General bulk-edge correspondence at positive temperature

Massimo Moscolari
Eberhard Karls Universität Tübingen

Joint works with H. Cornean, B. Støttrup (Aalborg University) and S. Teufel (Tübingen)

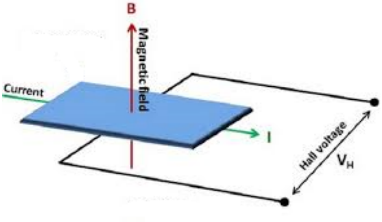
Arxiv: 2107.13456 and 2201.08803

Séminaire de Physique Mathématiques
@ Institut de Mathématiques de Bourgogne, Dijon
March 9, 2022

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

What is the bulk-edge correspondence? Part I

Topological insulator, $d = 2$: **Integer Quantum Hall Effect** K. von Klitzing 1981



The sample is an insulator.

Temperature ~ 0 K and $B \sim 5 - 10$ T.

Zero current $\parallel$ to the voltage drop V_H.

Current $I \perp$ to the voltage drop V_H.

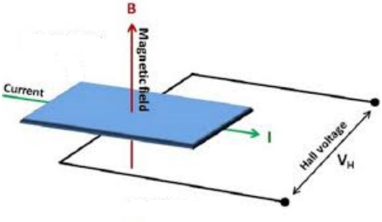
$$\boxed{\frac{I}{V_H} \text{ is quantized!}}$$

$\rightarrow$ Classical physics does not explain this behaviour ($V_H / I \propto B$).

Is there a rigorous quantum mechanical explanation from first principles? $\rightarrow$ ($\sim$) **yes!**

What is the bulk-edge correspondence? Part I

Topological insulator, $d = 2$: **Integer Quantum Hall Effect** K. von Klitzing 1981



The sample is an insulator.

Temperature ~ 0 K and $B \sim 5 - 10$ T.

Zero current $\parallel$ to the voltage drop V_H.

Current $I \perp$ to the voltage drop V_H.

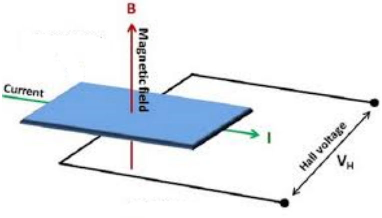
$\frac{I}{V_H}$ is quantized!

$\rightarrow$ Classical physics does not explain this behaviour ($V_H / I \propto B$).

Is there a rigorous quantum mechanical explanation from first principles? $\rightarrow$ ($\sim$) **yes!**

What is the bulk-edge correspondence? Part I

Topological insulator, $d = 2$: **Integer Quantum Hall Effect** K. von Klitzing 1981



The sample is an insulator.

Temperature ~ 0 K and $B \sim 5 - 10$ T.

Zero current $\|$ to the voltage drop V_H.

Current $I \perp$ to the voltage drop V_H.

$$\boxed{\frac{I}{V_H} \text{ is quantized!}}$$

$\rightarrow$ Classical physics does not explain this behaviour ($V_H/I \propto B$).

Is there a rigorous quantum mechanical explanation from first principles? $\rightarrow$ ($\sim$) **yes!**

Treating a finite simple in a rigorous way is beyond the status of present technology

→ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island ($(e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0))$).
  Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  → Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\,\sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    → Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island $((e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0)))$.
  Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\, \sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.

Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island $((e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0)))$. Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\, \sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island $((e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0)))$. Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\, \sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island ($(e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0))$). Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \to 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\, [H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\, [H_0, X_i] \rangle_{P_0} = \epsilon\, \sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.

Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island ($(e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0))$).
  Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i]\rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i]\rangle_{P_0} = \epsilon\,\sigma_{i2} + o(\epsilon)$$

- Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
- Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
  $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island ($(e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0))$).
  Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\, \sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different pictures to describe conduction: **bulk** and **edge**.
Important remark: independent electron approximation.

Bulk picture $T = 0$

- Hamiltonian H_0 acting on $L^2(\mathbb{R}^2)$.
- Spectrum of H_0 is usually purely a. c.
- Spectrum has an isolate spectral island ($(e_-, e_+) \subset (\mathbb{R} \setminus \sigma(H_0))$).
  Fermi energy $\mu \in (e_-, e_+)$.
  $\rightarrow P_0 = \chi_{(-\infty,\mu)}(H_0)$ spectral projection onto the spectral island below the gap.
- External perturbation: $H_0 + \epsilon X_2$ ($\epsilon \rightarrow 0$) .
  $\rightarrow$ Measure the current in the i-th direction:
  (Linear response theory using the adiabatic switching...)

$$\langle \mathrm{i}\,[H_0, X_i] \rangle_{\rho_\epsilon} - \langle \mathrm{i}\,[H_0, X_i] \rangle_{P_0} = \epsilon\,\sigma_{i2} + o(\epsilon)$$

  - Gap implies the system is insulating, no "direct" response! $\sigma_{22} = 0$.
  - Gap implies quantized transverse response! $\sigma_H =: \sigma_{12} \in \mathbb{Z}$.
    $\rightarrow$ Quantization has a topological origin: Chern number and non-commutative geometric generalization. P_0 has to be a projection!

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different approximations: **bulk** and **edge**.

Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\,[H_0^E, X_1] \rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different approximations: **bulk** and **edge**.

Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\left[H_0^E, X_1\right]\rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different approximations: **bulk** and **edge**.

Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\left[H_0^E, X_1\right]\rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different approximations: **bulk** and **edge**.
Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\left[H_0^E, X_1\right]\rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

Treating a finite simple in a rigorous way is beyond the status of present technology

$\rightarrow$ 2 different approximations: **bulk** and **edge**.

Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\left[H_0^E, X_1\right]\rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

Treating a finite simple in a rigorous way is beyond the status of present technology
$\rightarrow$ 2 different approximations: **bulk** and **edge**.
Important remark: independent electron approximation.

Edge picture $T = 0$

- H_0^E acting on $L^2(\mathbb{R} \times \mathbb{R}_+)$ with Dirichlet boundary condition.
- Spectrum of H_0^E is usually purely a.c..
- Spectrum in the gap of the bulk operator!
  $\Rightarrow (e_-, e_+) \in \sigma(H_0^E)$. $\rightarrow \chi_{(-\infty,\mu)}(H_0^E)$ depends on the position of μ!
- Generalized eigenfunction associated with the spectrum in the gap, decays exponentially far from the edge. $\rightarrow$ so-called edge modes.
- Edge current: "$I^E(\mu) := \langle \mathrm{i}\left[H_0^E, X_1\right]\rangle_{\chi_{(-\infty,\mu)}(H_0^E)}$". Edge conductance is defined by

$$-e\frac{\partial I^E(\mu)}{\partial \mu} =: \sigma_E$$

- $\sigma_E \in \mathbb{Z}$.

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of “edge modes/currents” at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

Our goal: Prove bulk-edge correspondence at any temperature.

$\Rightarrow$ Longer route than expected!

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of "edge modes/currents" at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

Our goal: Prove bulk-edge correspondence at any temperature.

$\Rightarrow$ Longer route than expected!

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of “edge modes/currents” at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

Our goal: Prove bulk-edge correspondence at any temperature.

$\Rightarrow$ Longer route than expected!

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of "edge modes/currents" at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

<u>All the results are for zero-temperature.</u>

Our goal: Prove bulk-edge correspondence at any temperature.

→ Longer route than expected!

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of “edge modes/currents” at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

All the results are for zero-temperature.

Our goal: Prove bulk-edge correspondence at any temperature.

→ Longer route than expected!

What is the bulk-edge correspondence? Part II

Physically:
Integer Quantum Hall Effect.

Linear response theory in the infinite volume limit gives $\sigma_H \in \mathbb{Z}$.

Analysis of “edge modes/currents” at the boundary of the sample gives $\sigma_E \in \mathbb{Z}$

Mathematically:

Bulk system defined on $L^2(\mathbb{R}^2)$.

Edge system defined by cutting the bulk one and imposing Dirichlet boundary conditions.

At zero temperature, with (mobility) gap

$$\sigma_H = \sigma_E \in \mathbb{Z}$$

Is there any mathematical relation/correspondence between the two systems?

Vast mathematical physics literature (2000-Today):
Kellendonk, Schulz-Baldes, Richter; Graf & collaborators; Prodan; Drouot; and many more...

Our goal: Prove bulk-edge correspondence at any temperature.

$\Rightarrow$ Longer route than expected!

Mathematical framework - The bulk-edge model

The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\gamma)$ the canonical action of $\mathbb{Z}^2$ on Θ ($T(\gamma)\omega = \{\omega_{\eta-\gamma}\}_{\eta\in\mathbb{Z}^2}$)

$$\Rightarrow \quad \tau_{b,\gamma} H_{\omega,b} \tau_{b,-\gamma} = H_{T(\gamma)\omega,b}, \quad \forall\, \gamma \in \mathbb{Z}^2 .$$

$\Rightarrow$ $(H_{\omega,b})_{\omega\in\Theta}$ is ergodic with respect to the lattice $\mathbb{Z}^2$

$\rightarrow$ No assumption on the spectrum of the model!

Mathematical framework - The bulk-edge model

The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

- Scalar potential V and magnetic potential $\mathcal{A}$ are smooth and $\mathbb{Z}^2$ periodic, namely
$$V(\mathbf{x}+\gamma) = V(\mathbf{x})\,, \quad \mathcal{A}(\mathbf{x}+\gamma) = \mathcal{A}(\mathbf{x}) \qquad \gamma \in \mathbb{Z}^2\,.$$
- $\mathbb{R} \ni b := -e\mathfrak{B}$ and $\mathbf{A}$ is the magnetic potential in the Landau gauge
$$A = (-x_2, 0)\,.$$
- The disordered background is modelled by the usual Anderson potential given by independent identically distributed random variables:
$$\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^2} = \omega \in \Theta = [-1,1]^{\mathbb{Z}^2}\,, \quad \mathbb{P} = \bigotimes_{\mathbb{Z}^2} \mu$$
$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma u(x-\gamma) \quad u \in C_0^\infty(\mathbb{R}^2)$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\omega)$ the canonical action of $\mathbb{Z}^2$ on Θ ...

Mathematical framework - The bulk-edge model

The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\gamma)$ the canonical action of $\mathbb{Z}^2$ on Θ ($T(\gamma)\omega = \{\omega_{\eta-\gamma}\}_{\eta\in\mathbb{Z}^2}$)

$$\Rightarrow \quad \tau_{b,\gamma} H_{\omega,b} \tau_{b,-\gamma} = H_{T(\gamma)\omega,b}, \quad \forall\,\gamma \in \mathbb{Z}^2\,.$$

$\Rightarrow$ $(H_{\omega,b})_{\omega\in\Theta}$ is ergodic with respect to the lattice $\mathbb{Z}^2$

$\to$ No assumption on the spectrum of the model!

The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\gamma)$ the canonical action of $\mathbb{Z}^2$ on Θ ($T(\gamma)\omega = \{\omega_{\eta-\gamma}\}_{\eta\in\mathbb{Z}^2}$)

$$\Rightarrow \quad \tau_{b,\gamma} H_{\omega,b} \tau_{b,-\gamma} = H_{T(\gamma)\omega,b}, \quad \forall\, \gamma \in \mathbb{Z}^2 .$$

$\Rightarrow$ $(H_{\omega,b})_{\omega\in\Theta}$ is ergodic with respect to the lattice $\mathbb{Z}^2$

$\rightarrow$ No assumption on the spectrum of the model!

Mathematical framework - The bulk-edge model

The bulk dynamics is described by a magnetic random Schroedinger operator on $L^2(\mathbb{R}^2)$:

$$H_{\omega,b} = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - bA\right)^2 + V + V_\omega$$

Let $\tau_{b,\gamma}$ be a family of magnetic translations compatible with the Landau gauge, and $T(\gamma)$ the canonical action of $\mathbb{Z}^2$ on Θ ($T(\gamma)\omega = \{\omega_{\eta-\gamma}\}_{\eta\in\mathbb{Z}^2}$)

$$\Rightarrow \quad \tau_{b,\gamma} H_{\omega,b} \tau_{b,-\gamma} = H_{T(\gamma)\omega,b}, \quad \forall\, \gamma \in \mathbb{Z}^2 .$$

$\Rightarrow$ $(H_{\omega,b})_{\omega\in\Theta}$ is ergodic with respect to the lattice $\mathbb{Z}^2$

$\rightarrow$ No assumption on the spectrum of the model!

Mathematical framework - The edge model

Consider the half-plane

$$E := \{(x_1, x_2) \in \mathbb{R}^2 |\, x_2 \geq 0\}.$$

The edge dynamics is described by the Hamiltonian $H^E_{\omega,b}$ living in $L^2(E)$ $\rightarrow$ $H^E_{\omega,b}$ is the natural choice given by the Dirichlet realization of $H_{\omega,b}$ in E $\rightarrow$ we cut the bulk system.

$(H^E_{\omega,b})_{\omega\in\Theta}$ is still ergodic with respect to the one-dimensional lattice generated by the vector $(1,0)$:

$$\tau_{b,\gamma} H^E_{\omega,b} \tau_{b,-\gamma} = H^E_{T(\gamma)\omega,b} \qquad \forall\, \gamma = (\gamma_1, 0) \in \mathbb{Z}^2 .$$

Mathematical framework - The edge model

Consider the half-plane

$$E := \{(x_1, x_2) \in \mathbb{R}^2 |\, x_2 \geq 0\}.$$

The edge dynamics is described by the Hamiltonian $H^E_{\omega,b}$ living in $L^2(E) \rightarrow$ $H^E_{\omega,b}$ is the natural choice given by the Dirichlet realization of $H_{\omega,b}$ in E $\rightarrow$ we cut the bulk system.

$(H^E_{\omega,b})_{\omega\in\Theta}$ is still ergodic with respect to the one-dimensional lattice generated by the vector $(1,0)$:

$$\tau_{b,\gamma} H^E_{\omega,b} \tau_{b,-\gamma} = H^E_{T(\gamma)\omega,b} \qquad \forall\, \gamma = (\gamma_1, 0) \in \mathbb{Z}^2 .$$

Mathematical framework - The edge model

Consider the half-plane

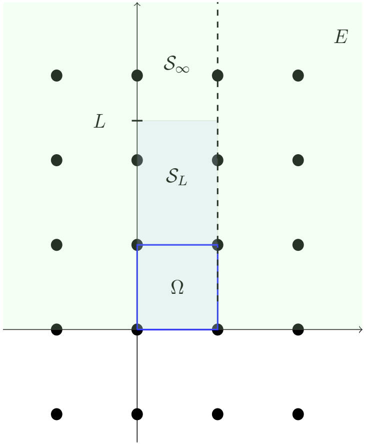
$$E := \{(x_1, x_2) \in \mathbb{R}^2 |\, x_2 \geq 0\}.$$

The edge dynamics is described by the Hamiltonian $H^E_{\omega,b}$ living in $L^2(E) \rightarrow$ $H^E_{\omega,b}$ is the natural choice given by the Dirichlet realization of $H_{\omega,b}$ in E $\rightarrow$ we cut the bulk system.

$(H^E_{\omega,b})_{\omega\in\Theta}$ is still ergodic with respect to the one-dimensional lattice generated by the vector $(1, 0)$:

$$\tau_{b,\gamma} H^E_{\omega,b} \tau_{b,-\gamma} = H^E_{T(\gamma)\omega,b} \qquad \forall\, \gamma = (\gamma_1, 0) \in \mathbb{Z}^2\,.$$

Mathematical framework - The edge model



$$\Omega := [0,1]^2$$

is the unit cell of the bulk Hamiltonian and χ_Ω is characteristic function of Ω.

$$\mathcal{S}_L := [0,1] \times [0,L]$$

χ_L characteristic function of $\mathcal{S}_L$.

$$\mathcal{S}_\infty := [0,1] \times [0,\infty]$$

χ_∞ characteristic function of $\mathcal{S}_\infty$.

(Edge) Thermodynamic pressure

Let $F_{\mu,T}(x) = -T \ln\left(1 + \mathrm{e}^{-(x-\mu)/T}\right)$ be the grandcanonical potential.
Remember that $F'_{\mu,T}(x) = \frac{1}{\mathrm{e}^{(x-\mu)/T}+1}$ is the Fermi-Dirac distribution

Bulk pressure

The bulk pressure is defined as the thermodynamic limit of the density of grandcanonical potential

$$p_{\mu,T}(b) := \mathbb{E}\left(\mathrm{Tr}\big(\chi_\Omega F_{\mu,T}(H_{\bullet,b})\big)\right) = \lim_{L\to\infty} \frac{1}{L^2} \mathrm{Tr}(\chi_{\Lambda_L} F_{\mu,T}(H_{\omega,b})).$$

What about the edge?

Edge pressure

$$p^{(E)}_{\mu,T}(b) := \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) := \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left(\chi_L F_{\mu,T}(H^E_{\omega,b})\right)$$

$\to$ These are the only two ingredients that we need!

Bulk-edge correspondence at positive temperature

Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P^{(L,\omega)}_{\mu,T}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p^{(E)}_{\mu,T}(b) = \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x})g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

- $(\star)$ holds true at every temperature.
- $(\star)$ holds independently of the spectrum of $H_{\omega,b}$.
- Purely analytic proof. No geometric/topological tools needed.

Bulk-edge correspondence at positive temperature

Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P^{(L,\omega)}_{\mu,T}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p^{(E)}_{\mu,T}(b) = \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x})g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

- $(\star)$ holds true at every temperature.
- $(\star)$ holds independently of the spectrum of $H_{\omega,b}$.
- Purely analytic proof. No geometric/topological tools needed.

Bulk-edge correspondence at positive temperature

Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P^{(L,\omega)}_{\mu,T}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p^{(E)}_{\mu,T}(b) = \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x})g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

- $(\star)$ holds true at every temperature.
- $(\star)$ holds independently of the spectrum of $H_{\omega,b}$.
- Purely analytic proof. No geometric/topological tools needed.

Bulk-edge correspondence at positive temperature

Theorem [H. Cornean, M.M., S.Teufel]

First, $p_{\mu,T}(\cdot)$ and $P^{(L,\omega)}_{\mu,T}(\cdot)$ are everywhere differentiable and, for a.e. $\omega \in \Theta$:

$$p^{(E)}_{\mu,T}(b) = \lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = p_{\mu,T}(b), \qquad \lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}}{\mathrm{d}b}(b) = \frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b).$$

Moreover, let $g \in C^1([0,1])$ be any function such that $g(0) = 1$ and $g(1) = 0$. Define $\widetilde{\chi}_L(\mathbf{x}) := \chi_L(\mathbf{x}) g(x_2/L)$. Then independently of g we have:

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

- $(\star)$ holds true at every temperature.
- $(\star)$ holds independently of the spectrum of $H_{\omega,b}$.
- Purely analytic proof. No geometric/topological tools needed.

Stability w.r.t. edge perturbation

The main result

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R}\times[0,d]$, $d>0$. Let $H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.
Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1,0)$.

$$\lim_{L\to\infty} \mathbb{E}(\mathrm{Tr}(\widetilde{\chi}_L \mathrm{i}\left[H^{E,W}_{\bullet,b}, X_1\right] F'(H^{E,W}_{\bullet,b}))) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'(H^E_{\bullet,b})\right)\right)$$

$\Rightarrow$ Stability w.r.t. boundary perturbations!

Stability w.r.t. edge perturbation

The main result

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R} \times [0, d]$, $d > 0$. Let $H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.

Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1, 0)$.

$$\lim_{L\to\infty} \mathbb{E}(\mathrm{Tr}(\widetilde{\chi}_L \mathrm{i}\left[H^{E,W}_{\bullet,b}, X_1\right] F'(H^{E,W}_{\bullet,b}))) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'(H^E_{\bullet,b})\right)\right)$$

$\Rightarrow$ Stability w.r.t. boundary perturbations!

Stability w.r.t. edge perturbation

The main result

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R}\times[0,d]$, $d>0$. Let $H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.
Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1,0)$.

$$\lim_{L\to\infty}\mathbb{E}(\mathrm{Tr}(\widetilde{\chi}_L \mathrm{i}\left[H^{E,W}_{\bullet,b}, X_1\right] F'(H^{E,W}_{\bullet,b}))) = \lim_{L\to\infty}\mathbb{E}\left(\mathrm{Tr}\left(\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'(H^E_{\bullet,b})\right)\right)$$

$\Rightarrow$ Stability w.r.t. boundary perturbations!

Stability w.r.t. edge perturbation

The main result

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R}\times[0,d]$, $d>0$. Let $H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.
Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1,0)$.

$$\lim_{L\to\infty} \mathbb{E}(\mathrm{Tr}(\widetilde{\chi}_L \mathrm{i}\left[H^{E,W}_{\bullet,b}, X_1\right] F'(H^{E,W}_{\bullet,b}))) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'(H^E_{\bullet,b})\right)\right)$$

⇒ Stability w.r.t. boundary perturbations!

Stability w.r.t. edge perturbation

The main result

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right). \qquad (\star)$$

still hold true in the case where the edge Hamiltonian is perturbed by a smooth potential W_ω supported in a finite strip near the edge.

Scalar potential W_ω, s.t. $\mathrm{supp}(W_\omega) \subseteq \mathbb{R} \times [0, d]$, $d > 0$. Let $H^{E,W}_{\omega,b} = H^E_{\omega,b} + W_\omega$, densely defined on $L^2(E)$ with Dirichlet boundary condition at $x_2 = 0$.
Assume that $(H^{E,W}_{\omega,b})_{\omega\in\Theta}$ is still ergodic on the one-dimensional lattice generated by $(1, 0)$.

$$\lim_{L\to\infty} \mathbb{E}(\mathrm{Tr}(\widetilde{\chi}_L \mathrm{i}\left[H^{E,W}_{\bullet,b}, X_1\right] F'(H^{E,W}_{\bullet,b}))) = \lim_{L\to\infty} \mathbb{E}\left(\mathrm{Tr}\left(\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'(H^E_{\bullet,b})\right)\right)$$

$\Rightarrow$ Stability w.r.t. boundary perturbations!

Physical interpretation: the bulk side

Left-hand side (the bulk):

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = m_{\mu,T}(b)$$

is just the definition of the bulk magnetization.

What about the right-hand side (the edge)?

$$\lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$$

$F'_{\mu,T}$ is the Fermi-Dirac distribution.

$\Rightarrow \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$ is *an* edge current!

($\rightarrow$ the limit is required because $\chi_\infty \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})$ is not trace class!)

Physical interpretation: the bulk side

Left-hand side (the bulk):

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = m_{\mu,T}(b)$$

is just the definition of the bulk magnetization.

What about the right-hand side (the edge)?

$$\lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$$

$F'_{\mu,T}$ is the Fermi-Dirac distribution.

$\Rightarrow \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$ is *an* edge current!

($\to$ the limit is required because $\chi_\infty \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})$ is not trace class!)

Physical interpretation: the bulk side

Left-hand side (the bulk):

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = m_{\mu,T}(b)$$

is just the definition of the bulk magnetization.

What about the right-hand side (the edge)?

$$\lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$$

$F'_{\mu,T}$ is the Fermi-Dirac distribution.

$\Rightarrow \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$ is *an* edge current!

($\rightarrow$ the limit is required because $\chi_\infty \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})$ is not trace class!)

Physical interpretation: the bulk side

Left-hand side (the bulk):

$$\frac{\mathrm{d}p_{\mu,T}}{\mathrm{d}b}(b) = m_{\mu,T}(b)$$

is just the definition of the bulk magnetization.

What about the right-hand side (the edge)?

$$\lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$$

$F'_{\mu,T}$ is the Fermi-Dirac distribution.

$\Rightarrow \lim_{L\to\infty} -\mathbb{E}\left(\mathrm{Tr}\left\{\widetilde{\chi}_L \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})\right\}\right)$ is *an* edge current!

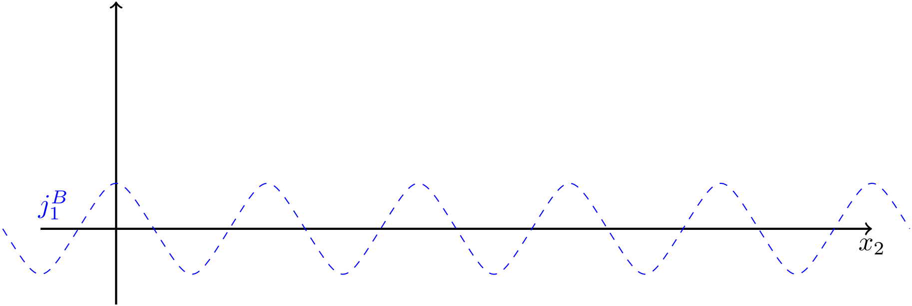
($\to$ the limit is required because $\chi_\infty \mathrm{i}\left[H^E_{\bullet,b}, X_1\right] F'_{\mu,T}(H^E_{\bullet,b})$ is not trace class!)

Analysis of the edge side

$$j_1^B(x_2) := -\int_0^1 \mathrm{d}x_1 \, \mathbb{E}\left(\mathrm{i}\,[H_{b,\bullet}, X_1]\, F'_{\mu,T}(H_{b,\bullet})\right)(x_1, x_2; x_1, x_2)\,.$$

$\rightarrow$ j_1^B is $\mathbb{Z}$-periodic.
$\rightarrow$ Vanishing of the persistent current:

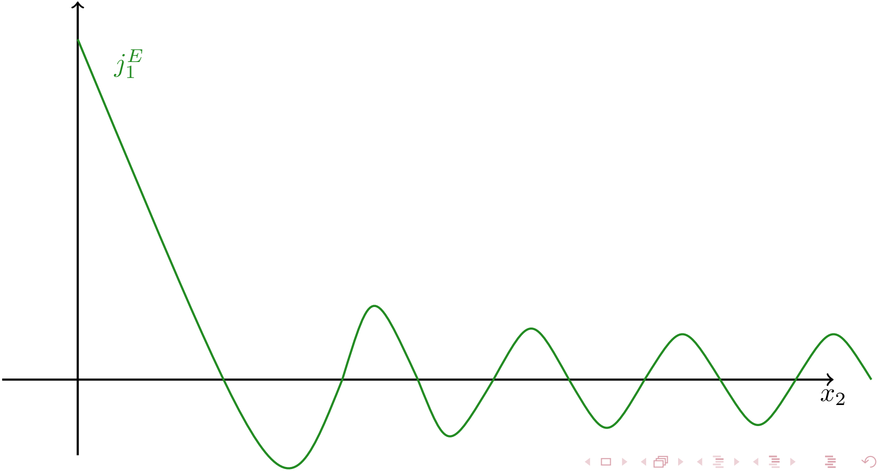
$$\int_0^1 \mathrm{d}x_2 j_1^B(x_2) = 0.$$



Analysis of the edge side

$$j_1^E(x_2) := -\int_0^1 \mathrm{d}x_1\, \mathbb{E}\left(\mathrm{i}\left[H_{b,\bullet}^E, X_1\right] F'_{\mu,T}(H_{b,\bullet}^E)\right)(x_1, x_2; x_1, x_2)\,.$$

$\rightarrow j_1^E$ is supported on E.

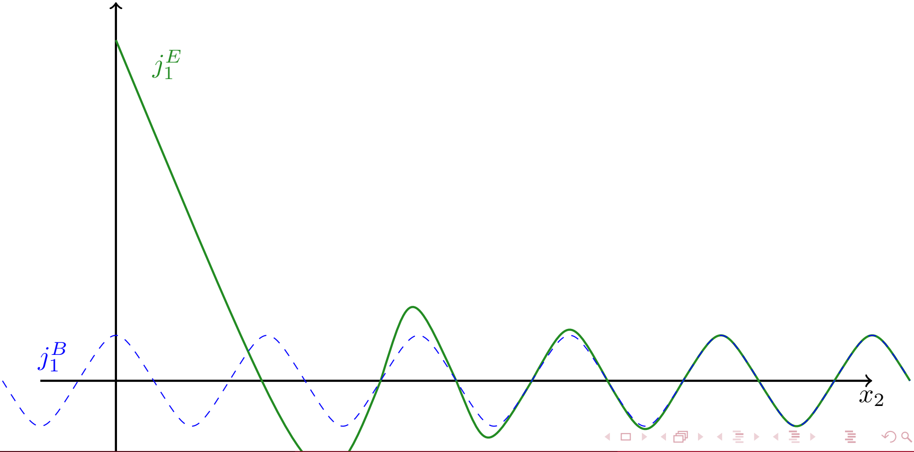


Analysis of the edge side

Theorem [M.M., B. Støttrup]

$$j_1^E(x_2) - j_1^B(x_2) = \mathcal{O}(x_2^{-\infty}) \qquad x_2 \to +\infty$$

$\Rightarrow j_1^E - j_1^B$ is integrable!



Analysis of the edge side

The total edge current is defined as

$$I_1^E(\mu, T, b) := \lim_{L\to\infty} \int_0^L \left(j_1^E(x_2) - \left(1 - g(x_2/L)\right) j_1^B(x_2)\right) \, \mathrm{d}x_2$$
$$= \lim_{L\to\infty} \int_0^L g(x_2/L) j_1^E(x_2) \, \mathrm{d}x_2 \, .$$

$\to$ we subtract j_1^B from j_1^E **only in the bulk** by using a C^1-function

$$g : [0,1] \to [0,1]$$

with $g(0) = 1$ and $g(1) = 0$.

Analysis of the edge side

The total edge current is defined as

$$I_1^E(\mu, T, b) := \lim_{L\to\infty} \int_0^L \left(j_1^E(x_2) - \big(1 - g(x_2/L)\big) j_1^B(x_2)\right) \, \mathrm{d}x_2$$
$$= \lim_{L\to\infty} \int_0^L g(x_2/L) j_1^E(x_2) \, \mathrm{d}x_2 \, .$$

The last equality follows by adding and subtracting $\lim_{L\to\infty} \int_0^L (1 - g(x_2/L)) \left(j_1^E(x_2) - j_1^B(x_2)\right) \, \mathrm{d}x_2 = 0.$
$\rightarrow$ we subtract j_1^B from j_1^E **only in the bulk** by using a C^1-function

$$g : [0, 1] \rightarrow [0, 1]$$

with $g(0) = 1$ and $g(1) = 0$.

Analysis of the edge side

The total edge current is defined as

$$I_1^E(\mu, T, b) := \lim_{L\to\infty} \int_0^L \big(j_1^E(x_2) - \big(1 - g(x_2/L)\big) j_1^B(x_2)\big)\, \mathrm{d}x_2$$
$$= \lim_{L\to\infty} \int_0^L g(x_2/L) j_1^E(x_2)\, \mathrm{d}x_2\,.$$

$\to$ we subtract j_1^B from j_1^E **only in the bulk** by using a C^1-function

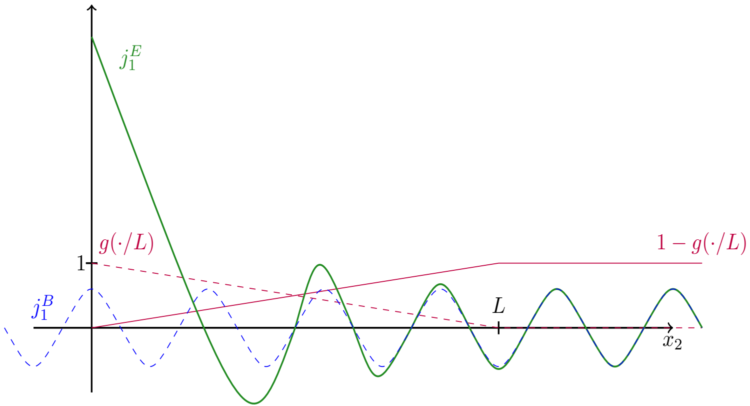
$$g : [0,1] \to [0,1]$$

with $g(0) = 1$ and $g(1) = 0$.

Analysis of the edge side

→ we subtract j_1^B form j_1^E **only in the bulk** by using a C^1-function $g : [0,1] \to [0,1]$ with $g(0) = 1$ and $g(1) = 0$.

$$I_1^E = \lim_{L\to\infty} \int_0^L g(x_2/L) j_1^E(x_2)\, \mathrm{d}x_2 \,.$$

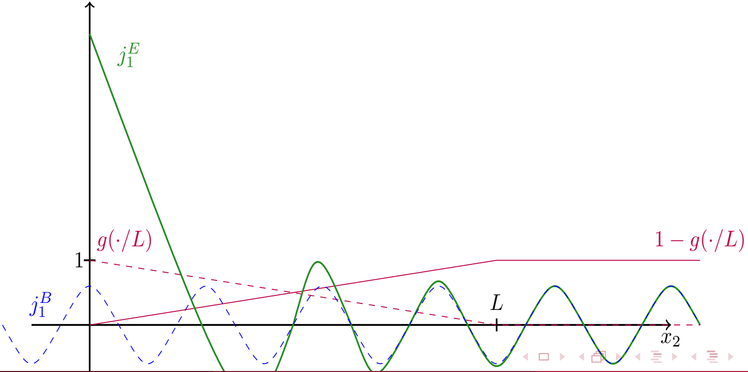


Analysis of the edge side

→ we subtract j_1^B form j_1^E **only in the bulk** by using a C^1-function $g : [0,1] \to [0,1]$ with $g(0) = 1$ and $g(1) = 0$.

$$I_1^E = \lim_{L\to\infty} \int_0^L g(x_2/L) j_1^E(x_2)\,\mathrm{d}x_2\,.$$

→ We show that the value of I_1^E is actually **independent of the specific cut-off function** g and **of the specific potential at the boundary** ! → It is a very robust quantity that lives near the edge!



Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

$\rightarrow$ What about the usual bulk-edge correspondence ($\sigma_H = \sigma_E$)?

Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Remarks:

- $m_{\mu,T}(b)$ is known as the orbital magnetization (spinless electrons).
- Classical system: orbital magnetization is always zero.
  $\rightarrow$ Bohr–Van Leeuwen theorem.
- $m_{\mu,T}(b) \neq 0$ is known as Landau diamagnetism.

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Remarks:

- $m_{\mu,T}(b)$ is known as the orbital magnetization (spinless electrons).
- Classical system: orbital magnetization is always zero.
  $\rightarrow$ Bohr–Van Leeuwen theorem.
- $m_{\mu,T}(b) \neq 0$ is known as Landau diamagnetism.

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Remarks:

- $m_{\mu,T}(b)$ is known as the orbital magnetization (spinless electrons).
- Classical system: orbital magnetization is always zero.
  $\rightarrow$ Bohr–Van Leeuwen theorem.
- $m_{\mu,T}(b) \neq 0$ is known as Landau diamagnetism.

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Remarks:

- $m_{\mu,T}(b)$ is known as the orbital magnetization (spinless electrons).
- Classical system: orbital magnetization is always zero.
  $\rightarrow$ Bohr–Van Leeuwen theorem.
- $m_{\mu,T}(b) \neq 0$ is known as Landau diamagnetism.

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

Physical interpretation II

Therefore we get the bulk-edge correspondence in the form:

$$m_{\mu,T}(b) = I_1^E(\mu, T, b).$$

Literature:

- Bulk side: Thorough analysis of the thermodynamic limit of the magnetization. Landau, Angelescu-Bundaru-Nenciu, Cornean-Briet-Savoie, Schulz-Baldes-Teufel, etc...
- Connection with edge current: Macris-Martan-Pulè (CMP 1988), Kunz (JSP 1994).
  Both restricted to pure Landau operator and high temperature (Maxwell-Boltzmann distribution).
  $\rightarrow$ our proof is far more general and allow to use the physically relevant Fermi-Dirac distribution (actually any Schwartz function!).

$\rightarrow$ What about the usual bulk-edge correspondence ($\sigma_H = \sigma_E$)?

Zero-temperature limit and bulk-edge correspondence

At positive temperature the pressure is C^2 in b and μ (Briet-Savoie RMP12):

$$\partial_\mu p_{\mu,T}(b) = n_{\mu,T}(b) = \mathbb{E}(\mathrm{Tr}(\chi_\Omega F_{\mu,T}(H_{\omega,b})))$$

where $n_{\mu,T}(b)$ is the particle density.

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$$

Assume that the **almost sure spectrum** $\Sigma(b_0)$ of the bulk Hamiltonian H_{ω,b_0}, $b_0 \in \mathbb{R}$, **has a gap** that includes the interval $[e_-, e_+]$ with $e_- < e_+$.
$\to \sigma_0(b) := \Sigma(b) \cap (-\infty, e_-)$, $P_{\omega,b}$ the spectral projection onto $\sigma_0(b)$.

Středa formula [Cornean,Monaco, M.M. JEMS 21, ...]

$$2\pi \partial_b n_{\mu,0}(b_0) = C_0 := 2\pi \mathbb{E}\Big(\mathrm{Tr}\left(\chi_\Omega P_{\bullet,b_0} \mathrm{i}\big[[X_1, P_{\bullet,b_0}],[X_2, P_{\bullet,b_0}]\big]\right)\Big) = \sigma_H \ (\in \mathbb{Z})$$

C_0 is the Chern character of the projection P_{ω,b_0}.

Zero-temperature limit and bulk-edge correspondence

At positive temperature the pressure is C^2 in b and μ (Briet-Savoie RMP12):

$$\partial_\mu p_{\mu,T}(b) = n_{\mu,T}(b) = \mathbb{E}(\mathrm{Tr}(\chi_\Omega F_{\mu,T}(H_{\omega,b})))$$

where $n_{\mu,T}(b)$ is the particle density.

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$$

Assume that the **almost sure spectrum** $\Sigma(b_0)$ of the bulk Hamiltonian H_{ω,b_0}, $b_0 \in \mathbb{R}$, **has a gap** that includes the interval $[e_-, e_+]$ with $e_- < e_+$.
$\to \sigma_0(b) := \Sigma(b) \cap (-\infty, e_-)$, $P_{\omega,b}$ the spectral projection onto $\sigma_0(b)$.

Středa formula [Cornean,Monaco, M.M. JEMS 21, ...]

$$2\pi \partial_b n_{\mu,0}(b_0) = C_0 := 2\pi \mathbb{E}\left(\mathrm{Tr}\left(\chi_\Omega P_{\cdot,b_0} \mathrm{i}\left[[X_1, P_{\cdot,b_0}], [X_2, P_{\cdot,b_0}] \right] \right) \right) = \sigma_H \, (\in \mathbb{Z})$$

C_0 is the Chern character of the projection P_{ω,b_0}.

Zero-temperature limit and bulk-edge correspondence

At positive temperature the pressure is C^2 in b and μ (Briet-Savoie RMP12):

$$\partial_\mu p_{\mu,T}(b) = n_{\mu,T}(b) = \mathbb{E}(\mathrm{Tr}(\chi_\Omega F_{\mu,T}(H_{\omega,b})))$$

where $n_{\mu,T}(b)$ is the particle density.

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$$

Assume that the **almost sure spectrum** $\Sigma(b_0)$ of the bulk Hamiltonian H_{ω,b_0}, $b_0 \in \mathbb{R}$, **has a gap** that includes the interval $[e_-, e_+]$ with $e_- < e_+$.
$\to \sigma_0(b) := \Sigma(b) \cap (-\infty, e_-)$, $P_{\omega,b}$ the spectral projection onto $\sigma_0(b)$.

Středa formula [Cornean,Monaco, M.M. JEMS 21, ...]

$$2\pi \partial_b n_{\mu,0}(b_0) = C_0 := 2\pi \mathbb{E}\Big(\mathrm{Tr}\left(\chi_\Omega P_{\bullet,b_0} \mathrm{i}\big[[X_1, P_{\bullet,b_0}], [X_2, P_{\bullet,b_0}]\big]\right) \Big) = \sigma_H \; (\in \mathbb{Z})$$

C_0 is the Chern character of the projection P_{ω,b_0}.

Consider a "smooth characteristic function": Any smooth function $0 \leq F_0 \leq 1$ which equals 1 on $(-\infty, e_-]$ and 0 on $[e_+, \infty) \Rightarrow f_0(H_{\omega,b}) = P_{\omega,b}$

$$\Rightarrow n^{(f_0)}_{\mu,0}(b) := \mathbb{E}\left(\mathrm{Tr}\big(\chi_\Omega f_0(H_{\bullet,b})\big)\right) = \lim_{L\to\infty} \frac{1}{L^2}\mathrm{Tr}(\chi_{\Lambda_L} f_0(H_{\omega,b}))$$

$$n^{(f_0)}_{\mu,0}(b) = n_{\mu,0}(b) = \mathbb{E}\left(\mathrm{Tr}\big(\chi_\Omega P_{\omega,b}\big)\right)$$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula**.

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\rightarrow$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative $(\partial_\mu I(\mu, T, b))$ but that **do not** contribute to transport!

Consider a "smooth characteristic function": Any smooth function $0 \leq F_0 \leq 1$ which equals 1 on $(-\infty, e_-]$ and 0 on $[e_+, \infty) \Rightarrow f_0(H_{\omega,b}) = P_{\omega,b}$

$$\Rightarrow n^{(f_0)}_{\mu,0}(b) := \mathbb{E}\left(\mathrm{Tr}\big(\chi_\Omega f_0(H_{\bullet,b})\big)\right) = \lim_{L\to\infty} \frac{1}{L^2}\,\mathrm{Tr}(\chi_{\Lambda_L} f_0(H_{\omega,b}))$$

$$n^{(f_0)}_{\mu,0}(b) = n_{\mu,0}(b) = \mathbb{E}\left(\mathrm{Tr}\big(\chi_\Omega P_{\omega,b}\big)\right)$$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E(= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula**.

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\rightarrow$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative $(\partial_\mu I(\mu, T, b))$ but that **do not** contribute to transport!

- Maxwell relation: $\partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$
- Our theorem : $m_{\mu,T}(b) = I(\mu, T, b)$
- Středa formula: $2\pi \partial_b n_{\mu,0}(b_0) = \sigma_H$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula.**

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\rightarrow$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative ($\partial_\mu I(\mu, T, b)$) but that **do not** contribute to transport!

- Maxwell relation: $\partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$
- Our theorem : $m_{\mu,T}(b) = I(\mu, T, b)$
- Středa formula: $2\pi \partial_b n_{\mu,0}(b_0) = \sigma_H$

Proposition [H. Cornean, M.M., S. Teufel]

There exist two constants $C_1, C_2 > 0$ such that

$$|\partial_b n_{\mu,T}(b_0) - \sigma_H| \leq C_1 \, \mathrm{e}^{-C_2/T}$$

$$|\partial_\mu I(\mu, T, b_0) - \sigma_E| \leq C_1 \, \mathrm{e}^{-C_2/T}.$$

Moreover, let χ_∞ denote the indicator function of the strip $\mathcal{S}_\infty := [0,1] \times (0, \infty)$, then, independently of the specific choice f_0, we have:

$$\lim_{T \searrow 0} \partial_b n_{\mu,T}(b_0) = \sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E.$$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula**.

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

- Maxwell relation: $\partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$
- Our theorem : $m_{\mu,T}(b) = I(\mu, T, b)$
- Středa formula: $2\pi \partial_b n_{\mu,0}(b_0) = \sigma_H$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula.**

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\rightarrow$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative ($\partial_\mu I(\mu, T, b)$) but that **do not** contribute to transport!

- Maxwell relation: $\partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$
- Our theorem : $m_{\mu,T}(b) = I(\mu, T, b)$
- Středa formula: $2\pi \partial_b n_{\mu,0}(b_0) = \sigma_H$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I_1^E)$$

Key ingredient: **Středa formula.**

$\to$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\to$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative ($\partial_\mu I(\mu, T, b)$) but that **do not** contribute to transport!

- Maxwell relation: $\partial_\mu m_{\mu,T}(b) = \partial_\mu \partial_b p_{\mu,T}(b) = \partial_b n_{\mu,T}(b)$
- Our theorem : $m_{\mu,T}(b) = I(\mu, T, b)$
- Středa formula: $2\pi \partial_b n_{\mu,0}(b_0) = \sigma_H$

$\Rightarrow$ We recover the usual bulk-edge correspondence at zero temperature!

$$\sigma_H = -\mathbb{E}\left(\mathrm{Tr}\left\{\chi_\infty \mathrm{i}\left[H^E_{\bullet,b_0}, X_1\right] f_0'(H^E_{\bullet,b_0})\right\}\right) = \sigma_E (= \partial_\mu I^E_1)$$

Key ingredient: **Středa formula.**

$\rightarrow$ Problem: Středa formula holds true only at zero temperature and with μ in a spectral gap!

$\rightarrow$ Starting point :

$$m_{\mu,T}(b) = I(\mu, T, b).$$

$\Rightarrow$ there might be states that contribute to the statistical derivative ($\partial_\mu I(\mu, T, b)$) but that **do not** contribute to transport!

(Simplified) physical picture

This has been first emphasized by Haidu-Gummich 83, Cooper-Halperin-Ruzin 96, Středa 96:

Splitting of the edge current density

$$j(x) = j_{mag}(x) + j_{tr}(x)$$

Splitting of the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j_{mag} is a pure "magnetization current density", that is

$$j_{mag}(x) = \nabla \times m^{(circ)}_{\mu,T}(x).$$

$\rightarrow$ Why this magnetization current influences only the edge?

$\Rightarrow \quad I_{mag} = \int_{-\infty}^{\infty} dx_2 j_{mag}(x) = \int_{-\infty}^{\infty} dx_2 \partial_2 \varphi(x_2) m_c = m^{(circ)}_{\mu,T}.$

$$\Rightarrow I = I_{mag} + I_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) = m_{\mu,T}(b)$$

In order to get the correct transport edge current we have to be able to split either the current or the magnetization!

(Simplified) physical picture

This has been first emphasized by Haidu-Gummich 83, Cooper-Halperin-Ruzin 96, Středa 96:

Splitting of the edge current density

$$j(x) = j_{mag}(x) + j_{tr}(x)$$

Splitting of the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j_{mag} is a pure "magnetization current density", that is

$$j_{mag}(x) = \nabla \times m^{(circ)}_{\mu,T}(x).$$

$\rightarrow$ Why this magnetization current influences only the edge?

$\Rightarrow \quad I_{mag} = \int_{-\infty}^{\infty} dx_2 j_{mag}(x) = \int_{-\infty}^{\infty} dx_2 \partial_2 \varphi(x_2) m_c = m^{(circ)}_{\mu,T}.$

$$\Rightarrow I = I_{mag} + I_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) = m_{\mu,T}(b)$$

In order to get the correct transport edge current we have to be able to split either the current or the magnetization!

(Simplified) physical picture

Splitting of the edge current density

$$j(x) = j_{mag}(x) + j_{tr}(x)$$

Splitting of the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j_{mag} is a pure "magnetization current density", that is

$$j_{mag}(x) = \nabla \times m^{(circ)}_{\mu,T}(x).$$

$\rightarrow$ Why this magnetization current influences only the edge? Smooth boundary $m^{(circ)}_{\mu,T}(x) := \varphi(x_2) m^{(circ)}_{\mu,T}$. $\rightarrow m^{(circ)}_{\mu,T}$ is the constant value for an homogeneous sample.

$$\Rightarrow \quad j_{mag}(x) = j_{mag}(x_2) = \partial_2 \varphi(x_2) m^{(circ)}_{\mu,T}.$$

$$\Rightarrow \quad I_{mag} = \int_{-\infty}^{\infty} dx_2 j_{mag}(x) = \int_{-\infty}^{\infty} dx_2 \partial_2 \varphi(x_2) m_c = m^{(circ)}_{\mu,T}.$$

$$\Rightarrow I = I_{mag} + I_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) \ldots m_{\mu,T}(b)$$

(Simplified) physical picture

Splitting of the edge current density

$$j(x) = j_{mag}(x) + j_{tr}(x)$$

Splitting of the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j_{mag} is a pure "magnetization current density", that is

$$j_{mag}(x) = \nabla \times m^{(circ)}_{\mu,T}(x).$$

$\rightarrow$ Why this magnetization current influences only the edge? Smooth boundary $m^{(circ)}_{\mu,T}(x) := \varphi(x_2) m^{(circ)}_{\mu,T}$. $\rightarrow$ $m^{(circ)}_{\mu,T}$ is the constant value for an homogeneous sample.

$$\Rightarrow \quad j_{mag}(x) = j_{mag}(x_2) = \partial_2\varphi(x_2) m^{(circ)}_{\mu,T}.$$

$$\Rightarrow \quad I_{mag} = \int_{-\infty}^{\infty} dx_2 j_{mag}(x) = \int_{-\infty}^{\infty} dx_2 \partial_2\varphi(x_2) m_c = m^{(circ)}_{\mu,T}.$$

$$\Rightarrow I = I_{mag} + I_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) = m_{\mu,T}(b)$$

(Simplified) physical picture

Splitting of the edge current density

$$j(x) = j_{mag}(x) + j_{tr}(x)$$

Splitting of the magnetization

$$m_{\mu,T}(b) = m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

j_{mag} is a pure "magnetization current density", that is

$$j_{mag}(x) = \nabla \times m^{(circ)}_{\mu,T}(x).$$

$\rightarrow$ Why this magnetization current influences only the edge?

$\Rightarrow \quad I_{mag} = \int_{-\infty}^{\infty} dx_2 j_{mag}(x) = \int_{-\infty}^{\infty} dx_2 \partial_2 \varphi(x_2) m_c = m^{(circ)}_{\mu,T}$.

$$\Rightarrow I = I_{mag} + I_{tr} = m^{(circ)}_{\mu,T}(b) + m^{(res)}_{\mu,T}(b) = m_{\mu,T}(b)$$

In order to get the correct transport edge current we have to be able to split either the current or the magnetization!

Bulk-edge correspondence $T \geq 0$: Landau case

Setting:
Bulk Hamiltonian $H_b = \frac{1}{2}(-\mathrm{i}\nabla - bA)^2$
Spectrum given by infinitely degenerate eigenvalue $\left\{E_{n,b} = b(n+\frac{1}{2}) \,|\, n \in \mathbb{N}\right\}$.
$\Pi_{n,b}$ spectral projection onto $E_{n,b}$.
Integrated density of states associated to each Landau level:

$$\lim_{L\to\infty} \frac{\mathrm{Tr}(\chi_{\Lambda_L}\Pi_{n,b})}{L^2} = \mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \frac{b}{2\pi}.$$

Hall conductivity for $T \geq 0$ (Evaluation of Kubo formula, Cornean-Nenciu-Pedersen 2006, physics paper...):

$$\sigma_H(\mu, T, b) = \frac{n_{\mu,T}(b)}{b}.$$

$\to$ The pressure is simply given by

$$p_{\mu,T}(b) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b})\,\mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b}) \frac{b}{2\pi}$$

$$\Rightarrow m_{\mu,T}(b) := \partial_b p_{\mu,T}(b) = \sum^{\infty} F'_{\mu,T}(E_{n,b}) \frac{dE_{n,b}}{} \frac{b}{} + \frac{p_{\mu,T}(b)}{}$$

Bulk-edge correspondence $T \geq 0$: Landau case

Setting:

$H_b = \frac{1}{2}(-\mathrm{i}\nabla - bA)^2$

$\{E_{n,b} = b(n + \frac{1}{2}) \mid n \in \mathbb{N}\}$. I.d.s.: $\mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \frac{b}{2\pi}$.

Hall conductivity for $T \geq 0$: $\sigma_H(\mu, T, b) = \frac{n_{\mu,T}(b)}{b}$.

$\rightarrow$ The pressure is simply given by

$$p_{\mu,T}(b) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b})\, \mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b}) \frac{b}{2\pi}$$

$$\Rightarrow m_{\mu,T}(b) := \partial_b p_{\mu,T}(b) = \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{dE_{n,b}}{db} \frac{b}{2\pi} + \frac{p_{\mu,T}(b)}{b}$$

$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \boxed{\sum_{n=1}^{\infty} F''_{\mu,T}(E_{n,b}) \frac{dE_{n,b}}{db} \frac{b}{2\pi}} + \sigma_H(\mu, T, b)$$

$\rightarrow$ Extra term $\rightarrow$ the weighted sum of the angular momentum of each states in the Landau levels!

Bulk-edge correspondence $T \geq 0$: Landau case

Setting:

$H_b = \frac{1}{2}(-\mathrm{i}\nabla - bA)^2$

$\left\{E_{n,b} = b(n + \frac{1}{2}) \,|\, n \in \mathbb{N}\right\}$. I.d.s.: $\mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \frac{b}{2\pi}$.

Hall conductivity for $T \geq 0$: $\sigma_H(\mu, T, b) = \frac{n_{\mu,T}(b)}{b}$.

$\rightarrow$ The pressure is simply given by

$$p_{\mu,T}(b) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b})\,\mathrm{Tr}(\chi_\Omega \Pi_{n,b}) = \sum_{n=0}^{\infty} F_{\mu,T}(E_{n,b}) \frac{b}{2\pi}$$

$$\Rightarrow m_{\mu,T}(b) := \partial_b p_{\mu,T}(b) = \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{dE_{n,b}}{db} \frac{b}{2\pi} + \frac{p_{\mu,T}(b)}{b}$$

$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

$$\Rightarrow \partial_\mu m_{\mu,T}(b) = \boxed{\sum_{n=1}^{\infty} F''_{\mu,T}(E_{n,b}) \frac{dE_{n,b}}{db} \frac{b}{2\pi}} + \sigma_H(\mu, T, b)$$

$\rightarrow$ Extra term $\rightarrow$ the weighted sum of the angular momentum of each states in the Landau levels!

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula $(\star)$

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

$$m^{res}_{\mu,T}(b) = m_{\mu,T}(b) - m^{circ}_{\mu,T}(b) = I(\mu, T, b) - I_{mag}(\mu, T, b) = I_{tr}(\mu, T, b)$$

Bulk-edge correspondence at $T \geq 0$ [Cornean, M.M., Teufel]

$$\sigma_H(\mu, T, b) = \frac{n_{\mu,T,b}}{b} = \partial_\mu m^{res}_{\mu,T}(b) = \partial_\mu I_{tr}(\mu, T, b) = \sigma_E(\mu, T, b)$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

- $\frac{b}{2\pi}$ is the number of states per unit area.
- $F'_{\mu,T}(E_{n,b})$ is the statistical weight $\rightarrow$ *occupation number.*
- What about $\frac{d}{db} E_{n,b} = n$?
  $\rightarrow$ "Hellmann-Feynman theorem", $\{\psi_{n,m}\}_{m \in \mathbb{Z}}$ o.n.b. for $\mathrm{Ran}\Pi_{n,b}$

$$\langle \psi_{n,m}, \frac{1}{2} \mathbf{x} \times \mathbf{p} \psi_{n,m} \rangle = \langle \psi_{n,m}, L \psi_{n,m} \rangle = n.$$

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula $(\star)$

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

- $\frac{b}{2\pi}$ is the number of states per unit area.
- $F'_{\mu,T}(E_{n,b})$ is the statistical weight $\rightarrow$ *occupation number*.
- What about $\frac{d}{db} E_{n,b} = n$?
  $\rightarrow$ "Hellmann-Feynman theorem", $\{\psi_{n,m}\}_{m \in \mathbb{Z}}$ o.n.b. for $\mathrm{Ran}\Pi_{n,b}$

$$\langle \psi_{n,m}, \frac{1}{2} \mathbf{x} \times \mathbf{p} \psi_{n,m} \rangle = \langle \psi_{n,m}, L \psi_{n,m} \rangle = n.$$

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula $(\star)$

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m_{\mu,T}^{circ}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

- $\frac{b}{2\pi}$ is the number of states per unit area.
- $F'_{\mu,T}(E_{n,b})$ is the statistical weight $\rightarrow$ *occupation number*.
- What about $\frac{d}{db} E_{n,b} = n$?
  $\rightarrow$ "Hellmann-Feynman theorem", $\{\psi_{n,m}\}_{m \in \mathbb{Z}}$ o.n.b. for $\mathrm{Ran}\Pi_{n,b}$

$$\langle \psi_{n,m}, \frac{1}{2} \mathbf{x} \times \mathbf{p} \psi_{n,m} \rangle = \langle \psi_{n,m}, L \psi_{n,m} \rangle = n.$$

$\Rightarrow$ $m_{\mu,T}^{circ}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m_{\mu,T}^{circ}(b) = I_{mag}(\mu, T, b)$$

Our formula $(\star)$

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

- $\frac{b}{2\pi}$ is the number of states per unit area.
- $F'_{\mu,T}(E_{n,b})$ is the statistical weight $\to$ *occupation number*.
- What about $\frac{d}{db} E_{n,b} = n$?
  $\to$ "Hellmann-Feynman theorem", $\{\psi_{n,m}\}_{m\in\mathbb{Z}}$ o.n.b. for $\mathrm{Ran}\Pi_{n,b}$

$$\langle \psi_{n,m}, \frac{1}{2}\mathbf{x} \times \mathbf{p}\psi_{n,m}\rangle = \langle \psi_{n,m}, L\psi_{n,m}\rangle = n.$$

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation

$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula ($\star$)

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula ($\star$)

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

$$m^{res}_{\mu,T}(b) = m_{\mu,T}(b) - m^{circ}_{\mu,T}(b) = I(\mu, T, b) - I_{mag}(\mu, T, b) = I_{tr}(\mu, T, b)$$

Bulk-edge correspondence at $T \geq 0$ [Cornean, M.M., Teufel]

$$\sigma_H(\mu, T, b) = \frac{n_{\mu,T,b}}{b} = \partial_\mu m^{res}_{\mu,T}(b) = \partial_\mu I_{tr}(\mu, T, b) = \sigma_E(\mu, T, b)$$

Bulk-edge correspondence $T \geq 0$: Landau case

$$m^{circ}_{\mu,T}(b) := \sum_{n=1}^{\infty} F'_{\mu,T}(E_{n,b}) \frac{d}{db} E_{n,b} \frac{b}{2\pi}$$

is magnetic moment per unit area.

$\Rightarrow$ $m^{circ}_{\mu,T}(b)$ is the part of the magnetization given by the local circulation
$\Rightarrow$ we have to subtract the associated edge current contribution:

$$m^{circ}_{\mu,T}(b) = I_{mag}(\mu, T, b)$$

Our formula $(\star)$

$$\Rightarrow \quad \boxed{m_{\mu,T}(b) = I(\mu, T, b)}$$

$$m^{res}_{\mu,T}(b) = m_{\mu,T}(b) - m^{circ}_{\mu,T}(b) = I(\mu, T, b) - I_{mag}(\mu, T, b) = I_{tr}(\mu, T, b)$$

Bulk-edge correspondence at $T \geq 0$ [Cornean, M.M., Teufel]

$$\sigma_H(\mu, T, b) = \frac{n_{\mu,T,b}}{b} = \partial_\mu m^{res}_{\mu,T}(b) = \partial_\mu I_{tr}(\mu, T, b) = \sigma_E(\mu, T, b)$$

Separated Bloch bands

In general situation the splitting of the magnetization is not clear at all.

Theorem [Teufel, Schulz-Baldes 12 ; Teufel,Stiepan 13]

In tight-binding model, with M isolated Bloch bands we have:

$$m_{\mu,T}(b) = \sum_{l=1}^{M} \int_{\mathbb{B}_b} \frac{dk}{(2\pi)^d} \left(F'_{\mu,T}\left(E_l(k)\right) R^{(l)}_{j+1,j+2}(k) + F_{\mu,T}(E_l(k))\Omega^{(l)}_{j+1,j+2}(k) \right)$$
$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

Conjecture

$$\sigma_H(\mu, T, b) = \partial_\mu m^{res}_{\mu,T}(b)$$

$\Rightarrow$ the conjecture coupled with our formula $(\star)$ would imply bulk-edge correspondence at $T \geq 0$.

Separated Bloch bands

In general situation the splitting of the magnetization is not clear at all.

Theorem [Teufel, Schulz-Baldes 12 ; Teufel,Stiepan 13]

In tight-binding model, with M isolated Bloch bands we have:

$$m_{\mu,T}(b) = \sum_{l=1}^{M} \int_{\mathbb{B}_b} \frac{dk}{(2\pi)^d} \left(F'_{\mu,T}\left(E_l(k)\right) R^{(l)}_{j+1,j+2}(k) + F_{\mu,T}(E_l(k))\Omega^{(l)}_{j+1,j+2}(k) \right)$$
$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

where $R^{(l)}_{i,j}$ is the Rammal-Wilkinson tensor and $\Omega^{(l)}_{i,j}(k)$ is the Berry curvature:

$$R^{(l)}_{i,j}(k) = \frac{\imath}{2}\left(\mathrm{Tr}\left(P_l(k)\partial_i P_l(k)\left(H(k) - E_l(k)\right)\partial_j P_l(k)\right) - (i \leftrightarrow j)\right)$$
$$\Omega^{(l)}_{i,j}(k) = \imath\,\mathrm{Tr}\left(P_l(k)\left[\partial_i P_l(k), \partial_j P_l(k)\right]\right)$$

Conjecture

$$\sigma_H(\mu, T, b) = \partial_\mu m^{res}_{\mu,T}(b)$$

Separated Bloch bands

In general situation the splitting of the magnetization is not clear at all.

Theorem [Teufel, Schulz-Baldes 12 ; Teufel,Stiepan 13]

In tight-binding model, with M isolated Bloch bands we have:

$$m_{\mu,T}(b) = \sum_{l=1}^{M} \int_{\mathbb{B}_b} \frac{dk}{(2\pi)^d} \left(F'_{\mu,T}\left(E_l(k)\right) R^{(l)}_{j+1,j+2}(k) + F_{\mu,T}(E_l(k))\Omega^{(l)}_{j+1,j+2}(k) \right)$$
$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

Conjecture

$$\sigma_H(\mu, T, b) = \partial_\mu m^{res}_{\mu,T}(b)$$

$\Rightarrow$ the conjecture coupled with our formula $(\star)$ would imply bulk-edge correspondence at $T \geq 0$.

Separated Bloch bands

In general situation the splitting of the magnetization is not clear at all.

Theorem [Teufel, Schulz-Baldes 12 ; Teufel,Stiepan 13]

In tight-binding model, with M isolated Bloch bands we have:

$$m_{\mu,T}(b) = \sum_{l=1}^{M} \int_{\mathbb{B}_b} \frac{dk}{(2\pi)^d} \left(F'_{\mu,T}\left(E_l(k)\right) R^{(l)}_{j+1,j+2}(k) + F_{\mu,T}(E_l(k)) \Omega^{(l)}_{j+1,j+2}(k) \right)$$

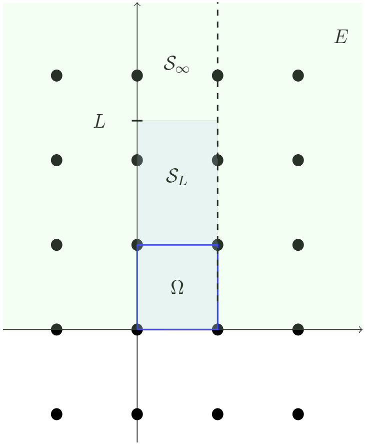
$$=: m^{circ}_{\mu,T}(b) + m^{res}_{\mu,T}(b)$$

Conjecture

$$\sigma_H(\mu, T, b) = \partial_\mu m^{res}_{\mu,T}(b)$$

$\Rightarrow$ the conjecture coupled with our formula ($\star$) would imply bulk-edge correspondence at $T \geq 0$.

Mathematical framework - The bulk-edge model



$$\Omega := [0,1]^2$$

is the unit cell of the bulk Hamiltonian and χ_Ω is characteristic function of Ω.

$$\mathcal{S}_L := [0,1] \times [0,L]$$

χ_L characteristic function of $\mathcal{S}_L$.

$$\mathcal{S}_\infty := [0,1] \times [0,\infty]$$

χ_∞ characteristic function of $\mathcal{S}_\infty$.

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current", that is

$$\mathbb{E}\left(\mathrm{Tr}\left(\chi_\Omega \mathrm{i}[H_{\bullet,b}, X_i]F'(H_{\bullet,b})\right)\right) = 0, \qquad i \in \{1,2\}.$$

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L}\,\mathrm{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

→ Main tool: geometric perturbation theory:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{\mathrm{x} \in E \,|\, \mathrm{dist}(\mathrm{x}, \partial E) \leq t\sqrt{L}\right\}, t > 0.$

$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathrm{x}) + \eta_L(\mathrm{x}) = 1$ for every $\mathrm{x} \in E$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1} \eta_L + \widetilde{\eta}_0 \left(H^E_{\omega,b} - z\right)^{-1} \eta_0.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1} \eta_L$$

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current", that is

$$\mathbb{E}\left(\mathrm{Tr}\left(\chi_\Omega \mathrm{i}[H_{\bullet,b}, X_i] F'(H_{\bullet,b})\right)\right) = 0, \qquad i \in \{1,2\}.$$

$\to$ Main difficulty/novelty: F does not have compact support (F is a Scwhartz function and the spectrum is only unbounded from below)!

$\to$ Main tool: Helffer-Sjöstrand formula

$$F(H_\omega^{E/}) = -\frac{1}{\pi}\int_{\mathbb{R}\times[-1,1]} \mathrm{d}z_1 \mathrm{d}z_2\, \bar{\partial} F_N(z)(H_\omega^{E/} - z)^{-1}, \quad z = z_1 + \mathrm{i}z_2,$$

F_N is an almost analytic extension of F, that is: Let $0 \leq g(y) \leq 1$ with $g \in C_0^\infty(\mathbb{R})$ such that $g(y) = 1$ if $|y| \leq 1/2$ and $g(y) = 0$ if $|y| > 1$. Fix some $N \geq 2$ and define

$$F_N(z_1 + \mathrm{i}z_2) := g(z_2)\sum_{j=0}^N \frac{1}{j!}\frac{\partial^j F}{\partial z_1{}^j}(z_1)(\mathrm{i}z_2)^j.$$

+ regularity and decay estimates on $\left(H_{\omega,b}^{/E} - z\right)^{-1}$.

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current"

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L}\,\mathrm{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

→ Main tool: <u>geometric perturbation theory</u>:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{ \mathbf{x} \in E \,|\, \mathrm{dist}(\mathbf{x}, \partial E) \leq t\sqrt{L} \right\}$, $t > 0$.

$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathbf{x}) + \eta_L(\mathbf{x}) = 1$ for every $\mathbf{x} \in E$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1} \eta_L + \widetilde{\eta}_0 \left(H^E_{\omega,b} - z\right)^{-1} \eta_0.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1}\eta_L$$

$$\left(-2\mathrm{i}\nabla\widetilde{\eta}_0 \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_0)\right)\left(H^E_{\omega,b} - z\right)^{-1}\eta_0.$$

The resolvent of the edge Hamiltonian obeys the identity:

Sketch of the proof

Step 0.: Trace class properties, regularities of integral kernels and vanishing of equilibrium "current"

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L}\,\mathrm{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

$\to$ Main tool: geometric perturbation theory:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{\mathrm{x} \in E \,|\, \mathrm{dist}(\mathrm{x}, \partial E) \leq t\sqrt{L}\right\}$, $t > 0$.

$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathrm{x}) + \eta_L(\mathrm{x}) = 1$ for every $\mathrm{x} \in E$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1} \eta_L + \widetilde{\eta}_0 \left(H^E_{\omega,b} - z\right)^{-1} \eta_0.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1} \eta_L$$

$$\left(-2\mathrm{i}\nabla\widetilde{\eta}_0 \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_0)\right)\left(H^E_{\omega,b} - z\right)^{-1} \eta_0.$$

The resolvent of the edge Hamiltonian obeys the identity:

Sketch of the proof

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L} \operatorname{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

$\to$ Main tool: geometric perturbation theory:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{\mathbf{x} \in E \,|\, \operatorname{dist}(\mathbf{x}, \partial E) \leq t\sqrt{L}\right\}$, $t > 0$.

$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathbf{x}) + \eta_L(\mathbf{x}) = 1$ for every $\mathbf{x} \in E$, and satisfying

$$\begin{aligned} &\operatorname{supp}(\eta_0) \subset \Xi_L(2), \\ &\operatorname{supp}(\eta_L) \subset E \setminus \Xi_L(1), \\ &\|\partial_2^n \eta_i\|_\infty \simeq L^{-\frac{n}{2}}, \quad n \geq 1, \quad i \in \{0, L\}\,. \end{aligned}$$

Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1} \eta_L + \widetilde{\eta}_0 \left(H^E_{\omega,b} - z\right)^{-1} \eta_0\,.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L \cdot (-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1} \eta_L$$

Sketch of the proof

Step 1. Edge pressure coincides in the limit with the bulk pressure :

$$\lim_{L\to\infty} P^{(L,\omega)}_{\mu,T}(b) = \lim_{L\to\infty} \frac{1}{L}\,\mathrm{Tr}\left(\chi_L F(H^E_{\omega,b})\right) = p_{\mu,T}(b)\,.$$

$\to$ Main tool: geometric perturbation theory:

$L \geq 1$; Strip near the edge: $\Xi_L(t) := \left\{\mathbf{x}\in E \,|\, \mathrm{dist}(\mathbf{x},\partial E) \leq t\sqrt{L}\right\}$, $t>0$.
$0 \leq \eta_0, \eta_L \leq 1$, smooth functions and only depending on x_2 such that $\eta_0(\mathbf{x}) + \eta_L(\mathbf{x}) = 1$ for every $\mathbf{x}\in E$
Take $0 \leq \widetilde{\eta}_0, \widetilde{\eta}_L \leq 1$ again only depending on x_2, such that they are a sort of stretched version of η_0 and η_L (no partition of unity in this case).

$$U_{L,\omega}(z) := \widetilde{\eta}_L \left(H_{\omega,b} - z\right)^{-1}\eta_L + \widetilde{\eta}_0\left(H^E_{\omega,b} - z\right)^{-1}\eta_0.$$

$$W_{L,\omega}(z) := \left(-2\mathrm{i}\nabla\widetilde{\eta}_L\cdot(-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_L)\right)\left(H_{\omega,b} - z\right)^{-1}\eta_L$$
$$\left(-2\mathrm{i}\nabla\widetilde{\eta}_0\cdot(-\mathrm{i}\nabla - \mathcal{A} - bA) - (\Delta\widetilde{\eta}_0)\right)\left(H^E_{\omega,b} - z\right)^{-1}\eta_0.$$

The resolvent of the edge Hamiltonian obeys the identity:

$$\left(H^E_{\omega,b} - z\right)^{-1} = U_{L,\omega}(z) - \left(H^E_{\omega,b} - z\right)^{-1} W_{L,\omega}(z). \tag{1}$$

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \qquad \text{for a.e. } \omega\in\Theta.$$

$\to$ Magnetic derivative and thermodynamic limit commute!

Step 3. The limit of the magnetic derivative of the edge pressure is an edge current :

$$\lim_{L\to\infty} -\mathbb{E}\left(\int_0^1 \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, g\left(\frac{x_2}{L}\right) \{\mathrm{i}[H^E_{\bullet,b}, X_1]F'(H^E_{\bullet,b})\}(x_1,x_2;x_1,x_2)\right)$$

$\to$ Exploit Step 2.

$\to$ Previous trace class estimates + vanishing of the equilibrium current allows to prove that the limit is independent from g.

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \qquad \text{for a.e. } \omega \in \Theta.$$

$\to$ Magnetic derivative and thermodynamic limit commute!

$\to$ **New mathematical tool**: We extended gauge covariant magnetic perturbation theory (Cornean-Nenciu '00) to operators defined on domains with boundary.

Modified asymmetric phase $\varphi(\mathbf{x},\mathbf{y}) := (y_1 - x_1)y_2$

$$(P_{x_1} - \epsilon A_1(\mathbf{x}))\mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})} = \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}(P_{x_1} - \epsilon A_1(\mathbf{x}-\mathbf{y})).$$

$\to S^E_{\omega,\epsilon,z}(\mathbf{x};\mathbf{y}) := \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}\left(H^E_{\omega,b} - z\right)^{-1}(\mathbf{x};\mathbf{y})$ is an "almost inverse" for the magnetically perturbed edge operator $\left(H^E_{\omega,b+\epsilon} - z\right)$:

$$\left(H^E_{\omega,b+\epsilon} - z\right) S^E_{\omega,\epsilon,z} = 1 + T^E_{\omega,\epsilon,z}$$

$\Rightarrow$ Explicit resolvent expansion in ϵ as $\epsilon \to 0$.

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \qquad \text{for a.e. } \omega \in \Theta.$$

$\to$ Magnetic derivative and thermodynamic limit commute!

$\to$ **New mathematical tool**: We extended gauge covariant magnetic perturbation theory (Cornean-Nenciu '00) to operators defined on domains with boundary.

Modified asymmetric phase $\varphi(\mathbf{x},\mathbf{y}) := (y_1 - x_1)y_2$

$$\big(P_{x_1} - \epsilon A_1(\mathbf{x})\big)\mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})} = \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}\big(P_{x_1} - \epsilon A_1(\mathbf{x}-\mathbf{y})\big).$$

$\to S^E_{\omega,\epsilon,z}(\mathbf{x};\mathbf{y}) := \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}\left(H^E_{\omega,b} - z\right)^{-1}(\mathbf{x};\mathbf{y})$ is an "almost inverse" for the magnetically perturbed edge operator $\left(H^E_{\omega,b+\epsilon} - z\right)$:

$$\left(H^E_{\omega,b+\epsilon} - z\right) S^E_{\omega,\epsilon,z} = 1 + T^E_{\omega,\epsilon,z}$$

$\Rightarrow$ Explicit resolvent expansion in ϵ as $\epsilon \to 0$.

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \qquad \text{for a.e. } \omega \in \Theta.$$

$\to$ Magnetic derivative and thermodynamic limit commute!

$\to$ **New mathematical tool**: We extended gauge covariant magnetic perturbation theory (Cornean-Nenciu '00) to operators defined on domains with boundary.

Modified asymmetric phase $\varphi(\mathbf{x},\mathbf{y}) := (y_1 - x_1)y_2$

$$\big(P_{x_1} - \epsilon A_1(\mathbf{x})\big)\mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})} = \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}\big(P_{x_1} - \epsilon A_1(\mathbf{x}-\mathbf{y})\big).$$

$\to S^E_{\omega,\epsilon,z}(\mathbf{x};\mathbf{y}) := \mathrm{e}^{\mathrm{i}\epsilon\varphi(\mathbf{x},\mathbf{y})}\left(H^E_{\omega,b} - z\right)^{-1}(\mathbf{x};\mathbf{y})$ is an "almost inverse" for the magnetically perturbed edge operator $\left(H^E_{\omega,b+\epsilon} - z\right)$:

$$\left(H^E_{\omega,b+\epsilon} - z\right) S^E_{\omega,\epsilon,z} = 1 + T^E_{\omega,\epsilon,z}$$

$\Rightarrow$ Explicit resolvent expansion in ϵ as $\epsilon \to 0$.

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P^{(L,\omega)}_{\mu,T}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \quad \text{for a.e. } \omega \in \Theta.$$

$\rightarrow$ Magnetic derivative and thermodynamic limit commute!

Step 3.The limit of the magnetic derivative of the edge pressure is an edge current :

$$\lim_{L\to\infty} -\mathbb{E}\left(\int_0^1 \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, g\left(\frac{x_2}{L}\right) \left\{\mathrm{i}[H^E_{\bullet,b}, X_1]F'(H^E_{\bullet,b})\right\}(x_1,x_2;x_1,x_2)\right)$$

$\rightarrow$ Exploit Step 2.

$\rightarrow$ Previous trace class estimates + vanishing of the equilibrium current allows to prove that the limit is independent from g.

Sketch of the proof

Step 2. The magnetic derivative of the edge pressure has a thermodynamic limit:

$$\lim_{L\to\infty} \frac{\mathrm{d}P_{\mu,T}^{(L,\omega)}(b)}{\mathrm{d}b}(b) = \lim_{L\to\infty} \mathbb{E}\left(\frac{\mathrm{d}P_{\mu,T}^{(L,\omega)}(b)}{\mathrm{d}b}(b)\right) = \frac{\mathrm{d}p_{\mu,T}(b)}{\mathrm{d}b} \quad \text{for a.e. } \omega\in\Theta.$$

→ Magnetic derivative and thermodynamic limit commute!

Step 3.The limit of the magnetic derivative of the edge pressure is an edge current :

$$\lim_{L\to\infty} -\mathbb{E}\left(\int_0^1 \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, g\left(\frac{x_2}{L}\right)\ \{\mathrm{i}[H^E_{\bullet,b},X_1]F'(H^E_{\bullet,b})\}(x_1,x_2;x_1,x_2)\right)$$

→ Exploit Step 2.

→ Previous trace class estimates + vanishing of the equilibrium current allows to prove that the limit is independent from g.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Recap:

- Bulk-edge correspondence in the form $m = I$ at every temperature.
- Usual bulk-edge correspondence for μ in a gap and limit $\searrow T = 0$.
- Bulk-edge correspondence for the Landau Hamiltonian at $T \geq 0$.

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case (see Elgart-Graf-Schenker 12)? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Summary & open questions

Open questions:

- General splitting of the magnetization and bulk-edge correspondence.
- What about higher order derivatives? Is there a bulk-edge correspondence for the bulk magnetic susceptibility ?
- Limit to zero temperature in the mobility gap case? Limit to zero temperature in the metallic case?
- Extension to Dirac operators and more general elliptic pseudo-differential operators.
- Interacting case? Work in progress with J. Lampart, S. Teufel, T. Wessel.

Thank you for your attention!

Cornean H.D., M. M., Teufel, S.: General bulk-edge correspondence at positive temperature. ArXiv: 2107.13456 (2021).

M.M, B. B. Støttrup: Regularity properties of bulk and edge current densities at positive temperature . ArXiv: 2201.08803 (2022).