# Realizability toposes as homotopy categories

We give a definition of a *category* $\mathbf{Set}\langle\mathcal{P}\rangle$ *of fibrant objects* [1] for any tripos [2] $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$, and show that its homotopy category is the topos $\mathbf{Set}[\mathcal{P}]$. As special cases we obtain ways to view *realizability toposes*, but also *localic toposes* as homotopy categories (since both can be constructed from triposes).

On the other hand, the construction generalizes to *existential hyperdoctrines* $\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ (i.e. indexed preorders having fiberwise finite meets and left adjoints to reindexing satisfying Beck-Chevalley and Frobenius conditions) on finite limit categories $\mathbb{C}$.

Let $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ be a tripos [2].

**Definition 1** The category $\mathbf{Set}\langle\mathcal{P}\rangle$ is defined as follows.

- Objects are pairs $(A \in \mathbf{Set}, \rho \in \mathcal{P}(A \times A))$ such that the judgments

  **(sym)** $\rho(x, y) \vdash_{x,y} \rho(y, x)$ and
  **(trans)** $\rho(x, y), \rho(y, z) \vdash_{x,y,z} \rho(x, z)$

  hold in the logic of $\mathcal{P}$.
- Morphisms from $(A, \rho)$ to $(B, \sigma)$ are functions $f : A \to B$ validating the judgment

  **(compat)** $\rho(x, y) \vdash_{x,y} \sigma(fx, fy)$.
- Composition and identities are inherited from $\mathbf{Set}$. $\diamondsuit$

Thus, objects of $\mathbf{Set}\langle\mathcal{P}\rangle$ are partial equivalence relations in $\mathcal{P}$ – just like in the topos $\mathbf{Set}[\mathcal{P}]$ – but the morphisms are *structure-preserving functions*, whereas in $\mathbf{Set}[\mathcal{P}]$ they are functional relations internal to $\mathbf{Set}[\mathcal{P}]$ that are compatible with the partial equivalence relations.

We will see that $\mathbf{Set}[\mathcal{P}]$ can be recovered from $\mathbf{Set}\langle\mathcal{P}\rangle$ as a *homotopy category* with respect to a structure of *category of fibrant objects*. To this end, we make the following definitions.

**Definition 2** A morphism $f : (A, \rho) \to (B, \sigma)$ in $\mathbf{Set}\langle\mathcal{P}\rangle$ is a *fibration* if the judgment

**(fib)** $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

is valid in $\mathcal{P}$. It is a *weak equivalence* if the judgments

**(inj)** $\rho x, \sigma(fx, fy), \rho y \vdash_{x,y} \rho(x, y)$ and
**(esurj)** $\sigma u \vdash_u \exists x \,.\, \rho x \wedge \sigma(fx, u)$

are valid in $\mathcal{P}$. $\diamondsuit$

With this definition we can show the following.

**Theorem 3**
1. $\mathbf{Set}\langle\mathcal{P}\rangle$ *with the classes of fibrations and weak equivalences as in the previous definition is a* category of fibrant objects *in the sense of Brown [1].*
2. *Its homotopy category is the topos* $\mathbf{Set}[\mathcal{P}]$. ■

The statement that $\mathbf{Set}[\mathcal{P}]$ is the homotopy category of $\mathbf{Set}\langle\mathcal{P}\rangle$ means that $\mathbf{Set}[\mathcal{P}]$ is obtained from $\mathbf{Set}\langle\mathcal{P}\rangle$ by *freely inverting weak equivalences*. In other words, the canonical functor $E : \mathbf{Set}\langle\mathcal{P}\rangle \to \mathbf{Set}[\mathcal{P}]$ inverts weak equivalences, and moreover has the property that for every other functor $F : \mathbf{Set}\langle\mathcal{P}\rangle \to \mathbb{D}$ which inverts weak equivalences, there exists a unique functor $\widetilde{F} : \mathbf{Set}[\mathcal{P}] \to \mathbb{D}$ with $\widetilde{F} \circ E = F$.

$$\begin{array}{ccc} \mathbf{Set}\langle\mathcal{P}\rangle & & \\ {\scriptstyle E}\big\downarrow & \searrow^{F} & \\ \mathbf{Set}[\mathcal{P}] & \xrightarrow[\widetilde{F}]{} & \mathbb{D} \end{array}$$

This statement can also be proved directly, but we believe that the structure of 'category of fibrant objects' on $\mathbf{Set}\langle\mathcal{P}\rangle$ is interesting in its own right.

## References

[1] K.S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[2] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–231, 1980.