# Probability sheaves

Alex Simpson

Faculty of Mathematics and Physics
University of Ljubljana, Slovenia

In [2], Tao observes that the probability theory concerns itself with properties that are "preserved with respect to extension of the underlying sample space", in much the same way that modern geometry concerns itself with properties that are invariant with respect to underlying symmetries. Reformulating this in category-theoretic language, probabilistic concepts organise themselves into *presheaves* over a category of sample spaces. In this talk, I observe that they further form *sheaves*, and I consider ramifications of this observation.

As a suitable category of sample spaces, I take the category of measure-preserving measurable maps (modulo almost sure equality) between *standard* (a.k.a. *Lebesgue-Rokhlin*) probability spaces. In this category, every cospan completes to a commutative square enjoying a universal conditional-independence property. As a consequence, the category carries an atomic Grothendieck topology, whose sheaves can themselves be characterised in terms of conditional independence. Examples of such *probability sheaves* include sheaf representations of standard probability spaces (given by representables), sheaves of random variables, sheaves of probability measures (given by a general coend construction), and sheaves of orbits of ergodic group actions.

In general, I argue that the resulting atomic topos of probability sheaves is a natural category of generalised probabilistic concepts. Moreover, as a boolean topos, it models a mathematical universe in which *random variable* occurs as a primitive rather than derived mathematical notion. I believe this model has the potential to inform the development of an alternative approach to probability theory founded on primitive random variables, somewhat along the lines envisaged by Mumford in [1].

## References

[1] D. Mumford. The dawning of the age of stochasticity. In *Mathematics: Frontiers and Perspectives*. AMS, 2000.

[2] T. Tao. A review of probability theory. Note 0 in 254A – Random Matrices. Lecture notes on blog at `terrytao.wordpress.com`, 2010.