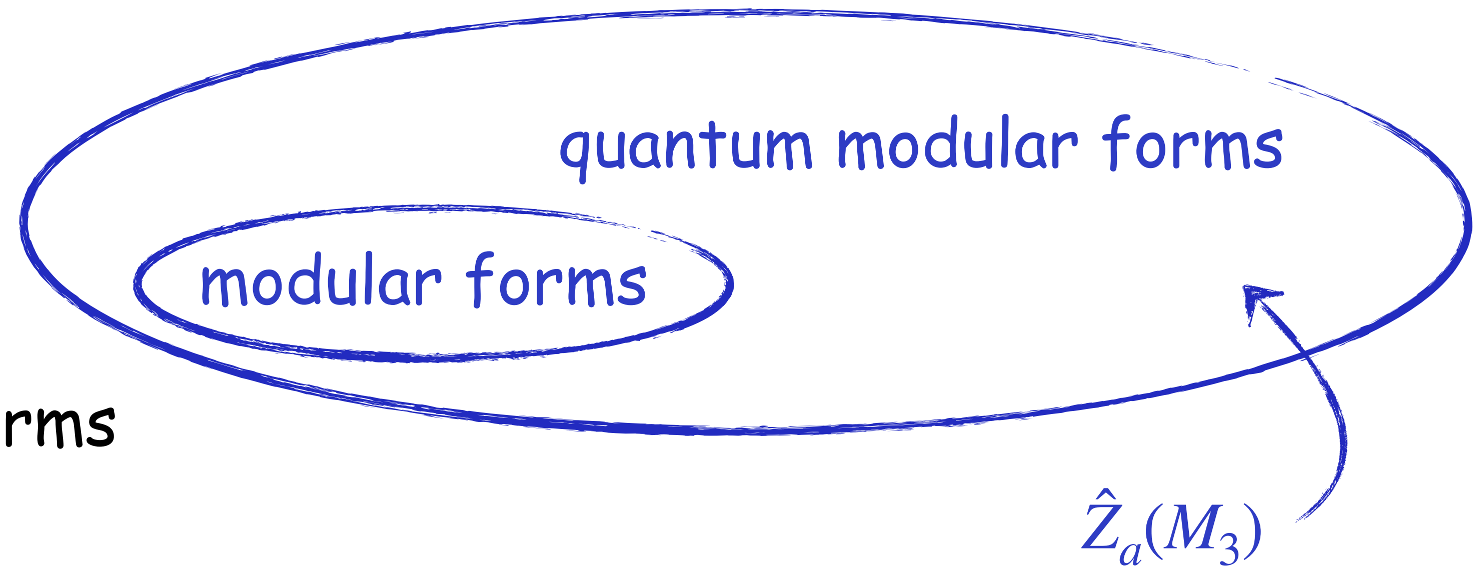

**Quantum modularity of $\hat{Z}_a(M_3)$**

[Cheng, Chun, Ferrari, Gukov, Harrison '18]
[Cheng, Ferrari, Sgroi '19]
[Bringmann, Kaszian, Milas '19]
[Bringmann, Kaszian, Milas, Nazaroglu '21] + …

… can provide various physical insights

eg. definition of $\hat{Z}$, hidden structures

# Main outcomes




Modular forms & quantum modular forms

The $\hat{Z}^{\mathfrak{g}}(M_3)$ invariants are defined for simply laced gauge groups G with Lie algebra $\mathfrak{g}$ and weakly negative definite Seifert 3-manifolds $M_3$

[Park '19], [Cheng, Chun, Fegin, Ferrari, Gukov, Harrison, Passaro '22]

Theorem For weakly negative definite Seifert 3-manifolds with 3 exceptional fibers & $\mathfrak{g} = A_2$

1. QMF: $\hat{Z}^{A_2}(q)$ is a sum of two depth-2 quantum modular forms
2. Recursion: If $\hat{Z}^{A_1}(q)$ has a certain $SL(2,\mathbb{Z})$ structure, this structure is also found in $\hat{Z}^{A_2}(q)$

[Cheng, Coman, Passaro, Sgroi, *to appear*]

# Modular forms

A modular form f($\tau$) of weight w, multiplier system $\chi$ with respect to $\Gamma \subseteq SL_2(\mathbb{Z})$

is a holomorphic function of $\tau \in \mathbb{H}$ if $f|_{w,\chi}\gamma(\tau) = f(\tau)$ for any $\gamma \in \Gamma$,

where $f|_{w,\chi}\gamma(\tau) := (c\tau + d)^{-w}\chi(\gamma)^{-1}f(\gamma\tau)$ .

$\gamma = \begin{pmatrix} a, b \\ c, d \end{pmatrix} \in \Gamma$ acts on $\mathbb{H}$ by a fractional linear transformation $\gamma\tau = \dfrac{a\tau + b}{c\tau + d}$


Modular forms include $\theta$-functions $\theta(\tau) = \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}}$ with expansion parameter $q = e^{2\pi i \tau}$

- Half integer weight $\theta$-functions relevant in relation to $\hat{Z}_a(M_3)$

$$\theta^0_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} q^{\frac{k^2}{4p}} \quad \text{weight } 1/2 \qquad \theta^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} k q^{\frac{k^2}{4p}} \quad \text{weight } 3/2$$

The radial limit $|q| \to 1 \Leftrightarrow \tau \to \alpha \in \mathbb{Q}$ defines a function on $\mathbb{Q}$ $\quad f(\alpha) := \lim_{t \to 0^+} f(\alpha + it)$

# Quantum modular forms

Quantum modular forms (QMF's) are defined at the boundary of $\mathbb{H}$, on $\mathbb{Q} \cup \{i\infty\}$

A quantum modular form of weight w, multiplier system $\chi$ with respect to $\Gamma \subseteq SL_2(\mathbb{Z})$

is a function $f : \mathbb{Q} \to \mathbb{C}$ such that, $\forall \gamma \in \Gamma$, the function $p_\gamma(x) : \mathbb{Q} \backslash \{\gamma^{-1}(\infty)\} \to \mathbb{C}$

defined by $p_\gamma(x) := f(x) - f|_{w,\chi}\gamma(x)$ has a better analytic behaviour than $f(x)$ . [Zagier '10]

Neither analyticity, nor modularity are required, but failure of one offsets the failure of the other.

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Neither analyticity, nor modularity are required, but failure of one offsets the failure of the other.

A strong quantum modular form is a QMF $f$ which associates to each element $x \in \mathbb{Q}$

a formal power series over $\mathbb{C}$, so that $p_\gamma(x) := \lim_{t\to 0^+} \left(f - f|_{w,\chi}\gamma\right)(x + it)$ $\gamma \in \Gamma$

has a power series expansion around each point $x \in \mathbb{Q}$ and extends holomorphically

to a neighbourhood of $\mathbb{P}^1(\mathbb{R})\backslash S_\gamma$ for $S_\gamma$ a finite set.

# Quantum modular forms

Eichler integrals allow to construct quantum modular forms from modular forms

Given a modular form g of weight w, its Eichler integrals

holomorphic $$\tilde{g}(\tau) = c_{(w)} \int_{\tau}^{i\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$$

non-holomorphic $$g^*(\tau, \bar{\tau}) = c_{(w)} \int_{-\bar{\tau}}^{i\infty} g(\tau')(\tau' + \tau)^{w-2} d\tau'$$

MF $g$ with $w \in \mathbb{Z}/2$ and Fourier expansion

$$g = \sum_{n>0} a_g(n) q^n \longleftrightarrow \tilde{g} = \sum_{n\geq 1} a_g(n) n^{1-w} q^n$$

$$c_{(w)} = \frac{(2\pi i)^{w-1}}{\Gamma(w-1)}$$

[Lawrence, Zagier '99]

are QMF's, since $\tilde{g} - \tilde{g}|_{2-w}\gamma$ and $g^* - g^*|_{2-w}\gamma$ are period integrals.

$$(\tilde{g} - \tilde{g}|_{2-w}\gamma)(\tau) = \int_{\gamma^{-1}(\infty)}^{\infty} g(\tau')(\tau' - \tau)^{w-2} d\tau'$$

♣ Example: false $\theta$-functions

$$\theta^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} k q^{\frac{k^2}{4p}}$$ weight 3/2 $\theta$-function → $$\widetilde{\theta}^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} \operatorname{sgn}(k)\, q^{\frac{k^2}{4p}}$$ false $\theta$-function

# Quantum modular forms - higher depth

More general quantum modularity can be defined recursively

A depth-N QMF is a function $f : \mathbb{Q} \to \mathbb{C}$ such that $p_\gamma := f - f|_w \gamma$ is a sum of QMF's of depth N'<N, multiplied by some real-analytic functions, $\forall \gamma \in \Gamma$.

♣ Example: Iterated non-holomorphic Eichler integral

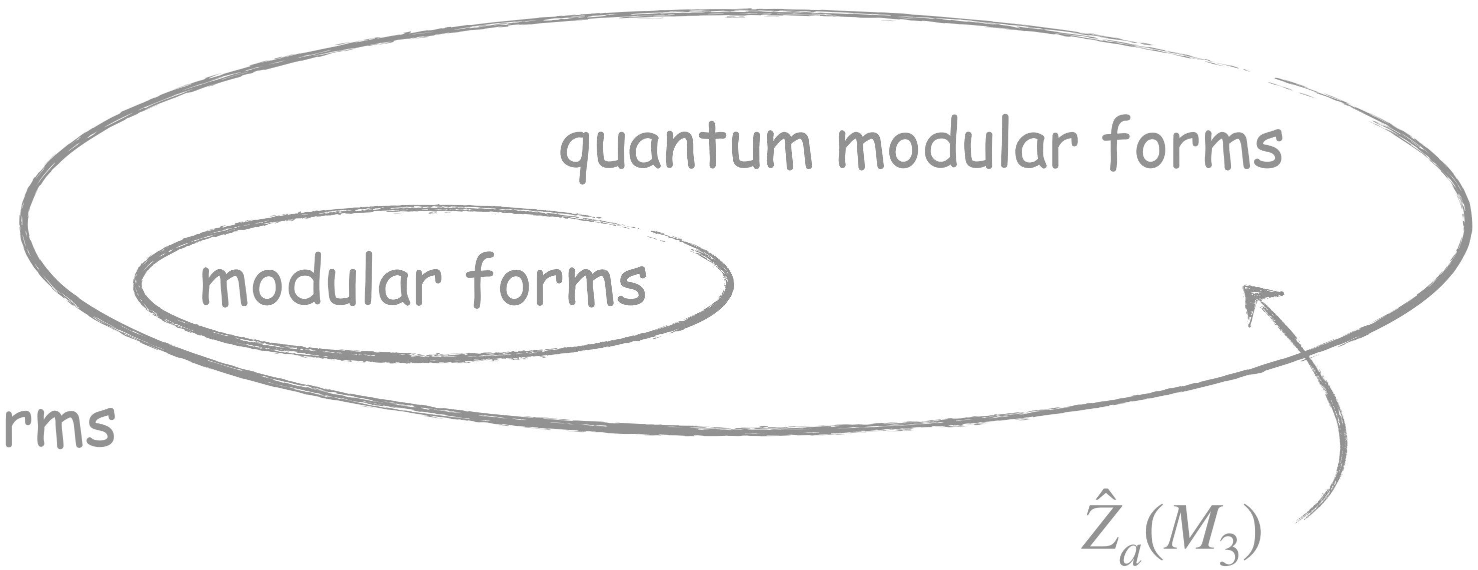
[Bringmann, Kaszian, Milas '17]
[Cheng, Coman, Passaro, Sgroi, *to appear*]

$$I_{f_1,f_2}(\tau,\bar{\tau}) := \int_{-\bar{\tau}}^{i\infty} dz_1 \int_{z_1}^{i\infty} dz_2 \, \frac{f_1(z_1) f_2(z_2)}{(-i(z_1+\tau))^{2-w_1}(-i(z_2+\tau))^{2-w_2}}$$

is a depth-2 QMF

$p_\gamma$ contains a regular non-holomorphic Eichler integral (depth-1 QMF) and analytic functions.

# Context and motivation

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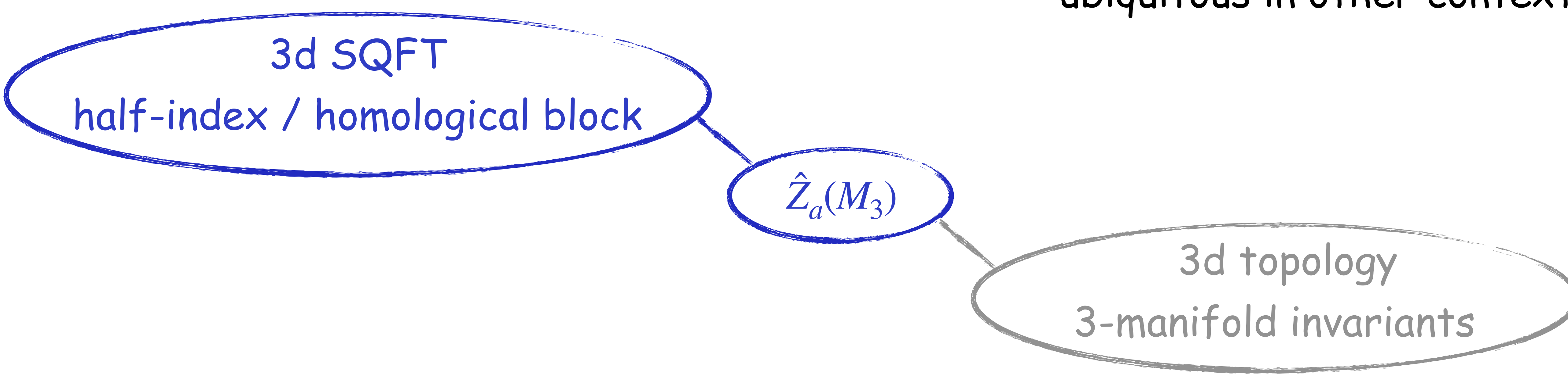


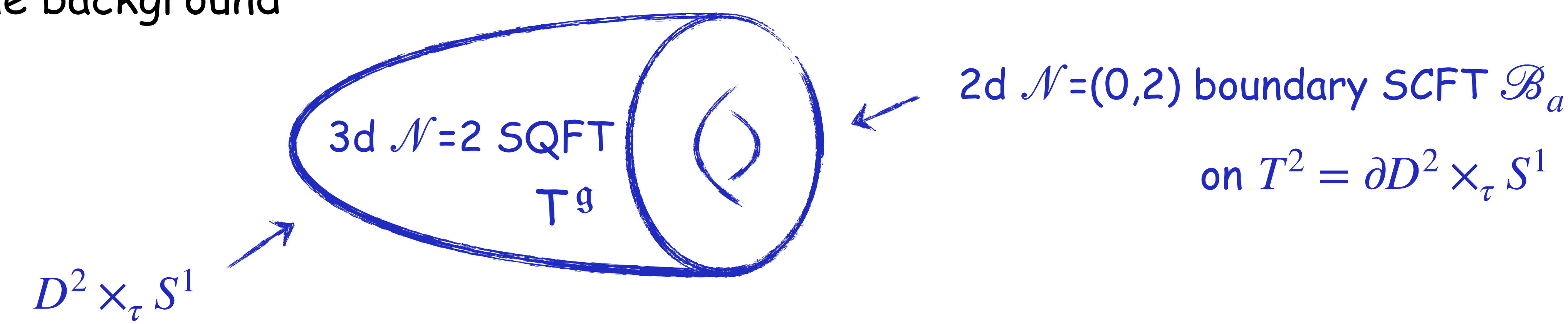
# Physical origin of $\hat{Z}_a(M_3)$

Definition of $\hat{Z}_a^{\mathfrak{g}}(M_3)$ q-series from 3d $\mathcal{N}$=2 SQFT $T^{\mathfrak{g}}$ with simply-laced gauge group G

[Gukov, Putrov, Vafa '16]
[Gukov, Pei, Putrov, Vafa '17]

Lie algebra $\mathfrak{g}$

The spacetime background




$\hat{Z}_a^{\mathfrak{g}}(M_3)$ is the supersymmetric index of $T^{\mathfrak{g}}$ or "half-index" counting of BPS states

Hilbert space of BPS states $\mathscr{H}_{BPS;a} = \oplus_{i,j} \mathscr{H}_a^{i,j}$ doubly graded by two U(1) symmetries

$$\hat{Z}_a(M_3) = Z_{T^{\mathfrak{g}}}(D^2 \times_\tau S^1; \mathscr{B}_a) = \sum_{i,j} (-1)^i q^j \dim \mathscr{H}_a^{i,j}$$

$q = e^{2\pi i \tau}$ $\quad \tau \in \mathbb{H}$

boundary condition label

$\hat{Z}_a^{\mathfrak{g}}(M_3)$ admits a q-series expansion with integer powers and integer coefficients.

# Physical origin of $\hat{Z}_a(M_3)$

Definition of $\hat{Z}^{\mathfrak{g}}_a(M_3)$ q-series from 3d $\mathcal{N}$=2 SQFT $T^{\mathfrak{g}}$ with simply-laced gauge group $G$

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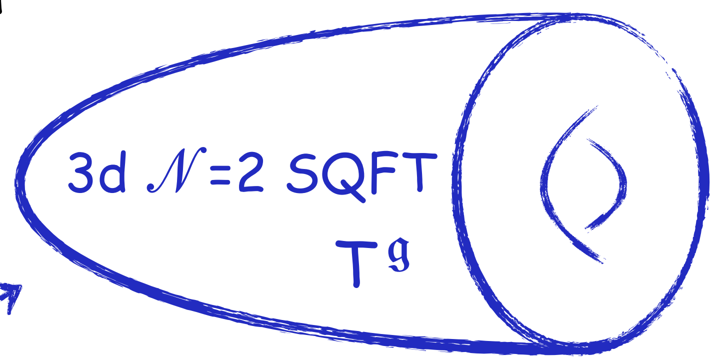


$D^2 \times_\tau S^1$

2d $\mathcal{N}$=(0,2) boundary SCFT $\mathscr{B}_a$

on $T^2 = \partial D^2 \times_\tau S^1$

$q = e^{2\pi i \tau}$ $\quad \tau \in \mathbb{H}$

$\hat{Z}^{\mathfrak{g}}_a(M_3)$ is the supersymmetric index of $T^{\mathfrak{g}}$ or "half-index" counting of BPS states

♣ when a Lagrangian description of $T^{\mathfrak{g}}$ is known, compute $\hat{Z}$ by localisation [Yoshida, Sugiyama '14] [Gukov, Putrov, Vafa '16]

$$\hat{Z}_a(q) = \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta^{(a)}_{2d}(x;q)$$

contains contributions from 3d bulk fields

$\theta$-function contains 2d boundary contribution

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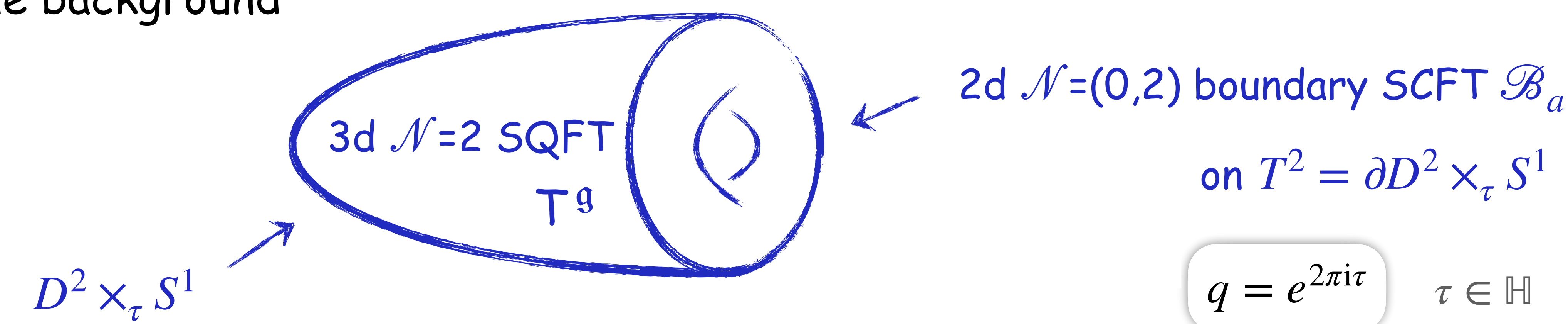


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$$\hat{Z}_a(q) = \int \frac{dx}{2\pi i x} F_{3d}(x)\, \Theta^{(a)}_{2d}(x;q)$$

$F_{3d}(x)$ trivial: $\hat{Z}_a(q)$ modular

$F_{3d}(x)$ non-trivial but small: $\hat{Z}_a(q)$ modularity distorted

$F_{3d}(x)$ non-trivial: $\hat{Z}_a(q)$ modularity compromised

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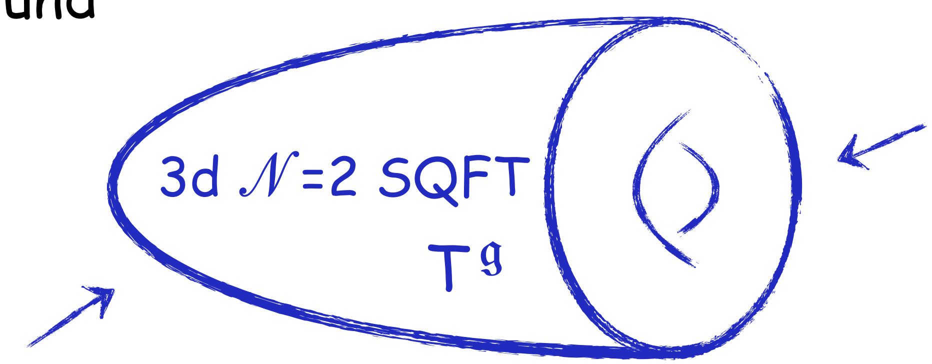
$D^2 \times_\tau S^1$




2d $\mathcal{N}$=(0,2) boundary SCFT $\mathscr{B}_a$

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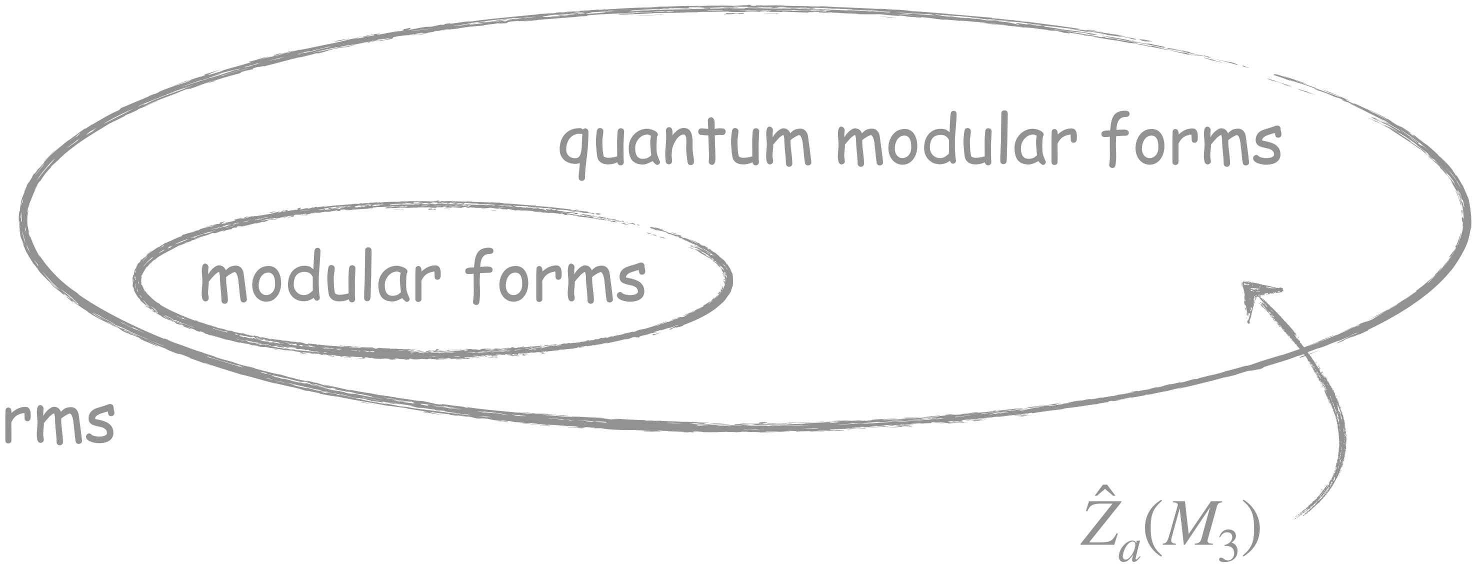
$\hat{Z}_a^{\mathfrak{g}}(M_3)$ is the supersymmetric index of $T^{\mathfrak{g}}$ or "half-index" counting of BPS states

is related to other supersymmetric quantities, for which it can be seen as a building block

Gluing two copies of $D^2 \times_\tau S^1$ into $S^2 \times_\tau S^1$ —> relates $\hat{Z}_a$ to the 3d $\mathcal{N}$=2 superconformal index

$$Z(S^2 \times_\tau S^1) = \sum_a |\mathscr{W}_a| \hat{Z}_a(M_3; q) \hat{Z}_a(M_3; q^{-1}) \quad \in \mathbb{Z}[[q]]$$ [Gukov, Pei, Putrov, Vafa '17]

# Context and motivation

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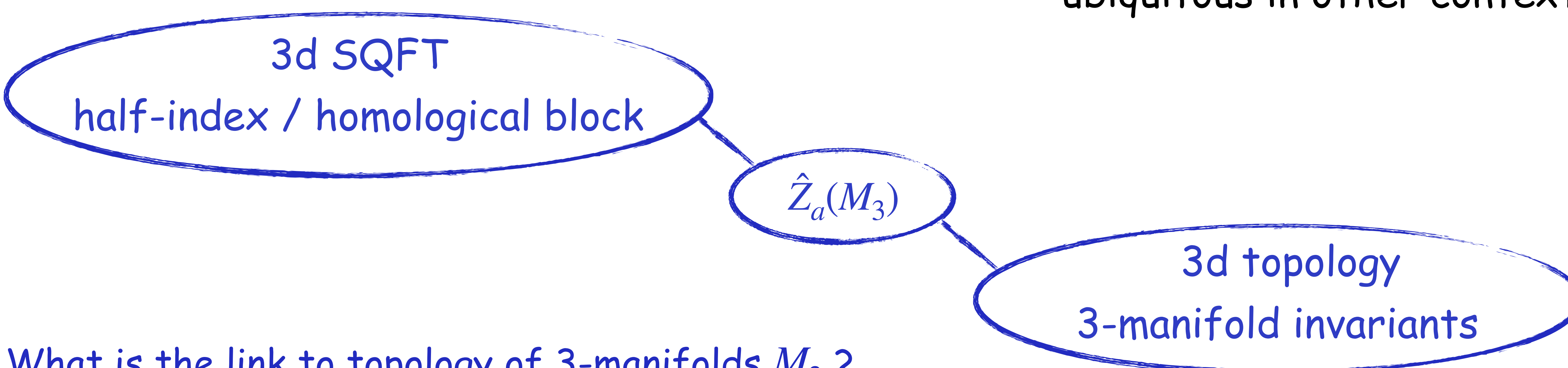



What is the link to topology of 3-manifolds $M_3$ ?

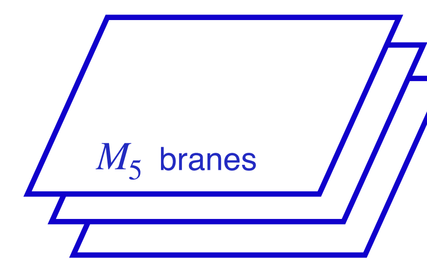
# Physical origin of $\hat{Z}_a(M_3)$ from M-theory

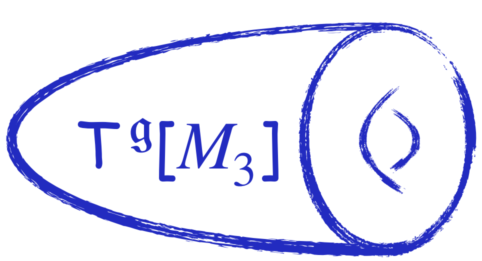
$\hat{Z}_a(M_3)$ as 3-manifold invariants … in the context of the 3d-3d correspondence

[Gukov, Putrov, Vafa '16], [Gukov, Pei, Putrov, Vafa '17]

- 3d SQFT $T^{\mathfrak{g}}[M_3]$ has an M-theory realisation by wrapping M5 branes on $M_3$

| | | | | | | |
|---|---|---|---|---|---|---|
| M-theory background | $TN$ | $\times$ | $S^1$ | $\times$ | $T^*M_3$ |
| | $\cup$ | | | | $\cup$ |
| N M5-branes on | $D^2$ | $\times$ | $S^1$ | $\times$ | $M_3$ |

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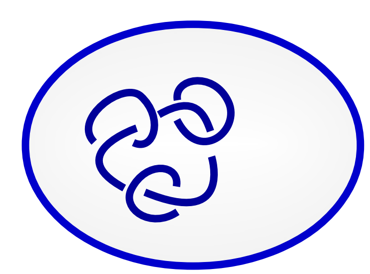

$M_3$

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6d $\mathcal{N}$=(2,0) $\mathfrak{g}$-SCFT on $D^2 \times_\tau S^1 \times M_3$ on the M5 branes

3d $\mathcal{N}$=2 gauge theory $T^{\mathfrak{g}}[M_3]$

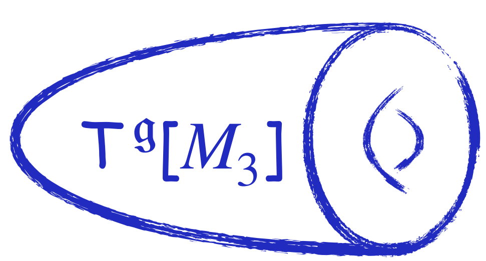
gauge group G, Lie algebra $\mathfrak{g}$,
choice of boundary condition

Topological quantum field theory

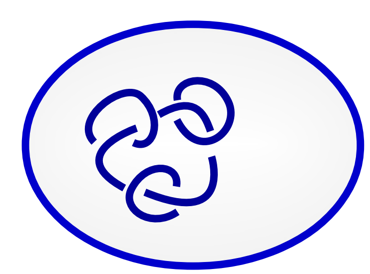
3d $G_{\mathbb{C}}$ Chern-Simons on $M_3$

abelian flat connections

index label $a$




2d $\mathcal{N}$=(0,2) boundary condition $\mathscr{B}_a$



$D^2 \times_\tau S^1$

$M_3$

# $\hat{Z}_a(M_3)$ and its relation to the WRT invariant of $M_3$

$\hat{Z}_a(M_3;q)$ as a convergent q-series with integer powers and integer coefficients in $|q|<1$

[Gukov, Pei, Putrov, Vafa '17]

↓ is related through a sum over "a", in the radial limit $|q| \to 1 \;\leftrightarrow\; \tau \to 1/k$ , to

The Witten-Reshetikhin-Turaev invariant $Z_{\text{CS}}(M_3)$ of $M_3$ [Witten'88; Reshetikhin, Turaev '90]

$$Z_{\text{CS}}(M_3;k) = \int_{\mathscr{A}} \mathscr{D}A e^{\frac{\mathrm{i}(k-h^\vee)}{4\pi}\int_{M_3}\text{Tr}(A\wedge dA + \frac{3}{2}A\wedge A\wedge A)}$$

3d Chern-Simons partition function

$k \in \mathbb{Z}$ shifted CS level

- A goal with defining the $\hat{Z}$-invariants was to make progress in the definition and categorification of topological 3-manifold invariants

- $$Z_{CS}(M_3;k) \sim \sum_{a,b\in\pi_0\mathscr{M}^{ab}_{flat}(M_3,G)} e^{2\pi i k\,\text{CS}(a)}\left[S_{ab}\hat{Z}_b(M_3;q)\right]_{\tau\to 1/k}$$

  (the bracketed term: $Z_a(e^{2\pi \mathrm{i}/k})$)

conjectured in [Gukov, Pei, Putrov, Vafa '17] , with proof in [Mori, Murakami '22] for examples

# $\hat{Z}_a(M_3)$ and its relation to the WRT invariant of $M_3$ *

$\hat{Z}_a(M_3;q)$ as a convergent q-series with integer powers and integer coefficients in $|q|<1$

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The Witten-Reshetikhin-Turaev invariant $Z_{\text{CS}}(M_3)$ of $M_3$ [Witten'88; Reshetikhin, Turaev '90]

## Modularity

♣ $\hat{Z}_a(M_3;q)$ from resurgence in 3d Chern Simons theory [Gukov, Marino, Putrov '16]

↪ is a Borel resummation of a perturbative series $\hat{Z}_a^{\text{pert}}\left(e^{2\pi i/k}\right)=\sum_{m\geq 1} N_m^b\left(2\pi i/k\right)^m \in \mathbb{Q}[[2\pi i/k]]$

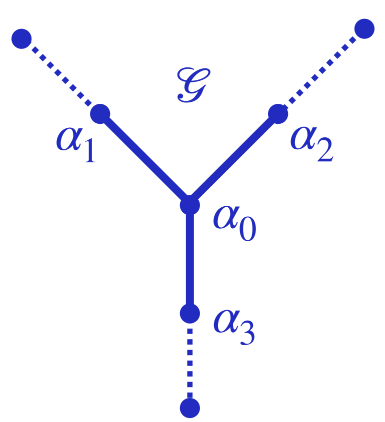
$$\hat{Z}_a\left(M_3;e^{2\pi i/k}\right)\sim S_{ab}\hat{Z}_b\left(M_3;e^{-2\pi i k}\right)+\text{perturbative series in } k^{-1}$$

(at rank-1, Seifert $M_3$)

# Topology of $M_3$ and mathematical definition of $\hat{Z}_a(M_3)$

Definition of the 3-manifold invariants $\hat{Z}_a(M_3)$ from the WRT inv. $Z_{\rm CS}(M_3;k)$

… when $M_3(\mathcal{G})$ is a plumbed 3-manifold, with plumbing graph $\mathcal{G}$ [Gukov, Pei, Putrov, Vafa '17]

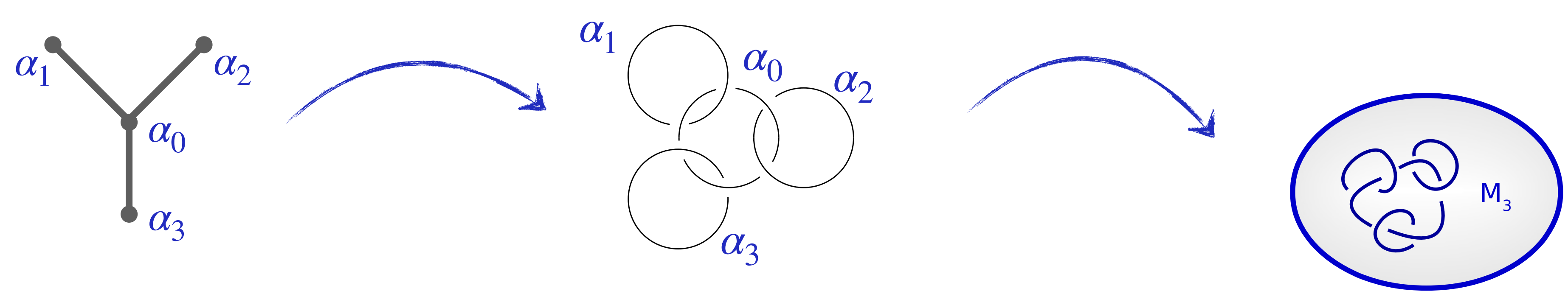



$\mathcal{G} := (V, E, \alpha)$ is a weighted graph $\alpha : V \to \mathbb{Z}$

… this data is encoded in the adjacency matrix $M$

$$M_{vv'} = \begin{cases} \alpha(v) & \text{if } v = v' \\ 1 & \text{if } (v,v') \in E \\ 0 & \text{otherwise} \end{cases}$$

$M_3(\mathcal{G})$ from Dehn surgery along the corresponding framed link




This class of manifolds includes the Seifert fibrations over $S^2$

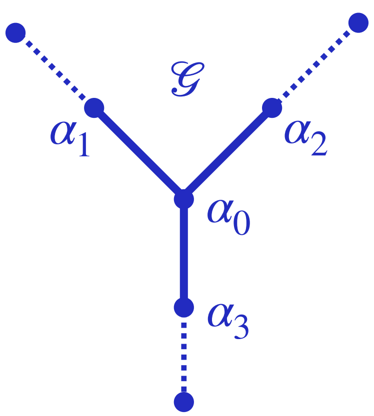
The definition of $\hat{Z}$ has been extended to cases where … $\mathcal{G}$ has loops [Chun, Gukov, Park, Sopenko '19]

… or $M_3$ is a knot complement from surgery along $K \subset S^3$ [Gukov, Manolescu '19]

# Examples

Definition of the 3-manifold invariants $\hat{Z}_a(M_3)$ from the WRT inv. $Z_{\mathrm{CS}}(M_3;k)$

… when $M_3(\mathscr{G})$ is a plumbed 3-manifold, with plumbing graph $\mathscr{G}$ [Gukov, Pei, Putrov, Vafa '17]




Seifert manifolds $X_{\mathscr{G}}\left(\alpha_0\,;\{q_i/p_i\}_{i=1}^{\text{n legs}}\right)$

$S^1$ fibered 2d orbifolds $\alpha_0 = e - \sum_{i=1}^{n} \frac{q_i}{p_i}$

Seifert invariants $(q_1,p_1),\ldots,(q_n,p_n)$ and orbifold Euler number $e$

$$\frac{q_i}{p_i} = -\cfrac{1}{\alpha_1^{(i)} - \cfrac{1}{\alpha_2^{(i)} - \cfrac{1}{\alpha_3^{(i)} - \cdots}}}$$

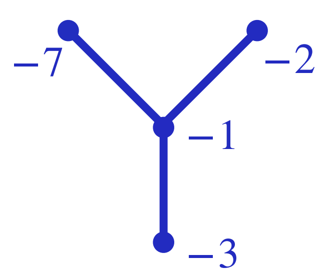
Brieskorn spheres $\Sigma(p_1,p_2,p_3)$ $p_i \in \mathbb{Z}$ coprime

adjacency matrix $|det(M)| = 1$

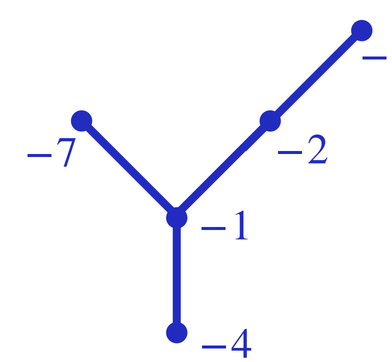
$$M_3(\mathscr{G}) = \Sigma(p_1,p_2,p_3) = \{(x,y,z) \in \mathbb{C}^3 \,|\, x^{p_1} + y^{p_2} + z^{p_3} = 0\} \cap S^5$$

$$b + \frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3} = -\frac{1}{p_1p_2p_3}$$

$$\Sigma(2,3,7) = M\left(-1;\frac{1}{2},\frac{1}{3},\frac{1}{7}\right)$$




$$\Sigma(4,5,7) = M\left(-1;\frac{1}{4},\frac{3}{5},\frac{1}{7}\right)$$



# Examples

Definition of the 3-manifold invariants $\hat{Z}_a(M_3)$ from the WRT inv. $Z_{\rm CS}(M_3;k)$

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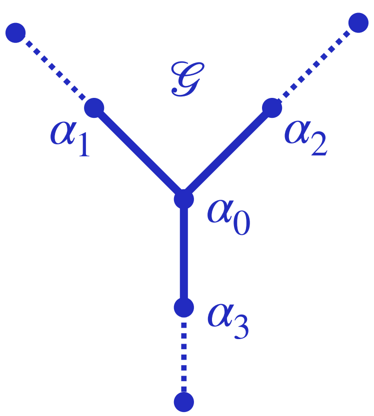


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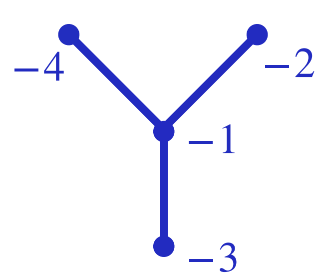
$$\frac{q_i}{p_i} = -\cfrac{1}{\alpha_1^{(i)} - \cfrac{1}{\alpha_2^{(i)} - \cfrac{1}{\alpha_3^{(i)} - \cdots}}}$$

More generally ( the adjacency matrix has $|det(M)| > 1$)

$$M\left(-1;\frac{1}{3},\frac{1}{2},\frac{1}{4}\right)$$

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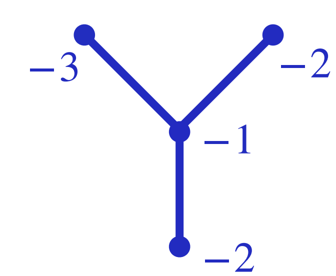
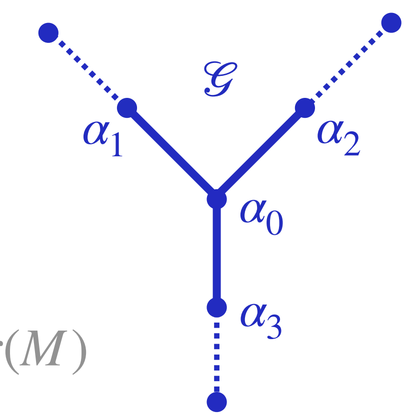


# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

$\mathfrak{g} = A_1$

Definition of the 3-manifold invariants $\hat{Z}^{\mathfrak{g}}_a(M_3)$ [Gukov, Pei, Putrov, Vafa '17]




$$\hat{Z}_a(M_3;q) = q^{\Delta_a} \oint \prod_{v\in V} \frac{dz_v}{2\pi \mathrm{i} z_v} \left(z_v - z_v^{-1}\right)^{2-\deg(v)} \underbrace{\sum_{\ell \in 2M\mathbb{Z}^{|V|}\pm a} q^{-\frac{\ell^T M^{-1} \ell}{4}} \mathbf{z}^{\ell}}_{\Theta^M_a(q;\mathbf{z})}$$

$\Delta_a \in \mathbb{Q}$

$a \in \mathrm{Coker}(M)$

- this form is reminiscent of the localisation result $\hat{Z}_a = \int \frac{dx}{2\pi \mathrm{i} x} F_{3d}(x)\, \Theta^{(a)}_{2d}(x)$
- the contour integral picks the $[z^0]$ term

$\hat{Z}_a(M_3;q)$ is well defined in this way as a convergent q-series only if $M_3(\mathscr{G})$ is weakly negative

the sum is over a positive definite lattice and $\Theta^M_a(q)$ converges for $|q|<1$

$M_3(\mathscr{G})$ weakly negative if $M^{-1}$ negative definite when restricted to subspace of high-valency vertices

... for 3-star graphs, this means $(M^{-1})_{00} < 0$

# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

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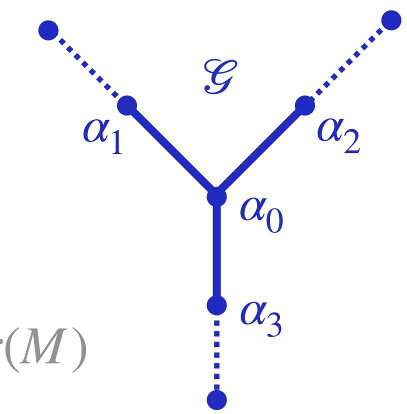
Definition of the 3-manifold invariants $\hat{Z}_a^{\mathfrak{g}}(M_3)$ [Gukov, Pei, Putrov, Vafa '17]

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$\Delta_a \in \mathbb{Q}$

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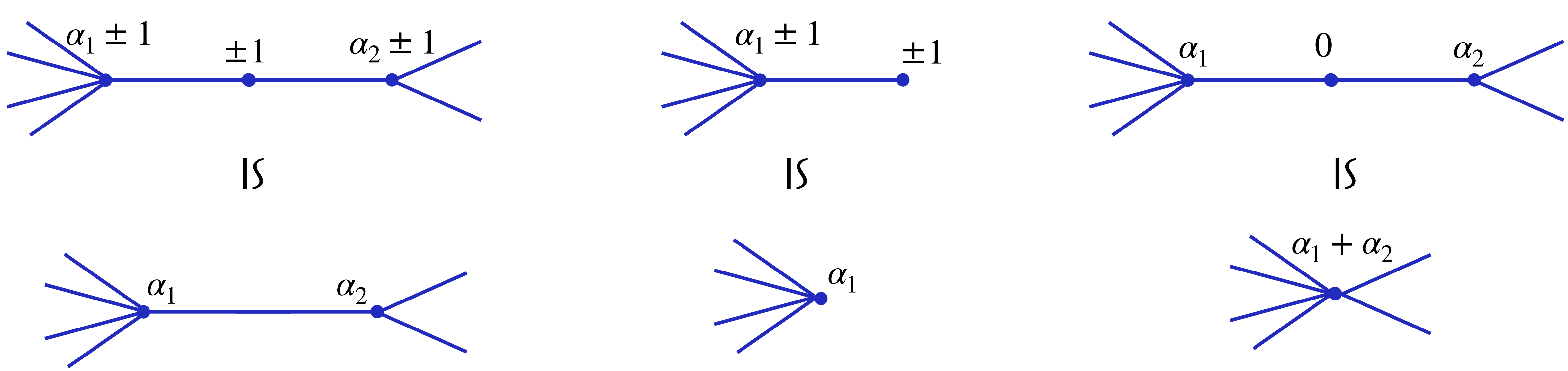
$\hat{Z}_a(M_3)$ is a topological invariant, unchanged by 3d Kirby moves on $\mathscr{G}$ [Gukov, Manolescu '19] [Neumann '81]



# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

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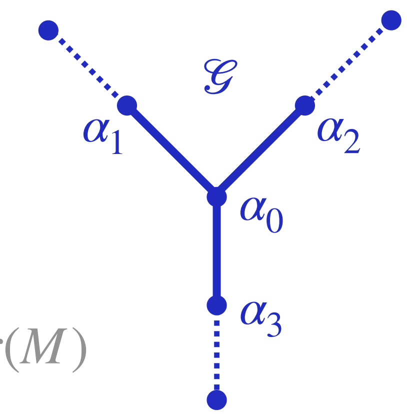


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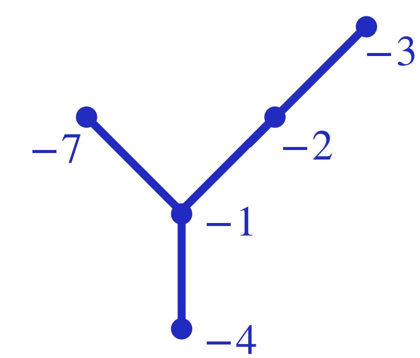


## Example: Brieskorn sphere $\Sigma(4,5,7)$ [Cheng, Chun, Ferrari, Gukov, Harrison '18]

$$\hat{Z}_0(\Sigma(4,5,7);q) = -\,q^{\Delta}(\widetilde{\theta}^1_{140,57} - \widetilde{\theta}^1_{140,97} - \widetilde{\theta}^1_{140,113} - \widetilde{\theta}^1_{140,127})$$

$$\widetilde{\theta}^1_{p,r}(\tau) = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} \mathrm{sgn}(k)\, q^{\frac{k^2}{4p}}$$

The $\hat{Z}$ invariant is ~ linear combination of false $\theta$-functions, so an example of QMF

# Hidden SL(2,Z) structure

$\hat{Z}_a(M_3)$ for Seifert manifolds with 3-star plumbing graphs $\mathscr{G}$ have the structure of a Weil orbit

$\theta_m = \left(\theta_{m,r}\right)_{r \bmod 2m}$, as a column vector, spans a 2m-dimensional representation $\Theta_m$ of $\widetilde{SL_2(\mathbb{Z})}$

$$\theta_{m,r}(\tau, z) = \sum_{l = r \bmod 2m} q^{l^2/4m} e^{2\pi i z l}$$

$\Theta_m$ is reducible for all m>1

Obtain sub-representations by considering eigenspaces of $a \in O_m$ orthogonal group

$O_m = \{a \in \mathbb{Z}/2m \,|\, a^2 = 1 \bmod 4m\}$ with action $\theta_{m,r} \xrightarrow{a} \theta_{m,ar}$ commuting with that of $\widetilde{SL_2(\mathbb{Z})}$

Use the isomorphism $Ex_m \simeq O_m$ to label such sub-representations

$\mathrm{Ex}_m = \{n \,|\, m, (n, m/n) = 1\}$ the group of exact divisors of $m$, has group action $n \star n' = nn'/(n, n')^2$

In particular for $K \subset \mathrm{Ex}_m$ with the property $\mathrm{Ex}_m = K \cup (m \star K)$ and $m \notin K$

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Weil representations $\Theta^{m+K}$ are irreducible sub-representations of $\Theta_m$ defined as the simultaneous eigenspaces of $a(n)$ for all $n \in K$ and which have eigenvalue 1.

A specific basis of $\Theta^{m+K}$ is given by $\{\theta_r^{m+K}\}$ for some set of indices $r$.

# Hidden SL(2,Z) structure

$\hat{Z}_a(M_3)$ for Seifert manifolds with 3-star plumbing graphs $\mathscr{G}$ have the structure of a Weil orbit

[Cheng, Chun, Ferrari, Gukov, Harrison '18]

Brieskorn sphere $\Sigma(p_1, p_2, p_3)$

$$\hat{Z}_0(\Sigma(p_1,p_2,p_3);q) \sim \widetilde{\theta}^1_{p,r_1} - \widetilde{\theta}^1_{p,r_2} - \widetilde{\theta}^1_{p,r_3} - \widetilde{\theta}^1_{p,r_4} := \widetilde{\theta}^{1,p+K}_{r_1}$$

$$p = p_1p_2p_3\,, \qquad K = \{1,\, p_1p_2,\, p_2p_3,\, p_1p_3\}$$

$$r_1 = p - p_1p_2 - p_1p_3 - p_2p_3$$
$$r_2 = p + p_1p_2 - p_1p_3 - p_2p_3$$
$$r_3 = p - p_1p_2 + p_1p_3 - p_2p_3$$
$$r_4 = p - p_1p_2 - p_1p_3 + p_2p_3$$

Weil representations $\Theta^{m+K}$ are irreducible sub-representations of $\Theta_m$ defined as the simultaneous eigenspaces of $a(n)$ for all $n \in K$ and which have eigenvalue 1.

A specific basis of $\Theta^{m+K}$ is given by $\{\theta^{m+K}_r\}$ for some set of indices $r$.

# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

$\mathfrak{g} = A_{N-1}$

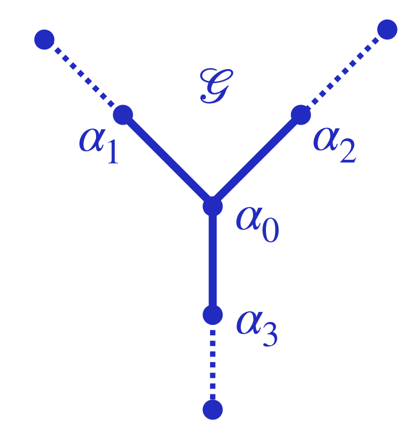
Definition of the 3-manifold invariants $\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3)$ [Park '19], [Cheng, Chun, Fegin, Ferrari, Gukov, Harrison, Passaro '22]

$$\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3;q) \sim \oint d\underline{\vec{\xi}} \prod_{v\in V} \left(\Delta(\vec{\xi}_v)^{2-\deg v}\right) \sum_{w\in W} \sum_{\underline{\vec{\ell}}\in\Gamma_{M,G}+w(\underline{\vec{a}})} q^{-\frac{1}{2}||\underline{\vec{\ell}}||^2} \left(\prod_{v'\in V} e^{\langle \vec{\ell}_{v'}, \vec{\xi}_{v'}\rangle}\right)$$

$$\oint d\underline{\vec{\xi}} = \mathrm{p.v.}\oint \prod_{v\in V} \prod_{i=1}^{\mathrm{rank}\,G} \frac{dz_{v,i}}{2\pi i z_{v,i}}$$

by extending $\mathfrak{g} = A_1$ the definition

$$\hat{Z}_a(M_3;q) \sim \oint \prod_{v\in V} \frac{dz_v}{2\pi i z_v} \left(z_v - z_v^{-1}\right)^{2-\deg(v)} \sum_{\ell\in 2M\mathbb{Z}^{|V|}\pm a} q^{-\frac{\ell^T M^{-1}\ell}{4}} \mathbf{z}^{\ell}$$



# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

$\mathfrak{g} = A_2$

Definition of the 3-manifold invariants $\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3)$

$$\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3;q) \sim \oint d\underline{\vec{\xi}} \prod_{v\in V} \left(\Delta(\vec{\xi}_v)^{2-\deg v}\right) \sum_{w\in W} \sum_{\underline{\vec{\ell}}\in\Gamma_{M,G}+w(\underline{\vec{a}})} q^{-\frac{1}{2}||\underline{\vec{\ell}}||^2} \left(\prod_{v'\in V} e^{\langle \vec{\ell}_{v'}, \vec{\xi}_{v'}\rangle}\right)$$

Rank 2 $\hat{Z}$-invariant

$$\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3;q) \sim \sum_{\mathrm{s}\in\mathcal{T}} (-1)^{l_{\mathrm{s}}} F_\varrho(q)$$

$\varrho = \{\vec{s}, \vec{\kappa}, m, D\}$ $\quad m, D \in \mathbb{Z}_+$

$\vec{s}, \vec{\kappa}$ root vectors

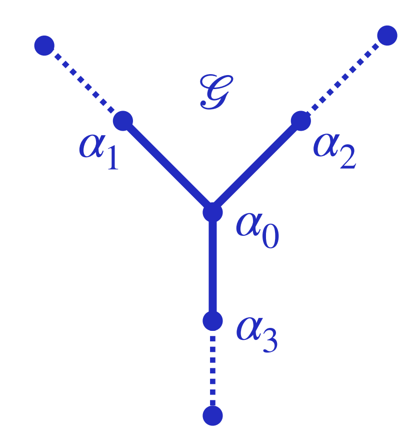
Generalised $A_2$ false $\theta$-function

$$F_\varrho(q) = \sum_{w\in W} \sum_{\substack{\vec{n}\in w^{-1}(\vec{\kappa}+\vec{\rho})+D\Lambda \\ \vec{n}\in\Lambda}} (-1)^{l(w)} \min(n_1,n_2) q^{\frac{1}{2Dm}|-\vec{s}+mw(\vec{n})|^2}$$

$\vec{\rho}$ Weyl vector

$W = A_2$ Weyl group $\quad \Lambda = A_2$ root lattice

[Cheng, Chun, Fegin, Ferrari, Gukov, Harrison, Passaro '22]



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Rank 2 $\hat{Z}$-invariant

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$\vec{s}, \vec{\kappa}$ root vectors

Generalised $A_2$ false $\theta$-function

$$F_\varrho(q) = F_0^{(\varrho)}(Dm\tau) + DF_1^{(\varrho)}(Dm\tau)$$

$$F_1^{(\varrho)}(\tau) = \sum_{w\in W_+} \sum_{i\in\{1,2\}} F_{1,\alpha_w^{(i)}}(\tau)$$

$$F_0^{(\varrho)}(\tau) = \frac{1}{m} \sum_{w\in W_+} \sum_{i\in\{1,2\}} w(\vec{s})|_i F_{0,\alpha_w^{(i)}}(\tau)$$

[Cheng, Coman, Passaro, Sgroi, *to appear*]

$$F_{0,\alpha}(\tau) = \left(\sum_{\mathbf{n}\in\alpha+\mathbb{N}_0^2} + \sum_{\mathbf{n}\in\mathbf{1}-\alpha+\mathbb{N}_0^2}\right) q^{Q(\mathbf{n})}, \quad F_{1,\alpha} = \left(\sum_{\mathbf{n}\in\alpha+\mathbb{N}_0^2} - \sum_{\mathbf{n}\in\mathbf{1}-\alpha-\mathbb{N}_0^2}\right) n_2\, q^{Q(\mathbf{n})}$$

$$Q(n_1,n_2) = (3n_1^2 + 3n_1n_2 + n_2^2)$$

cf. [Bringmann, Kaszian, Milas '17]

# Mathematical definition of $\hat{Z}_a(M_3)$ from the topology of $M_3$

$\mathfrak{g} = A_2$

Definition of the 3-manifold invariants $\hat{Z}^{\mathfrak{g}}_{\underline{\vec{a}}}(M_3)$

Rank 2 $\hat{Z}$-invariant

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$\varrho = \{\vec{s}, \vec{\kappa}, m, D\}$ $\quad m, D \in \mathbb{Z}_+$

$\vec{s}, \vec{\kappa}$ root vectors

Generalised $A_2$ false $\theta$-function

$$F_{\varrho}(q) = F_0^{(\varrho)}(Dm\tau) + DF_1^{(\varrho)}(Dm\tau)$$

$$F_1^{(\varrho)}(\tau) = \sum_{w \in W_+} \sum_{i \in \{1,2\}} F_{1,\boldsymbol{\alpha}_w^{(i)}}(\tau)$$

$$F_0^{(\varrho)}(\tau) = \frac{1}{m} \sum_{w \in W_+} \sum_{i \in \{1,2\}} w(\vec{s})|_i \, F_{0,\boldsymbol{\alpha}_w^{(i)}}(\tau)$$

[Cheng, Coman, Passaro, Sgroi, *to appear*]

<u>Lemma</u> Let $\boldsymbol{\beta} = \boldsymbol{\alpha} + (\delta\alpha_1, \delta\alpha_2)$ for $\delta\alpha_1, \delta\alpha_2 \in \mathbb{Z}$ and consider $F_{\varepsilon,\boldsymbol{\alpha}}(\tau)$ for $\varepsilon = 0,1$. Then

$$F_{\varepsilon,\boldsymbol{\beta}}(\tau) - F_{\varepsilon,\boldsymbol{\alpha}}(\tau)$$

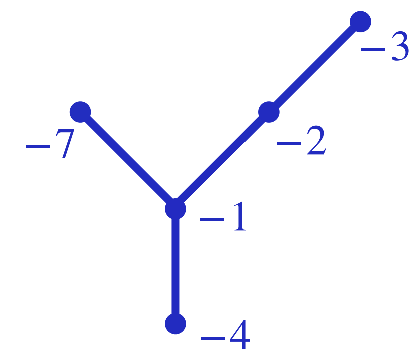
is a linear combination of the Eichler integrals of weight 1/2 & 3/2 $\theta$-functions.

So $\alpha_1, \alpha_2$ can be brought into the range [0,1]

# Examples

$\mathfrak{g} = A_2$

Brieskorn sphere $\Sigma(4,5,7) = M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$




$$\hat{Z}_0^{A_2}(\Sigma(4,5,7);q) \sim -6q^{24} + 12q^{35} + 12q^{41} + 12q^{47} - 12q^{48} + \mathcal{O}\left(q^{50}\right) = \sum_{i=0}^{1} F_i^{1D} + F_i^{2D}$$

$F_i^{1D} \sim$ false $\theta$-function $\quad$ $F_i^{2D} \sim$ generalised $A_2$ false $\theta$-function

parameters $\alpha_1, \alpha_2 \in [0,1]$

$$\tilde{F} = -\frac{9}{14}q^2 - \frac{18}{35}q^3 - \frac{33}{35}q^5 - \frac{81}{70}q^6 - \frac{57}{35}q^8 - \frac{39}{35}q^{11} - \frac{81}{35}q^{12} - \frac{261}{70}q^{14} - \frac{3}{35}q^{17} + \frac{123}{35}q^{20} + \frac{69}{35}q^{23}$$

$$F_0^{1D} \sim \tilde{F} + \frac{99}{35}q^{24} + \frac{141}{35}q^{26} - \frac{18}{7}q^{35} - \frac{39}{35}q^{38} - \frac{81}{35}q^{39} - \frac{51}{7}q^{41} - \frac{9}{2}q^{47} + \frac{36}{35}q^{48} + \mathcal{O}\left(q^{50}\right)$$

$$F_0^{2D} \sim -\tilde{F} - \frac{447}{70}q^{24} - \frac{141}{35}q^{26} + \frac{309}{35}q^{35} + \frac{39}{35}q^{38} + \frac{81}{35}q^{39} + \frac{66}{5}q^{41} + \frac{354}{35}q^{47} - \frac{213}{35}q^{48} + \mathcal{O}\left(q^{50}\right)$$

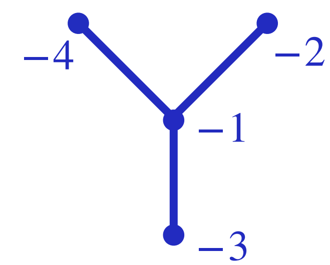
$$F_1^{1D} \sim -\tilde{F} - \frac{99}{35}q^{24} - \frac{141}{35}q^{26} + \frac{18}{7}q^{35} + \frac{39}{35}q^{38} + \frac{81}{35}q^{39} + \frac{51}{7}q^{41} + \frac{9}{2}q^{47} - \frac{36}{35}q^{48} + \mathcal{O}\left(q^{50}\right)$$

$$F_1^{2D} \sim \tilde{F} + \frac{27}{70}q^{24} + \frac{141}{35}q^{26} + \frac{111}{35}q^{35} - \frac{39}{35}q^{38} - \frac{81}{35}q^{39} - \frac{6}{5}q^{41} + \frac{66}{35}q^{47} - \frac{207}{35}q^{48} + \mathcal{O}\left(q^{50}\right)$$

# Examples

$\mathfrak{g} = A_2$

Seifert (more general) $M\left(-1; \frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$




$$\hat{Z}_0^{A_2}(M\left(-1; \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{4}\right); q) \sim 6q - 12q^2 - 6q^3 + 12q^4 - 30q^5 + 12q^6 + 24q^7 + \mathcal{O}\left(q^9\right) = \sum_{i=0}^{1} F_i^{1D} + F_i^{2D}$$

$F_i^{1D} \sim$ false $\theta$-function $\quad F_i^{2D} \sim$ generalised $A_2$ false $\theta$-function

parameters $\alpha_1, \alpha_2 \in [0,1]$

$$F_0^{1D} \sim \frac{1}{2}q - 35q^2 + \frac{5}{2}q^3 - 5q^4 + \frac{31}{2}q^5 - 7q^6 + 24q^7 + \mathcal{O}\left(q^9\right)$$

$$F_0^{2D} \sim 6q + 24q^2 - 6q^3 + 12q^4 - 30q^5 + 12q^6 - 12q^7 + \mathcal{O}\left(q^9\right)$$

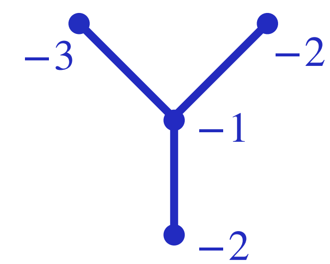
$$F_1^{1D} \sim -\frac{1}{2}q - q^2 - \frac{5}{2}q^3 + 5q^4 - \frac{31}{2}q^5 + 7q^6 + 12q^7 + \mathcal{O}\left(q^9\right)$$

$$F_1^{2D} \sim \mathcal{O}\left(q^9\right)$$

# Examples

$\mathfrak{g} = A_2$

Seifert (more general) $M\left(-1;\frac{1}{2},\frac{1}{2},\frac{1}{3}\right)$




$$\hat{Z}_0^{A_2}(M\left(-1;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{3}\right);q) \sim 6q - 12q^2 - 6q^3 + 12q^4 - 30q^5 + 12q^6 + 24q^7 + \mathcal{O}\left(q^9\right) = \sum_{i=0}^{1} F_i^{1D} + \cancel{F_i^{2D}}$$

$F_i^{1D} \sim$ false $\theta$-function $\quad F_i^{2D} \sim$ generalised $A_2$ false $\theta$-function

parameters $\alpha_1, \alpha_2 \in [0,1]$

$$F_0^{1D} \sim 4q^{-1} + 18q - 18q^3 - 18q^5 + 36q^9 + \mathcal{O}\left(q^{10}\right)$$

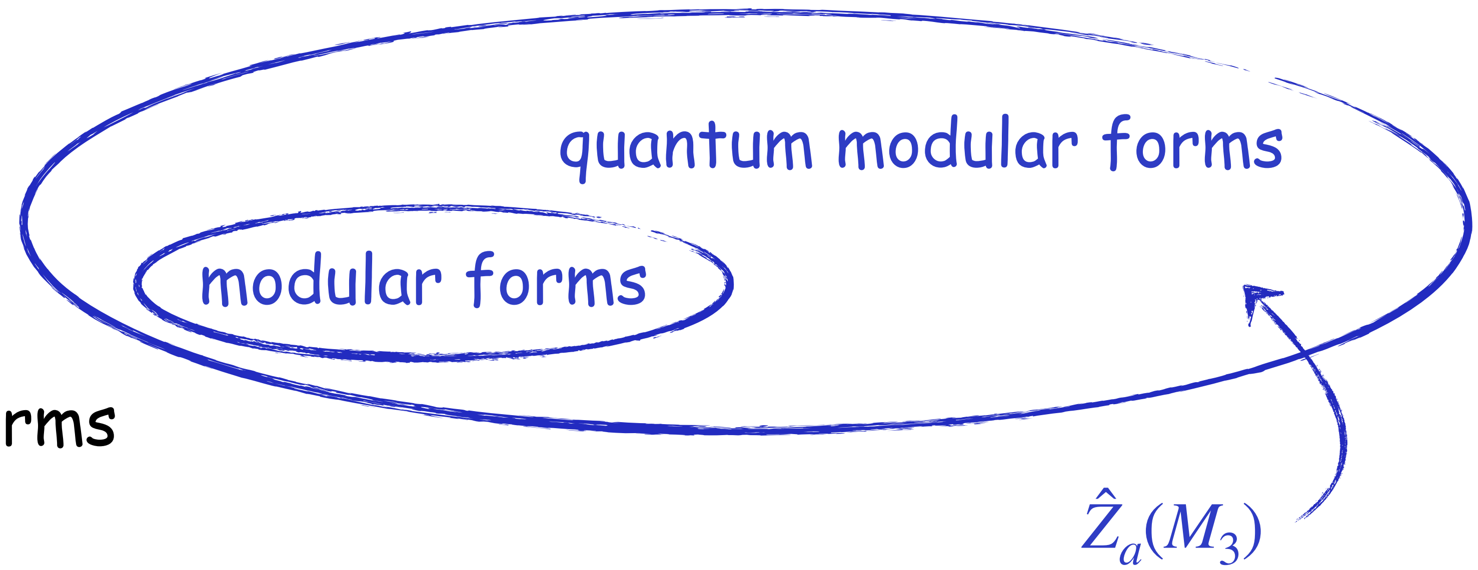
$$F_0^{2D} \sim \mathcal{O}\left(q^{10}\right)$$

$$F_1^{1D} \sim -2q^{-1} + 9q - 9q^3 + 9q^5 + \mathcal{O}\left(q^{10}\right)$$

$$F_1^{2D} \sim \mathcal{O}\left(q^{10}\right)$$

The $\hat{Z}$-invariant is here proportional to a linear combination of 1D false $\theta$-functions

# Have seen…*




Modular forms & quantum modular forms

- Mathematical definition of $\hat{Z}^{\mathfrak{g}}_a(M_3)$ from the plumbing data of $M_3$

  $\hat{Z}^{A_1}_a(M_3)$ are proportional to linear combinations of false $\theta$-functions, so ~ QMF

  Brieskorn spheres $\quad q^{-\Delta}\hat{Z}_0(\Sigma(p_1,p_2,p_3);q) = \widetilde{\theta}^1_{p,r_1} - \widetilde{\theta}^1_{p,r_2} - \widetilde{\theta}^1_{p,r_3} + \widetilde{\theta}^1_{p,r_4} := \widetilde{\theta}^{1,p+K}_{r_1}$

  [Cheng, Chun, Ferrari, Gukov, Harrison '18]

- What is there to gain from knowing this?

  Quantum modularity of $\hat{Z}_a(M_3)$ provides various insights

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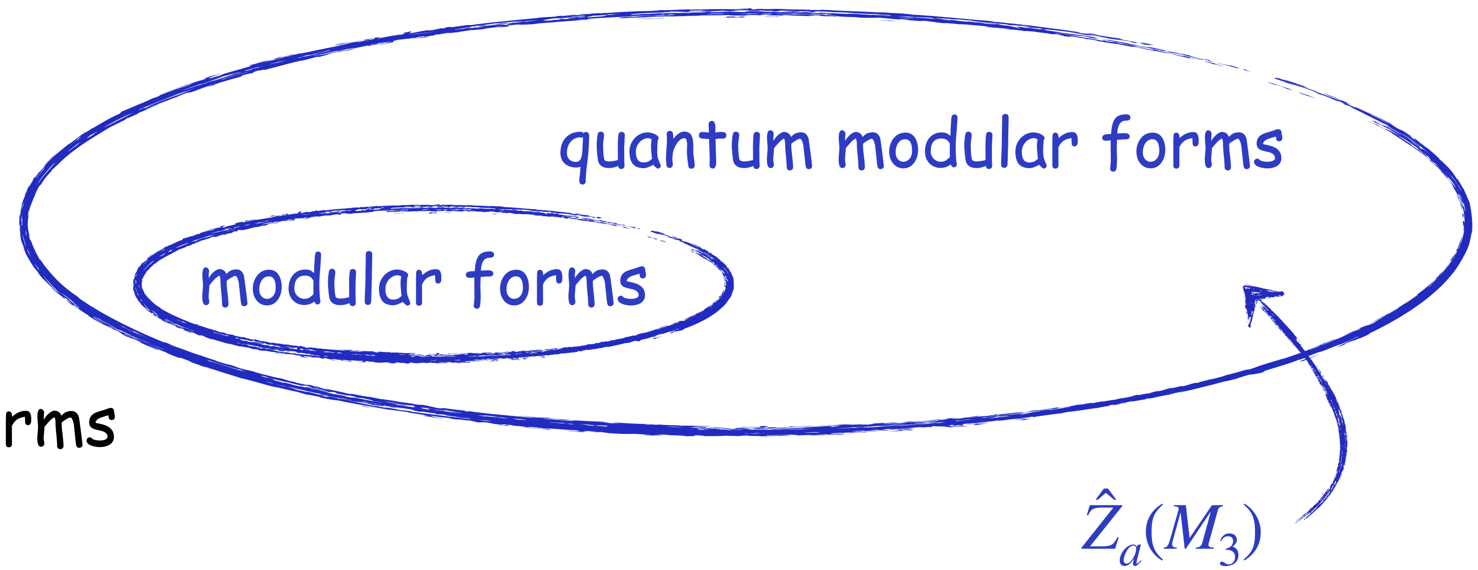


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  [Cheng, Chun, Ferrari, Gukov, Harrison '18]

- What is there to gain from knowing this?

  Quantum modularity of $\hat{Z}_a(M_3)$ provides various insights

- $\hat{Z}_a$ invariants have been calculated for $\tau \in \mathbb{H}$, but what happens for $\tau \in \mathbb{H}_-$ ?

$$Z(S^2 \times_\tau S^1) = \sum_a |\mathscr{W}_a| \hat{Z}_a(M_3;q)\, \hat{Z}_a(M_3;q^{-1})$$

- QMFs have appeared in other contexts in physics

# $\hat{Z}_a(M_3)$ when $M_3$ not weakly negative*

The 3-manifold invariants $\hat{Z}_a(M_3; q)$ were defined for $M_3$ weakly negative, but $\hat{Z}_a(M_3; q^{-1})$ ?

[Gukov, Pei, Putrov, Vafa '17]

From $Z_{\rm CS}(M_3; k) \sim \sum_{a,b} e^{2\pi i k\, {\rm lk}(a,a)} \left[ S_{ab} \hat{Z}_b(M_3; \tau) \right]_{\tau \to 1/k}$ & $Z_{\rm CS}(-M_3; k) = Z_{\rm CS}(M_3; -k)$ $\quad k \to \infty$

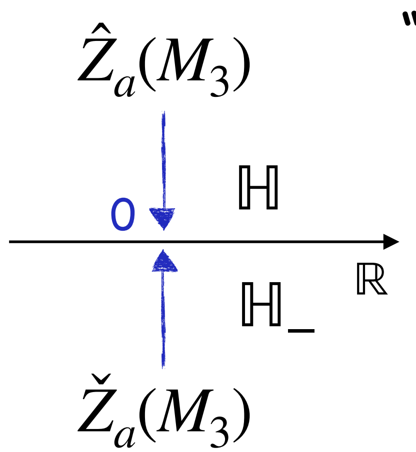
expect $\hat{Z}_a(-M_3; q) = \hat{Z}_a(M_3; q^{-1})$ ♣

but this means defining $\hat{Z}_a(M_3)$ to be a convergent q-series both for $|q| \lessgtr 1$

! But $\hat{Z}_a(-M_3; q)$ does not converge for $|q| > 1$ as defined from $-M_3$ plumbing data

♣ Need asymptotic agreement in radial lim, so use quantum modularity to find companion $\check{Z}_a(M_3)$

[Cheng, Chun, Ferrari, Gukov, Harrison '18], [Cheng, Ferrari, Sgroi '19] rank 1




"companions" in the sense

$$F\left(\frac{h}{k} + \frac{it}{2\pi}\right) \sim \sum_{m \geq 0} a_{h,k}(m)\, t^m \quad \underset{t \to 0^+}{\longleftrightarrow} \quad E\left(\frac{h}{k} + \frac{it}{2\pi}\right) \sim \sum_{m \geq 0} a_{-h,k}(m)\, (-t)^m$$

coprime $h, k \in \mathbb{Q}$

# Companion $\check{Z}_a(M_3)$ at higher rank

✔ $\hat{Z}_a^{A_2}(M_3; q)$ are ~ linear combinations of generalised $A_2$ false $\theta$-functions "2d sums + 1d sums"

✔ setting $\tau = \frac{h}{k} + \frac{it}{2\pi}$ for coprime $h, k \in \mathbb{Q}$, taking the limit $t \to 0^+$ and sending $h \to -h$

companions of $F_i^{(Q)}(q)$ are ~ $\sum$ iterated Eichler integrals $\theta^{\ell}_{p,r} = \sum_{\substack{k \in \mathbb{Z} \\ k = r \bmod 2p}} k^{\ell} q^{\frac{k^2}{2p}}$

depth-2 QMF $I_{f_1,f_2}(\tau,\bar{\tau}) := \int_{-\bar{\tau}}^{i\infty} dz_1 \int_{z_1}^{i\infty} dz_2 \, \frac{f(z_1)f(z_2)}{(-i(z_1+\tau))^{2-w_1}(-i(z_2+\tau))^{2-w_2}}$ ← $\prod \theta^1_{p,r}\theta^1_{3p,r'}$ and $\prod \theta^1_{p,r}\theta^0_{3p,r'}$

[Cheng, Coman, Passaro, Sgroi, *to appear*]
cf. expectation from earlier results
[Bringmann, Kaszian, Milas '17] + …

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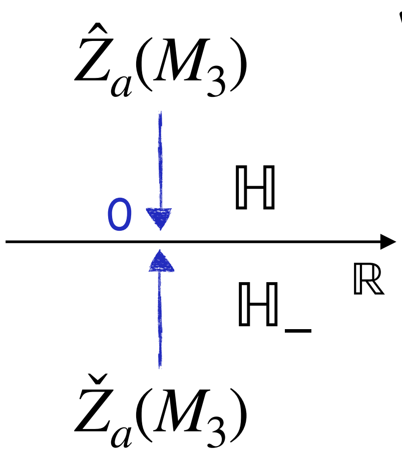


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$F_0^{(Q)}(q)$ : $E_0^{(Q)}(\tau) \sim \int_{-\bar{\tau}}^{i\infty}\int_{z_1}^{i\infty} \sum_{\substack{w\in W^+ \\ \delta\in\{0,1\}}} \frac{\frac{\Delta w(\vec{s})}{m}\theta^1_{mD,mD\delta+\frac{\Delta w(\vec{\sigma})}{3}}\left(\frac{3z_2}{mD}\right)\theta^1_{mD,mD\delta-w(\vec{\sigma})_{12}}\left(\frac{z_1}{mD}\right)}{\sqrt{-i(z_1+\tau)}\sqrt{-i(z_2+\tau)}} dz_2 dz_1$

$\xrightarrow{\tau \to -\tau}$

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# Companion $\check{Z}_a(M_3)$ at higher rank

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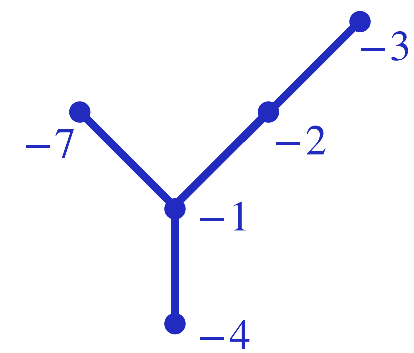
… for $M_3$ Brieskorn spheres

# Examples

$\mathfrak{g} = A_2$

Brieskorn sphere $\Sigma(4,5,7) = M\left(-1; \frac{1}{4}, \frac{3}{5}, \frac{1}{7}\right)$




$K = \{1,\ 20,\ 28,\ 35\}$

$\{1,\ p_1p_2,\ p_1p_3,\ p_2p_3\}$

Companion $\check{Z}_a^{A_2}(\Sigma(4,5,7); q) \ni \sum_{\hat{w}\in W^{\otimes 3}} (-1)^{\ell(\hat{w})} E_0^{(\varrho)}(\tau) = -\frac{\sqrt{3}}{4\,(140)^2} \int_{-\bar{\tau}}^{i\infty} \int_{z_1}^{i\infty} \frac{\mathbb{I}_0^{(\varrho)}}{\sqrt{-i(z_1+\tau)}\sqrt{-i(z_2+\tau)}} dz_1 dz_2$

$$\mathbb{I}_0^{(\varrho)} = 6\left[27\,\theta^1_{140,113}\left(\frac{3z_2}{140}\right) + 13\,\theta^1_{140,127}\left(\frac{3z_2}{140}\right) + 83\,\theta^1_{140,57}\left(\frac{3z_2}{140}\right) + 43\,\theta^1_{140,97}\left(\frac{3z_2}{140}\right)\right]\theta^{1,140+K}_{57}\left(\frac{z_1}{140}\right)$$

$$+48\left[\theta^1_{140,132}\left(\frac{3z_2}{140}\right) + 6\,\theta^1_{140,48}\left(\frac{3z_2}{140}\right) + \theta^1_{140,8}\left(\frac{3z_2}{140}\right) + 3\,\theta^1_{140,92}\left(\frac{3z_2}{140}\right)\right]\theta^{1,140+K}_{118}\left(\frac{z_1}{140}\right)$$

$$-6\left[13\,\theta^1_{140,13}\left(\frac{3z_2}{140}\right) + 27\,\theta^1_{140,27}\left(\frac{3z_2}{140}\right) + 43\,\theta^1_{140,43}\left(\frac{3z_2}{140}\right) + 83\,\theta^1_{140,83}\left(\frac{3z_2}{140}\right)\right]\theta^{1,140+K}_{83}\left(\frac{z_1}{140}\right) + \ldots$$

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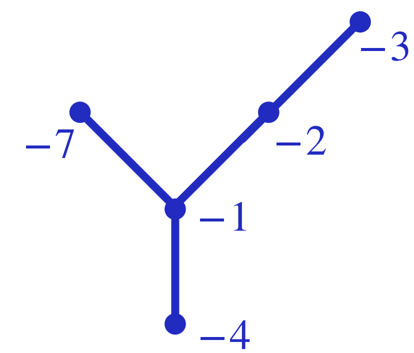


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cf. $\hat{Z}_a^{A_1}(\Sigma(4,5,7); q) \sim \widetilde{\theta}^1_{140,57} - \widetilde{\theta}^1_{140,97} - \widetilde{\theta}^1_{140,113} - \widetilde{\theta}^1_{140,127} = \widetilde{\theta}^{1,140+K}_{57}$

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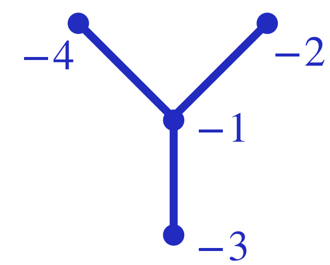


$K = \{1, 9\}$

Companion $\check{Z}_a^{A_2}(M; q) \ni \sum_{\hat{w} \in W^{\otimes 3}} (-1)^{\ell(\hat{w})} E_0^{(\varrho)}(\tau) = \frac{\sqrt{3}}{4(12)^3} \int_{-\bar{\tau}}^{i\infty} \int_{z_1}^{i\infty} \frac{\mathbb{I}_0^{(\varrho)}}{\sqrt{-i(z_1 + \tau)}\sqrt{-i(z_2 + \tau)}} dz_1 dz_2$

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$$+ \left[\theta^1_{12,1}\left(\frac{z_2}{4}\right) - \theta^1_{12,7}\left(\frac{z_2}{4}\right) + \theta^1_{12,9}\left(\frac{z_2}{4}\right)\right] \theta_{7}^{1,12+K}\left(\frac{z_1}{12}\right)$$

whereas $\sum_{\hat{w} \in W^{\otimes 3}} (-1)^{\ell(\hat{w})} E_1^{(\varrho)} = 0$ since $F_1^{2D}$ vanishes.

# Examples

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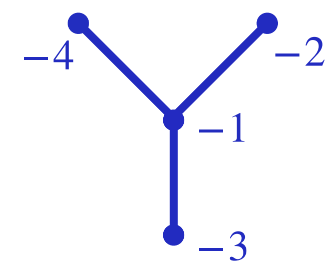


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$\hat{Z}_a^{A_1}(M;q) \sim \widetilde{\theta}_{11}^{1,12+K}$

whereas $\sum_{\hat{w} \in W^{\otimes 3}} (-1)^{\ell(\hat{w})}\, E_1^{(\varrho)} = 0$ since $F_1^{2D}$ vanishes.

# Companion $\check{Z}_a(M_3)$ at higher rank

✓ $\hat{Z}_a^{A_2}(M_3;q)$ are ~ linear combinations of generalised $A_2$ false $\theta$-functions "2d sums + 1d sums"

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[Cheng, Coman, Passaro, Sgroi, *to appear*]

✓ $\check{Z}_a^{A_2}(M_3;q)$ has a nice structure with respect to $SL_2(\mathbb{Z})$ and the plumbing data of $M_3$

... for $M_3$ Brieskorn spheres

<u>Theorem</u> For weakly negative definite Seifert 3-manifolds with 3 exceptional fibers & $\mathfrak{g} = A_2$

1. QMF: $\hat{Z}^{A_2}(q)$ is a sum of two depth-2 quantum modular forms
2. Recursion: If $\hat{Z}^{A_1}(q)$ has companion $g^*$, then $\check{Z}_a^{A_2}(q)$ is in the linear span of $I_{g',f}$ and $(g'')^*$ for simple $f$ and where $g'$, $g''$ are modular forms in the $SL(2,\mathbb{Z})$-orbit of $g$.

# Iterated Eichler integrals elsewhere

$b_2^+ = 1$

Topologically twisted $\mathcal{N}$=4 SU(N) / U(N) SYM theory on compact $M_4$ (Vafa-Witten)

partition function $Z_N$, for G=SU(N), is not modular under an S-transformation

[Vafa, Witten '94]

complexified gauge coupling $\tau \to -1/\tau$

for pure SU(2) SYM and $M_4 = \mathbb{P}^2$

- modular anomaly is an integral of a MF; can be traded for a holomorphic anomaly

[Minahan, Nemeschansky, Vafa, Warner '98] [Manschot '17]

by adding a non-holomorphic period integral to the partition function

- higher rank: modular transformation includes a shift by iterated integrals of $\theta$-series

[Manschot '17]

- holomorphic anomaly of $Z_N$ factorises into partition functions at lower rank

[Minahan, Nemeschansky, Vafa, Warner '98] [Manschot '17]

$$\partial_{\bar{\tau}} Z_N \sim \sum_k k(N-k) Z_k Z_{N-k}$$

constrains $Z_N$

- interpretation: the non-holomorphic contributions are generated by Q-exact terms due to boundaries of the moduli space

[Vafa, Witten '94] proposed, [Dabholkar, Putrov, Witten '20] verified by example

# What conclusions can be drawn

✓ $\hat{Z}_a^{A_2}(M_3;q)$ are ~ linear combinations of generalised $A_2$ false $\theta$-functions

✓ companions $\check{Z}_a^{A_2}(M_3;q)$ are ~ iterated Eichler integrals

$\hat{Z}_0^{SU(2)}(q) \sim \tilde{\theta}_{r_1}^{1,p+K}(q)$

$$\hat{Z}_a^{A_2}(q) \ni \sum_s (-1)^{l_s} F_\varrho^{2D}(\tau) \xrightarrow{\tau \to -\tau} \sum_s (-1)^{l_s} E_\varrho(q) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta_r^{1,p+K}\, \mathscr{B}_{r'}^{p+K_{r'}}\right)^*(\tau,\bar{\tau})\,;\, r,r' \in \mathbb{Z}/2p\right]$$

$$\# \mathscr{B}_r^{p+K_r}(\tau,z) \in \mathrm{Span}_{\mathbb{Z}}\left[\left(\theta_{3p,3r}^1\right)^*, \left(\theta_{3p,3r}^0\right)^*\right]$$

✓ Recursive and combinatorial structure ~ topological data

depth-2 QMF

<u>To do …</u>

- ☐ What happens for more general families of 3-manifolds
- ☐ Extract prediction for generic building blocks
- ☐ Explore links to Log VOAs
- ☐ What insights can be obtained about T[$M_3$] from the quantum modularity of $\hat{Z}(M_3)$

Thank you!