De la beauté des mathématiques aux collisions des particules élémentaires

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DESY.

CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Séminaire Math-Physique, Institut de Mathématiques de Bourgogne
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PRL 126, 091603 (2021) avec D. Chicherin, J. M. Henn
PRL 124, 161603 (2020) avec B. Basso et L. J. Dixon
JHEP 08 (2020) 005 & JHEP 10 (2021) 007 avec N. Henke
JHEP 03 (2015) 072 avec J. M. Drummond et M. Spradlin

Introduction: Sonder le microcosme des particules élémentaires

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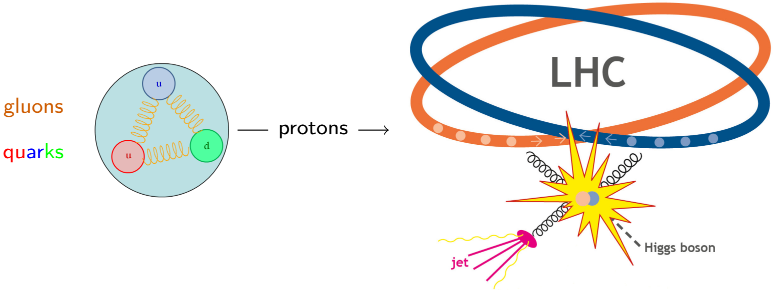

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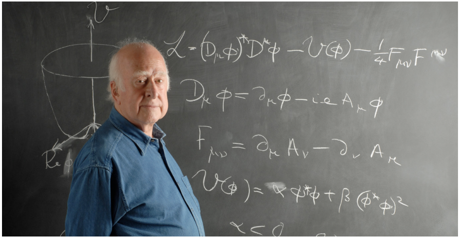
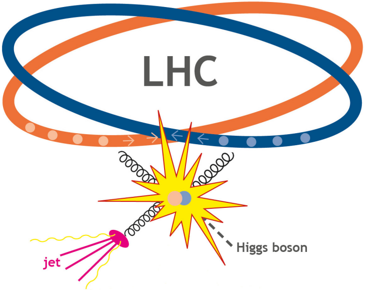
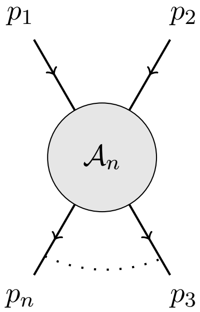




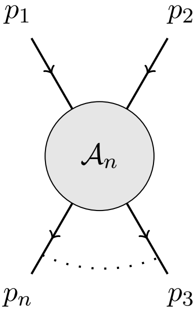
- Percée majeure : La découverte du boson de Higgs, donnant de la masse à d'autres particules élémentaires (2012).

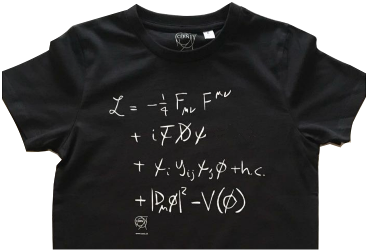
Amplitudes de diffusion $\mathcal{A}_n$:



Quand la théorie confronte l'expérience

Amplitudes de diffusion $\mathcal{A}_n$ en théorie quantique des champs (TQC):





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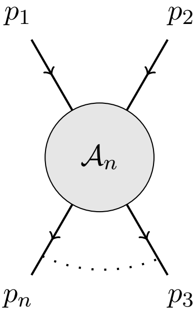

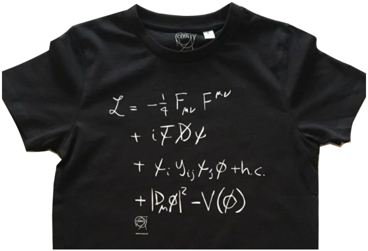

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$$|\mathcal{A}_n|^2 = \text{Probabilité de résultat de collision}$$

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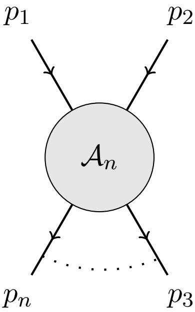

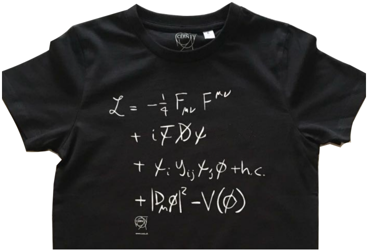

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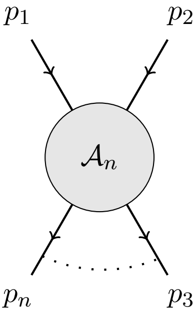

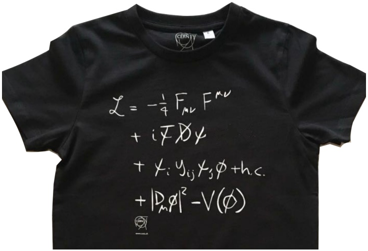

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- Structure mathématique remarquable, fertiles interactions avec nombreux branches des maths. [Brown;Goncharov;Postnikov;Sturmfels;Williams,...]

Diagrammes de Feynman

Pour une intensité d'interaction ou *couplage* g faible,
$g \ll 1$, calculer de manière *perturbative*,

$$\mathcal{A}_n = g^0 \mathcal{A}_n^{(0)} + g\mathcal{A}_n^{(1)} + g^2 \mathcal{A}_n^{(2)} + \dots$$

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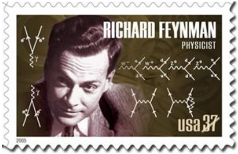

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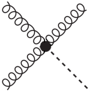
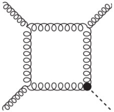
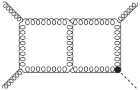
$\mathcal{A}_4^{(0)} =$  $+ \ldots ,$ $\mathcal{A}_4^{(1)} =$  $+ \ldots ,$ $\mathcal{A}_4^{(2)} =$  $+ \ldots$

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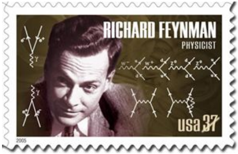

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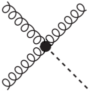
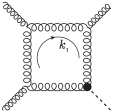
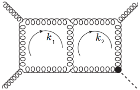
$\mathcal{A}_4^{(0)} =$  $+ \dots,$ $\mathcal{A}_4^{(1)} =$  $+ \dots,$ $\mathcal{A}_4^{(2)} =$  $+ \dots$

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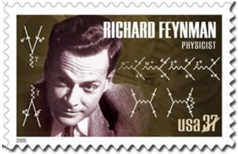
Diagrammes de Feynman & et leurs limitations



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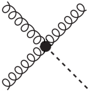
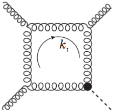
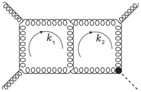
$\mathcal{A}_4^{(0)} =$  $+ \dots ,\quad \mathcal{A}_4^{(1)} =$ 

$+ \dots ,\quad \mathcal{A}_4^{(2)} =$ 

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⛔ Difficile à intégrer : Génériquement divergent, classe de fonctions inconnue au-delà d'une boucle.

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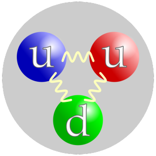
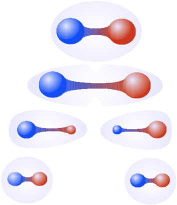


Confinement:

Les quarks et les gluons ne peuvent pas être observés individuellement



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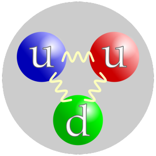

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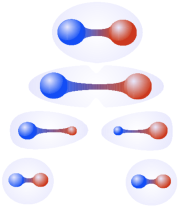

Pas prouvé analytiquement en ChromoDynamique Quantique (QCD), la théorie les décrivant.

Millenium Problem

CLAY MATHEMATICS INSTITUTE

Surmonter les limitations: La stratégie

Démarrant du cousin le plus symétrique de la QCD, nous nous appuierons sur des outils issus de domaines actifs de la recherche en mathématiques.

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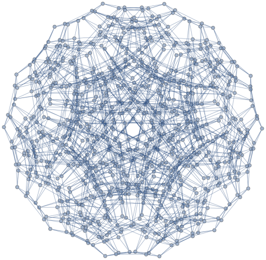

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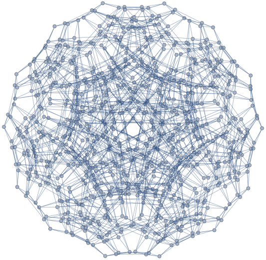
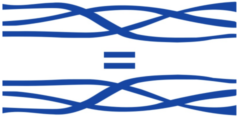
- *exactement* quel que soit le couplage → intégrabilité → compréhension conceptuelle





Des modèles simples pour des problèmes difficiles

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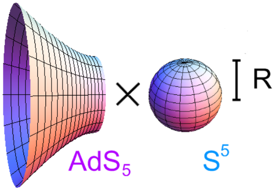
- YMSM $\Leftrightarrow$ IIB supercordes sur fond géometrique $AdS_5 \times S^5$
  fortement $\Leftrightarrow$ faiblement couplé
  [Maldacena'97]



- Source d'inspiration pour l'étude de pléthore de phénomènes physiques

Importance pratique de la théorie YMSM

Laboratoire théorique avéré pour développer de nouveaux paradigmes menant à des applications pratiques importantes. Par exemple:

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- Calcul conventionnel de $\mathcal{A}_6^{(2)}$: 16 pages de fonctions compliquées
  [Duhr,Del Duca,Smirnov'09]

$$
\begin{aligned}
&R_{6,WL}^{(2)}(u_1,u_2,u_3) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
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&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+\\
&G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\\
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\end{aligned}
$$

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\end{aligned}
$$

Importance pratique de la théorie YMSM

Laboratoire théorique avéré pour développer de nouveaux paradigmes menant à des applications pratiques importantes. Par exemple:

- Calcul conventionnel de $\mathcal{A}_6^{(2)}$: 16 pages de fonctions compliquées
  [Duhr,Del Duca,Smirnov'09]

$$\mathcal{A}_6^{(2)} = \mathrm{Li}_4\left(-\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}\right) - \mathrm{Li}_4\left(-\frac{\langle 14\rangle\langle 56\rangle}{\langle 45\rangle\langle 16\rangle}\right) - \frac{1}{2}\,\mathrm{Li}_4\left(-\frac{\langle 35\rangle\langle 26\rangle}{\langle 23\rangle\langle 56\rangle}\right) + \text{cyclique} + \text{produits}$$

- Réduit à deux lignes avec la **méthode des symboles**
  [Goncharov,Spradlin,Vergu,Volovich'10]

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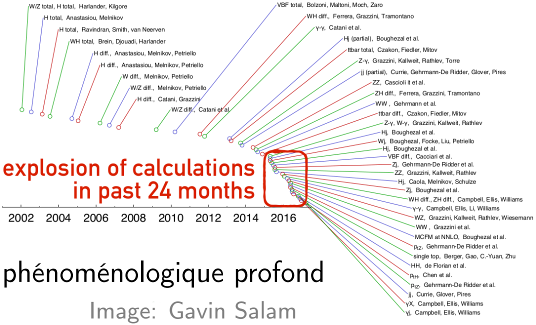

Impact phénoménologique profond

Image: Gavin Salam

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Plan

Algèbres amassées (cluster algebras) [Fomin,Zelevinsky'02]

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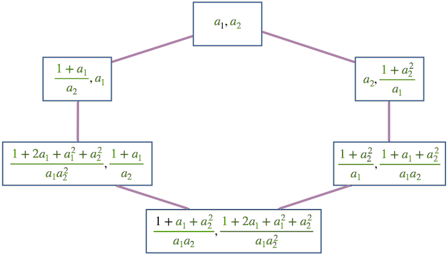

Graphe d'échange: Amas=sommets, mutations=arêtes

Algèbres amassées et amplitudes de diffusion YMSM

L'espace fonctionnel des *polylogarithmes multiples* (PLM)

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Collection de ϕ_α de chaque étape récursive: L'**alphabet (de symbole)**

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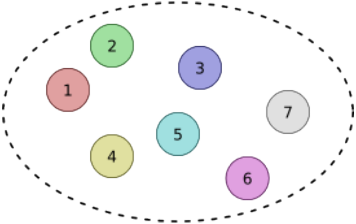

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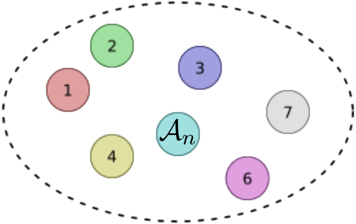

- *Nombre fini* de PLM avec alphabet et poids donnés
- Identifier $\mathcal{A}_n$ parmi eux!

Application: Le «Steinmann Cluster Bootstrap»pour des amplitudes YMSM

Éviter les diagrammes de Feynman en exploitant la structure analytique

Propriété TQC	Calcul

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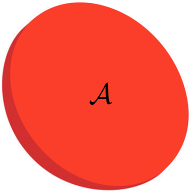

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Coupures physiques [Gaiotto,Maldacena, Sever,Vieira]	$\mathcal{A}_6^{(L)}, L = 3, 4$ [Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]

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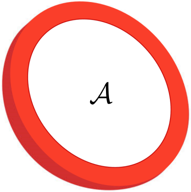

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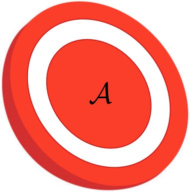

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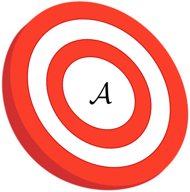

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«Cluster Adjacency» [Drummond,Foster, Gurdogan]	$\mathcal{A}^{(4)}_{7,\text{NMHV}}$ [Drummond,Foster, Gurdogan, **GP**]
«Extended Steinmann» ⇔ «Coaction Principle»	$\mathcal{A}_6^{(6)}, \mathcal{A}^{(7)}_{6,\text{MHV}}$ [Caron-Huot,Dixon,Dulat, McLeod,Hippel,**GP**]

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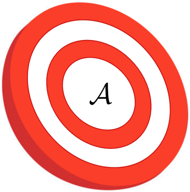

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Proprieté TQC	Calcul
Coupures physiques [Gaiotto,Maldacena, Sever,Vieira]	$\mathcal{A}_6^{(L)}, L = 3, 4$ [Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]
Algèbres amassées [Golden,Goncharov, Spradlin,Vergu,Volovich]	$\mathcal{A}^{(3)}_{7,\text{MHV}}$ [Drummond,**GP**, Spradlin]
Relation de Steinmann [Steinmann]	$\mathcal{A}_6^{(5)}, \mathcal{A}^{(3)}_{7,\text{NMHV}}, \mathcal{A}^{(4)}_{7,\text{MHV}}$ [Caron-Huot,Dixon,. . .] [Dixon,. . . , **GP**,Spradlin]
«Cluster Adjacency» [Drummond,Foster, Gurdogan]	$\mathcal{A}^{(4)}_{7,\text{NMHV}}$ [Drummond,Foster, Gurdogan, **GP**]
«Extended Steinmann» $\Leftrightarrow$ «Coaction Principle»	$\mathcal{A}_6^{(6)}, \mathcal{A}^{(7)}_{6,\text{MHV}}$ [Caron-Huot,Dixon,Dulat, McLeod,Hippel,**GP**]

[Revue:Caron-Huot,Dixon,Drummond,Dulat,Foster,Gurdogan,Hippel,McLeod,**GP**;CORFU'19]

Algèbres amassées pour le **monde physique**

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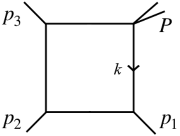

$$= \int \frac{d^D k}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-P)^2}$$

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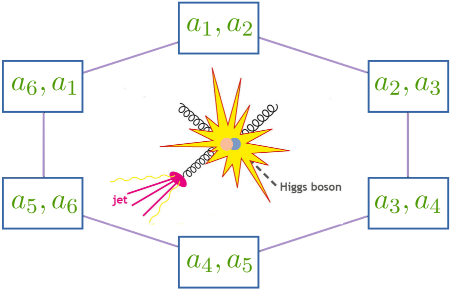

Découvert que les algèbres amassées *décrivent les singularités* de nombreux exemples pertinents, comme la production de Higgs+jet en QCD!

[Chicherin,Henn,**GP**;PRL 126, 091603]

Informations Complémentaires:

https://www.qu.uni-hamburg.de/activities/news/21-03-18-cluster-algebras.html

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

QUANTUM UNIVERSE

Cluster algebras promise simplifications in collider physics calculations

Modern mathematical tools may constrain the complexity of Feynman integrals

18 March 2021

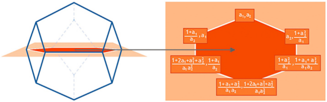

Photo: Max-Planck-Institut für Physik/Carolin Leyck

Theoretical particle physicists compare Feynman integrals to wild beasts that are difficult to tame. In a recent scientific publication, the authors provide evidence that cluster algebras, a new and vibrant area of mathematics, could serve as a principle to control the analytic form of Feynman integrals. This has the potential to simplify theoretical calculations for elementary particle experiments.

To test and to extend the current understanding of Nature, precise theoretical predictions of elementary particle collisions are needed. When the particles interact weakly during the collision, these predictions can be computed by so-called Feynman diagrams. The latter are a method to visualize how the colliding particles will transform to the ones observed after the collision by producing, annihilating and exchanging other unobserved, 'virtual' particles. As the energy and momentum of these virtual particles may be anything that the laws of Nature allow, one additionally has to integrate over integrals a challenge for scientists is that it is in general not known what they evaluate to as functions of the momenta and quantum numbers of the observed particles before and after the collision.

"More efficient predictions of particle collisions in idealized models of quantum theories have been achieved thanks to the beautiful mathematics of cluster algebras, which for that reason have been at the heart of my research for several years. Our current findings open for the first time the exciting prospect of applications of cluster algebras to the real world", says theoretical physicist Georgios Papathanasiou (DESY), key researcher at the Cluster of Excellence Quantum Universe. Together with his colleagues, he discovered that cluster algebras underlie the analytic structure of Feynman integrals contributing to realistic multi-scale processes, such as Higgs production in association with a jet. Cluster algebras are mathematical objects consisting of a set of variables, arranged in overlapping sets known as clusters. In the simplest setting, each cluster can be geometrically understood as a different triangulation of a polygon, with its cluster variables corresponding to the diagonals responsible for this triangulation.

In the setting of Feynman integrals, cluster variables were found to govern their singularities. This result, which generalizes previous observations about the relevance of cluster algebras in a simplified toy model, may facilitate obtaining results at unprecedented precision: The singularity information they provide has in several cases been essential in determining not only the Feynman integrals, but also the physical quantities they contribute to, via novel bootstrap methods that evade the formidable task of direct integration.

More about cluster algebras

Surprising cluster algebraic structures found in Higgs amplitudes

CONTACT

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Key Researcher
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22607 Hamburg
Tel: +49 40 8998-2412
Email: georgios.papathanasiou@desy.de

ORIGINAL PUBLICATION

Dmitry Chicherin, Johannes M. Henn, Georgios Papathanasiou: Cluster Algebras for Feynman Integrals
Phys. Rev. Lett. 126, 091603

For this publication, Georgios Papathanasiou was awarded a Best Paper Award by the Cluster of Excellence in September 2021.

Au-delà des algèbres amassées

Elles ne sont pas le fin mot de l'histoire même en théorie de YMSM: Pour amplitudes à $n \geq 8$ pattes, l'algèbre amassée associée devient infinie!

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Récemment, une proposition pour rémedier avec cette infinité a été avancé, en accord avec toutes données connues [Henke, Papathanasiou'19+'21][Arkani-Hamed,Lam,Spradlin'19][Drummond,Foster et. al.'19]

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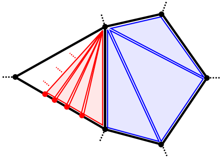
basée sur la relation entre des algèbres amassées et des Grassmanniennes tropicales. [Sturmfels,Speyer] [Speyer,Williams]

[Conférence: "Cluster Algebras & Geometry of Scatt.Amplitudes" Workshop, Edinburgh U.]



Outline

Au-delà du régime perturbatif: Intégrabilité

La propriété d'un système physique d'avoir autant de quantités conservées que de degrés de liberté.

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La propriété d'un système physique d'avoir autant de quantités conservées que de degrés de liberté. En 2 dimensions

[Zamolodchikov^2'79]

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$$M = 2 \qquad P = p_1 + p_2$$
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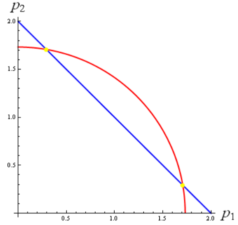

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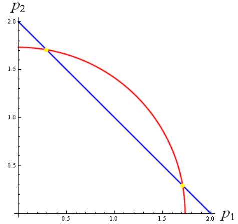

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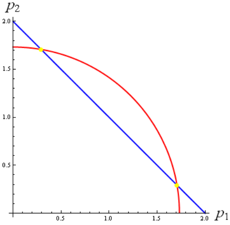

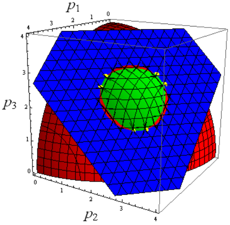

Intégrabilité en théorie des jauges et des cordes

Intégrabilité en théorie des jauges et des cordes

- Émerges en systèmes physiques qui décrivent

Dimensions d'échelle des operateurs Δ	$\Leftrightarrow$	Énergies des états de cordes E
YMSM		Théorie des cordes

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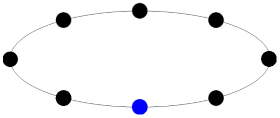
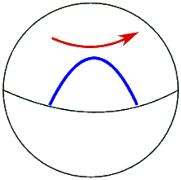


chaînes de spins

$\Leftrightarrow$



solitons

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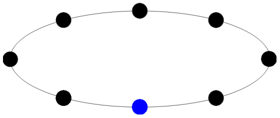

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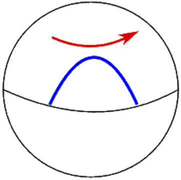

chaînes de spins — solitons

Permet de déterminer $\Delta = E$ comme fonction exacte du couplage g!

[Minahan,Zarembo'02] [Beisert,Eden,Staudacher'06] [Gromov,Kazakov,Leurent,Volin'13]

[Arutyunov,. . . ,GP,. . . ,Zarembo]

Intégrabilité en théorie des jauges et des cordes

Un exemple célèbre

La dimension anormale «cusp » [Beisert,Eden,Staudacher]

$$\Gamma_{\rm cusp} = 4g^2\left[\frac{1}{1+\mathbb{K}}\right]_{11} = 4g^2\left[1-\mathbb{K}+\mathbb{K}^2+\dots\right]_{11} \quad \leftarrow \text{élément matriciel}$$

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}\,, \quad i,j=1,2,\dots \quad J_i(x): \text{f}^{\text{n}} \text{ de Bessel}$$

qui contrôle des operateurs $SL(2)$ de spin $S \gg 1$, [Korchemsky]

$$\Delta - S = 2\Gamma_{\rm cusp}\log S + \mathcal{O}(\log^0 S)\,.$$

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Espoir de progrès similaire pour des amplitudes de diffusion?

Intégrabilité pour des amplitudes de diffusion

Initialement, établi principalement dans la *limite colinéaire*.
[Alday,Gaiotto,Maldacena,Sever,Vieira'10] [Basso,Sever,Vieira'13] [**GP**'13+14]

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Néanmoins, des simplifications très intéressantes apparaissent à de nouveaux régimes, p.e.x. la limite d'«origine» $u_1, u_2, u_3 \to 0$
[Caron-Huot,Dixon,Dulat,McLeod,Hippel,**GP**'19]

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Explication? S'applique à $\Gamma_{\pi/3}, C_0$ aussi?

Limite d'origine de l'amplitude à six pattes à *couplage fini*

Grâce à la resommation du limite colinéaire, arriver à la limite d'origine $u_1, u_2, u_3 \to 0$, où

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avec

[Basso,Dixon,**GP**'20; PRL 124, 161603]

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$$\mathbb{K}(\alpha) = 2\cos\alpha \begin{bmatrix} \cos\alpha\, \mathbb{K}_{\circ\circ} & \sin\alpha\, \mathbb{K}_{\circ\bullet} \\ \sin\alpha\, \mathbb{K}_{\bullet\circ} & \cos\alpha\, \mathbb{K}_{\bullet\bullet} \end{bmatrix}, \qquad \begin{array}{l} \mathbb{K}_{\circ\circ} = \mathbb{K}_{2n+1,2m+1} \,, \\ \mathbb{K}_{\circ\bullet} = \mathbb{K}_{2n+1,2m} \text{ etc,} \end{array}$$

$$C_0 = -\frac{\zeta_2}{2}\Gamma_{\pi/4} + D_{\pi/4} - D_{\pi/3} - \frac{1}{2} D_0 \,, \qquad D_\alpha \equiv \ln\det\left[1 + \mathbb{K}(\alpha)\right].$$

$\mathbb{K}_{ij}$: Noyau Beisert-Eden-Staudacher kernel, déformé par un paramètre «tilt» α!

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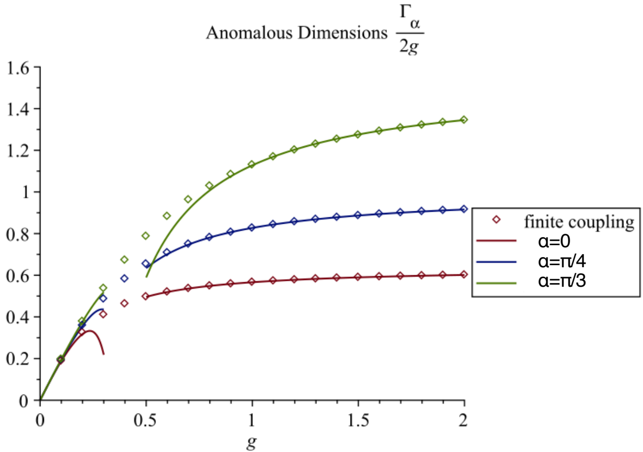
$\mathbb{K}_{ij}$: Noyau Beisert-Eden-Staudacher kernel, déformé par un paramètre «tilt» α! Joue un rôle aussi aux généralisations du limite d'origin à $n = 7, 8$

[Basso,Dixon,Liu, **GP**, en cours]

Limite d'origine de l'amplitude à six pattes à *couplage fini*

[Basso,Dixon,**GP**'20; PRL 124, 161603]



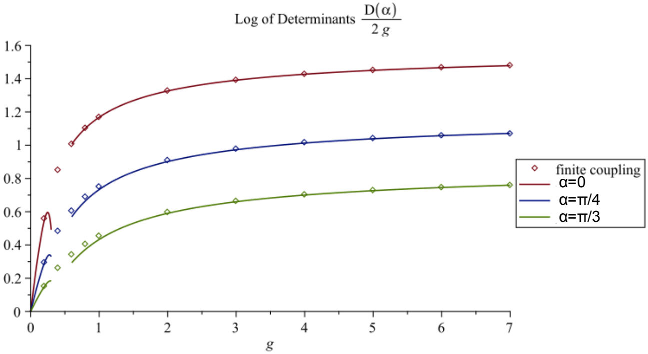
[Conférence: "Integrability in Gauge & String Theory 2020"]

Comparaison: Évaluation numérique / développements analytiques $g^{\pm 1} \ll 1$

Limite d'origine de l'amplitude à six pattes à *couplage fini*

[Basso,Dixon,**GP**'20; PRL 124, 161603]

$$C_0 = -\frac{\zeta_2}{2}\Gamma_{\pi/4} + D(\pi/4) - D(\pi/3) - \frac{1}{2}D(0)\,.$$



De même, $\mathcal{A}_n$ peut être déterminée à tout ordre dans la limite multi-Regge.
[Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,**GP**,Verbeek'20; PRL 124, 161602]

Conclusions

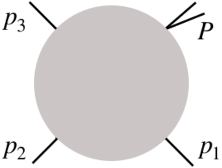
1. Amplitudes de diffusion: L'arène où la physique théorique des particules rencontre l'expérience.
2. De belles mathématiques d'algèbres amassées contrôlent la structure analytique de plusieurs intégrales de Feynman et des processus réels!
3. Puissante propriété d'intégrabilité → description exacte de la diffusion dans certaines limites du cousin le plus simple de la QCD.

Perspective d'avenir

- Application des algèbres amassées pour simplifier des futurs calculs nécessaires au «High Luminocity LHC»2027-2037.
- Diffusion exacte quelle que soit la multiplicité, le couplage et la cinématique. Leçons à tirer pour des théories plus réalistes.
- Connexion avec des généralisations des algèbres amassées apparaissant dans d'autres domaines de la math-physique: systèmes-Y, compte d'états BPS des théories de champ supersymétriques, symétrie miroir.

Cluster Algebras and Feynman Integrals

Main Example: Four-point functions with one leg offshell/massive



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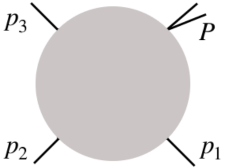

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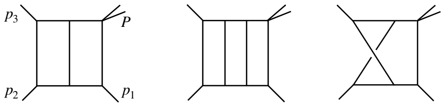

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- Alphabet of all known master integrals: [Gehrmann,Remiddi] [Di Vita, Mastrolia, Schubert, Yundin]

$$\Phi_{\text{2dHPL}} = \{z_1, z_2, z_3, 1 - z_1, 1 - z_2, 1 - z_3\}\,,$$

  "2-dimensional HPLs" [Gehrmann,Remiddi]

Identifying Candidate Cluster Algebras

	2dHPL		
# independent variables	2		

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	2dHPL	A_2
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$\boxed{\text{2dHPLs} = C_2 \text{ polylogarithms!}}$

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Physical Significance: C_2 Cluster Algebra Underlies Higgs Amplitudes!

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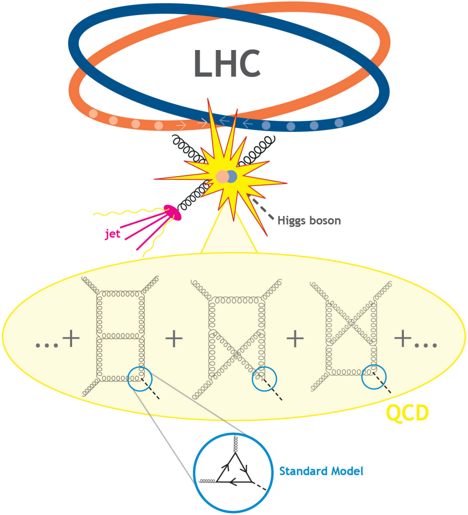
- $pp \to$ Higgs + jet
  [Gehrmann, Jaquier, Glover, Koukoutsakis] [Duhr]

  in heavy top mass limit



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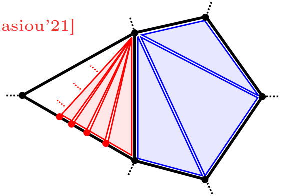
- In principle any n, explicitly $n=9$, [Henke, Papathanasiou'21]
  see also [Ren,Spradlin,Volovich'21]



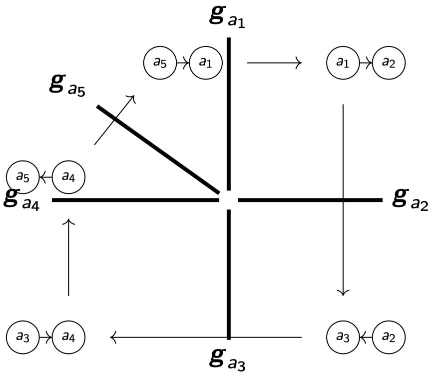
In agreement with all known amplitude data.

The Dual Cluster Fan

Equivalent description of exchange graph

Take normal vectors (of undetermined length) to maximal dimension faces

- Give rise to *rays* (half-lines emanating from origin) ↔ cluster variables
- Grouped in *cones* ↔ clusters

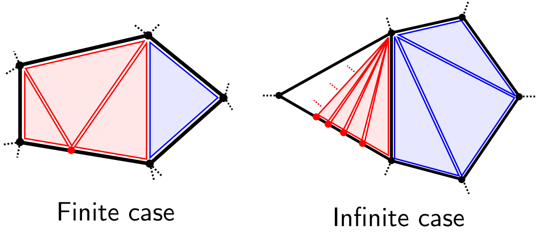


Collection of cones = *(polyhedral) fan* [Fomin,Zelevinsky'01B'02]

Tropical Grassmannians $Tr(k,n)$ and Cluster Algebras

- Finite $Gr(k,n)$ cluster algebras *triangulate* $Tr(k,n)$! [Speyer,Williams]

Illustration: Intersections of 3D cones with sphere ~ locally screen plane



· : $(p)Tr + Gr$ rays

– : $(p)Tr + Gr$ boundaries

· : Gr rays

– : Gr boundaries

- Triangulation used to compute generalized biadjoint scalar amplitudes [Drummond,Foster,Gurdogan,Kalousios'19B][Cachazo,Early,Guevara,Mizera'19]

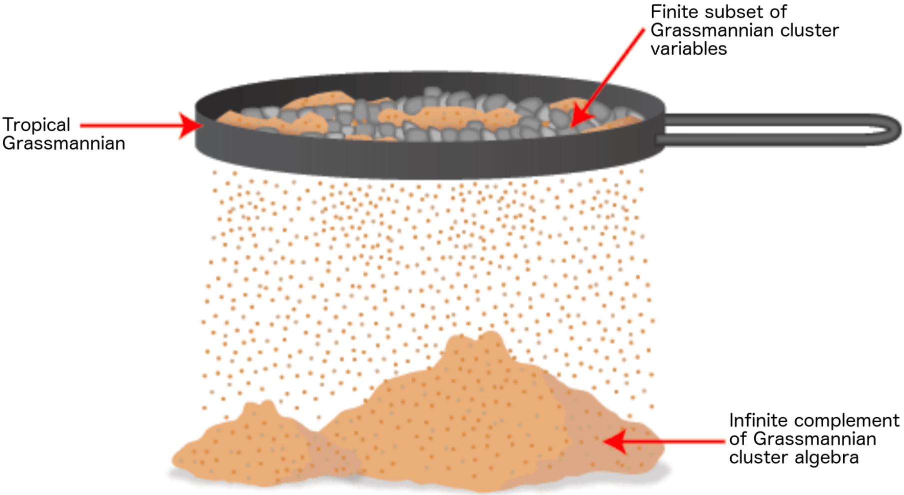
  Sometimes *redundant* (cluster but not tropical – in red) rays produced

> Idea: Cluster algebra ∞ due to infinitely redundant triangulations!
> Select *finite subset* of cluster variables corresponding to tropical rays

[Henke, Papathanasiou'19][Arkani-Hamed,Lam,Spradlin'19][Drummond,Foster,Gurdogan,Kalousios'19B]

Tropical Grassmannians and Cluster Algebras

Schematic relation



Origin at Strong coupling: Expansion & Comparison with String Theory

Expanded Γ_α to four orders in $1/g$, and C_0 to two. For example,

$$\Gamma_\alpha = \frac{8\alpha g}{\pi \sin(2\alpha)} + \mathcal{O}(g^0)\,, \quad D(\alpha) = 4\pi g\left[\frac{1}{4} - \frac{\alpha^2}{\pi^2}\right] + \mathcal{O}(g^0)\,.$$

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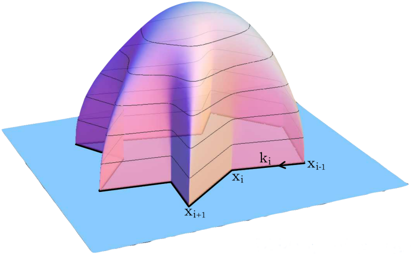

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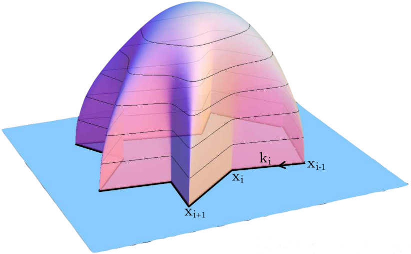

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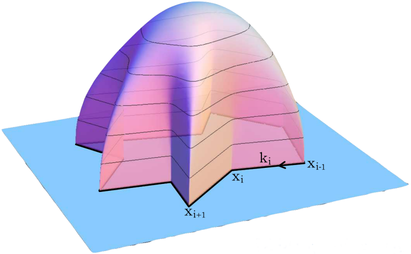
Perfect agreement!



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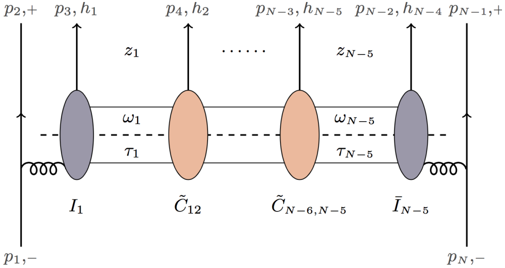

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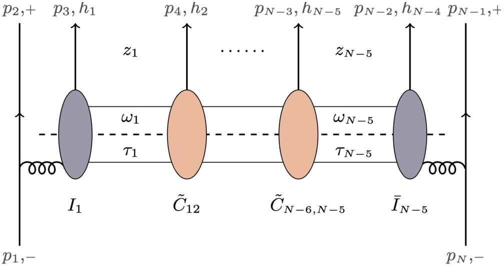

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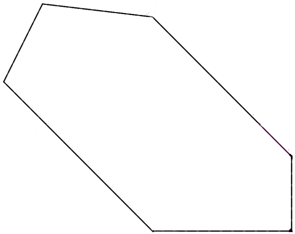
All-order amplitudes in MRL at *any multiplicity*!

[Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek'19]

Integrability in Scattering Amplitudes/Wilson Loops?



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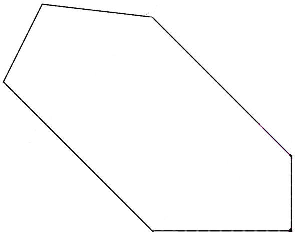

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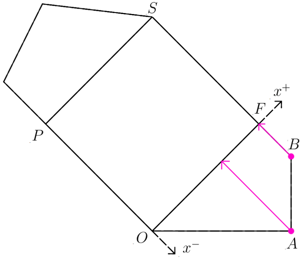

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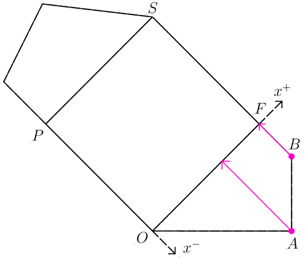

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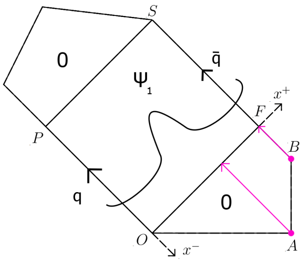

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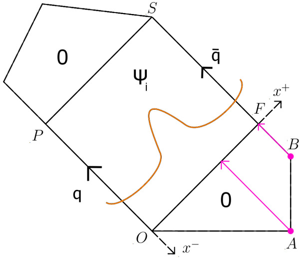

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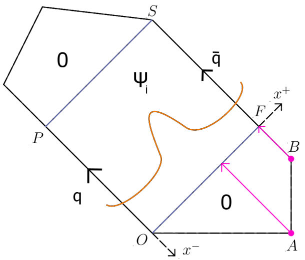

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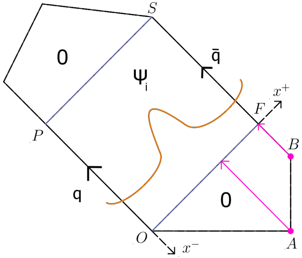

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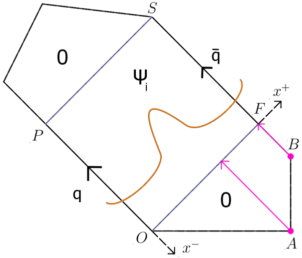

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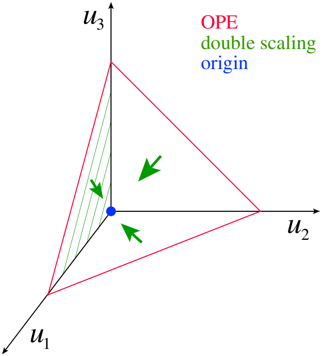
[Belitsky,Bonini,Bork,Caetano,Cordova,Drummond,Fioravanti,Hippel,Lam,Onishchenko, GP,Piscaglia,Rossi...

A Path to Originality



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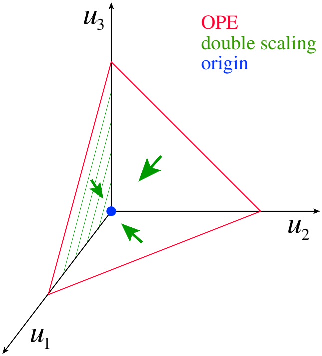

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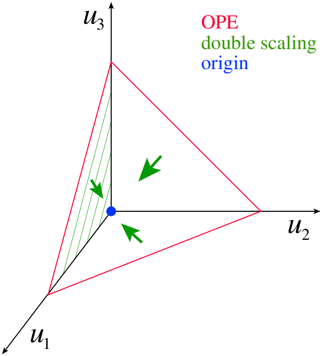

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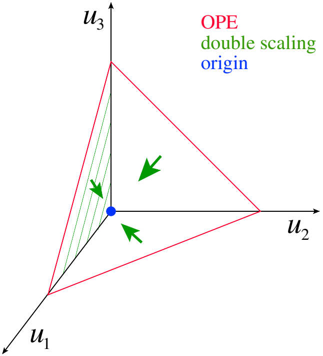

- Origin does not intersect collinear limit
- However part of double scaling limit:
  Only simpler, gluon flux tube excitations contribute, [Basso,Sever,Vieira][Drummond,GP]

$$\mathcal{W}_6^{\mathsf{DS}} = \sum_{N=1}^{\infty} \mathcal{W}_{6[N]}\,, \text{ e.g.}$$

$$\mathcal{W}_{6[1]} = \sum_{a=1}^{\infty} e^{a\phi} \int \frac{du}{2\pi}\, \mu_a(u) e^{-E_a(u)\tau + p_a(u)\sigma}\,.$$

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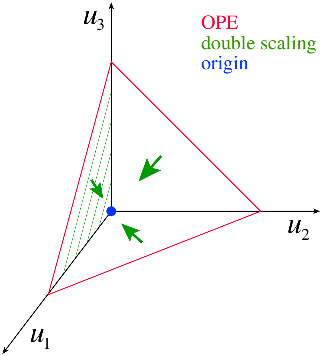

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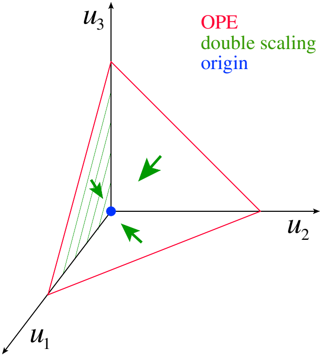

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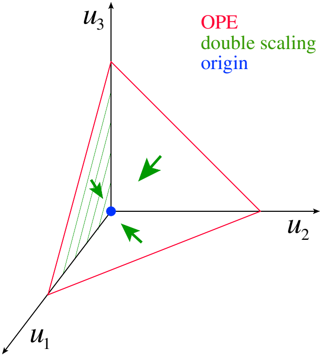

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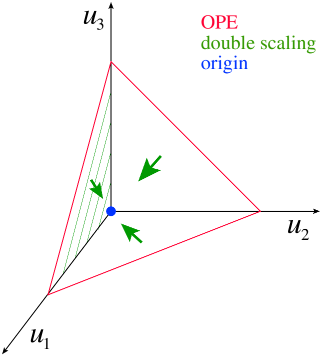

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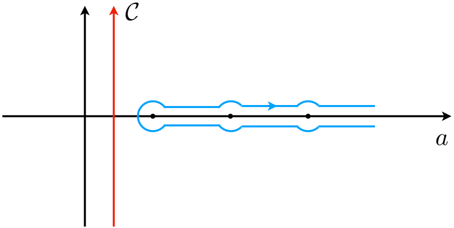
Sommerfeld-Watson Transform

Similar to Regge theory, where it amounts to analytic continuation in spin,

$$\sum_{a\geqslant 1}(-1)^a f(a) \to \int_{+\infty-i\epsilon}^{+\infty+i\epsilon} \frac{if(a)da}{2\sin(\pi a)}\,,$$

provided $f(z)$ decays faster than $1/z$ as $z\to\infty$.



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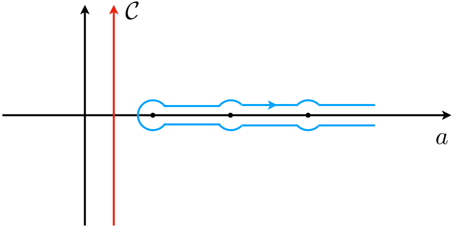

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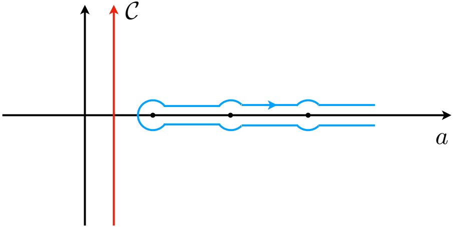

Finally, closing contour around $a=0$ on the left-hand side yields all nonvanishing terms at origin at finite coupling!

Cluster Algebras [Fomin,Zelevinsky'02]

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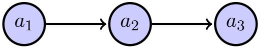

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$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[r] & a_3 \end{tikzcd}$$

Initial Cluster

$$\begin{tikzcd} a_1 \arrow[rr, bend left] & a_2' \arrow[l] & a_3 \arrow[l] \end{tikzcd}$$

Mutate a_2: New cluster

General rule for mutation at node k:

1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.

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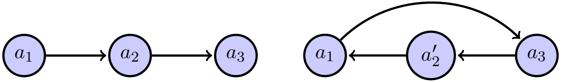

$a_2' = (a_1 + a_3)/a_2$

Initial Cluster

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and so on...

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2. In new quiver/cluster, $a_k \to a_k' = \Big(\prod_{\text{arrows } i \to k} a_i + \prod_{\text{arrows } k \to j} a_j \Big) / a_k$

Geometric Interpretation of Cluster Algebras

Finite cluster algebras classified by Dynkin diagrams. For A_n:

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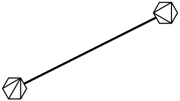

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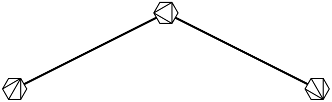

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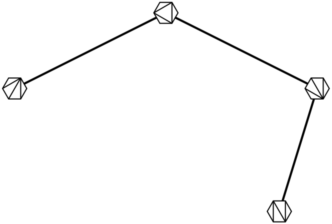

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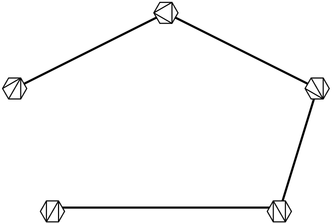

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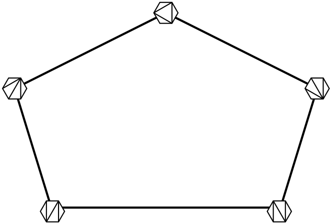

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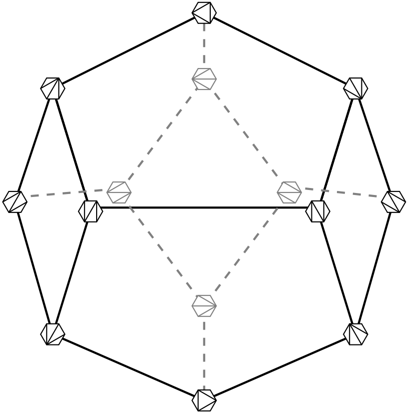
Example: A_3 = hexagon exchange graph



Adapted from 1810.08149

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Can thus use helicity $h = \vec{S} \cdot \hat{p}$ to classify on-shell particle content,

$$\begin{array}{ccccccccc} h: -1 & & -1/2 & & 0 & & 1/2 & & 1 \\ G^- & \xrightarrow{Q^1} & \bar{\Gamma}^A & \xrightarrow{Q^2} & \Phi_{AB} & \xrightarrow{Q^3} & \Gamma_A & \xrightarrow{Q^4} & G^+ \end{array}$$

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For the gluons $G^{\pm}$, the gluinos $\Gamma, \bar{\Gamma}$, and the scalars Φ. For n gluons,

$$\mathcal{A}_n^{L-\mathsf{loop}}(\{k_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\ A_n^{(L)}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n}))$$

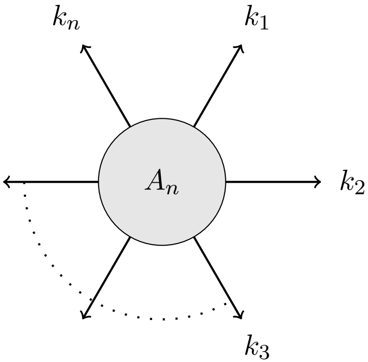
+multitrace terms, subleading by powers of $1/N^2$.

$A_n^{(L)}$: color-stripped amplitude, all color factors removed.

Maximally Helicity Violating (MHV) Amplitudes

These are the simplest amplitudes: $A_n(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)$

They have remarkable properties, namely they



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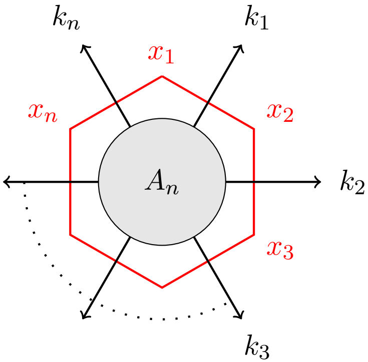

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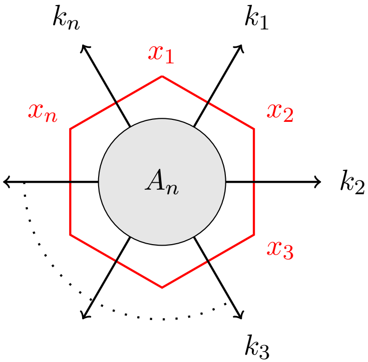

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- exhibit (formally) dual conformal invariance (DCI) under $x_i^\mu \to \frac{x_i^\mu}{x_i^2}$

Dual Conformal Invariance

- In reality DCI broken by divergences, (IR in massless $\mathcal{N} = 4$/UV in cusped WL). Breaking controlled by conformal Ward identity. [Drummond,Henn,Korchemsky,Sokatchev]

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$$\boxed{A_n = A_n^{\text{BDS-like}} \mathcal{E}_n(u_1, \ldots, u_m)}$$

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- e.g. $n=6$,

$$u_1 = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2}\,, \qquad u_2 = \frac{x_{24}^2\, x_{51}^2}{x_{25}^2\, x_{41}^2}\,, \qquad u_3 = \frac{x_{35}^2\, x_{62}^2}{x_{36}^2\, x_{52}^2}\,,$$

Special Conformal Ward Identity

[Drummond,Henn,Korchemsky,Sokatchev]

$$\begin{aligned}\mathbb{K}^{\nu} \ln W_n &= \sum_{i=1}^{n} (2x_i^{\nu} x_i \cdot \partial_i - x_i^2 \partial_i^{\nu}) \ln W_n \\ &= -\sum_{l \geq 1} g^{2l} \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{l\epsilon} + \Gamma^{(l)} \right) \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \mu^2 \right)^{l\epsilon} x_i^{\nu} + O(\epsilon),\end{aligned}$$

Γ: collinear anomalous dimension

Six-particle BDS(-like) Ansatz

[Bern,Dixon,Smirnov; Alday,Maldacena]

$$A_6^{\mathrm{BDS-like}} = \exp\left[\sum_{L=1}^{\infty}(g^2)^L\Big(f^{(L)}(\epsilon)\hat{M}_6(L\epsilon) + C^{(L)}\Big)\right],$$

where

$$f(\epsilon) = \sum_{L=1}^{\infty}(g^2)^L f^{(L)}(\epsilon) = \frac{1}{4}\Gamma_{\mathrm{cusp}} + \mathcal{O}(\epsilon).$$

and

$$\begin{aligned}\hat{M}_6(\epsilon) =&(4\pi e^{-\gamma_E})^{\epsilon}\sum_{i=1}^{6}\Bigg[-\frac{1}{\epsilon^2}\left(1+\epsilon\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)+\frac{\epsilon^2}{2}\ln^2\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)\\ &+\frac{1}{2}\ln^2\left(\frac{s_{i,i+1}}{s_{i+1,i+2}}\right)-\frac{1}{4}\ln^2\left(\frac{s_{i,i+1}}{s_{i+3,i+4}}\right)+\frac{3}{2}\zeta_2\Bigg]+\mathcal{O}(\epsilon),\end{aligned}$$

Relation to original, BDS ansatz:

$$A_6^{\mathrm{BDS}} = A_6^{\mathrm{BDS-like}} e^{\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}_6^{(1)}}, \quad \mathcal{E}_6^{(1)} = \sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)$$

Multiple Polylogarithms (MPLs) and Symbols

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_{\alpha} f_{k-1}^{(\alpha)} \, d\log \phi_\alpha$$

over some set of *letters* ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

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Convenient tool for describing them: **Symbol** $\mathcal{S}(f_k)$ encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_k} \mathcal{S}(f_{k-1}^{(\alpha_k)}) \otimes \phi_{\alpha_k}$$

Example: $d\,\mathrm{Li}_n(x) = \mathrm{Li}_{n-1}(x) d\log x\,, \quad \mathrm{Li}_1(x) = -\log(1-x)$

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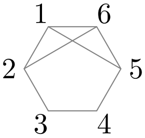
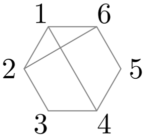
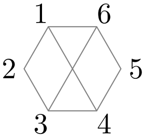
E.g. A_3: Crossing diagonals forbidden,

$\cancel{\ldots \otimes (15) \otimes (26) \otimes \ldots}$, $\cancel{\ldots \otimes (14) \otimes (26) \otimes \ldots}$, $\cancel{\ldots \otimes (14) \otimes (36) \otimes \ldots}$







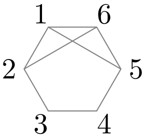
Cluster Adjacency ~ Extended Steinmann Relations

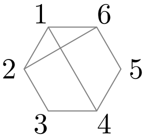
So far seen only cluster coordinates=alphabet relevant for scattering amplitudes. How about clusters, do they also play a role? YES: [Drummond,Foster,Gurdogan]

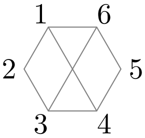
> Two distinct $\mathcal{A}$-coordinates can appear consecutively in a symbol only if there exists a cluster where they both appear.

E.g. A_3: Crossing diagonals forbidden,

$\cancel{\ldots \otimes (15) \otimes (26) \otimes \ldots}$, $\cancel{\ldots \otimes (14) \otimes (26) \otimes \ldots}$, $\cancel{\ldots \otimes (14) \otimes (36) \otimes \ldots}$







For physical $n = 6, 7$ functions, equivalent to *extended Steinmann relations*. Massively reduces size of function space. [Caron-Huot,Dixon,DulatMcLeod,Hippel,**GP**]