Quantum Trajectories, Quantum Jumps, and Classical Probability

Michel Bauer
Joint work with Denis Bernard and Antoine Tilloy

Le monde quantique, IHES, Bures-sur-Yvette,
February 25, 2015

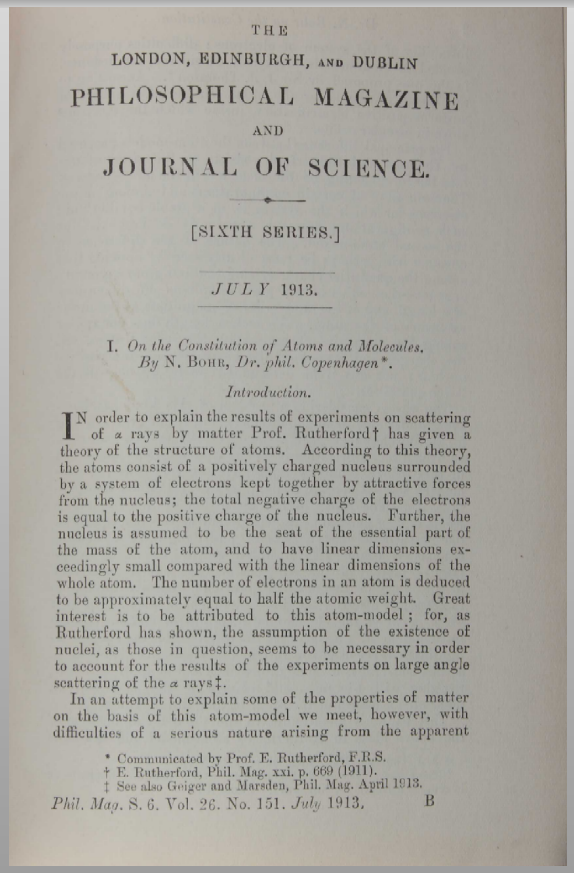
Plan

Introduction

1913: Bohr on Quantum Jumps

On the constitution of atoms and molecules

- ... the emission lines correspond to a radiation emitted during the passing of the system between two different stationary states ...



THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JULY 1913.

I. *On the Constitution of Atoms and Molecules.* *By* N. Bohr, *Dr. phil. Copenhagen**.

Introduction.

In order to explain the results of experiments on scattering of α rays by matter Prof. Rutherford† has given a theory of the structure of atoms. According to this theory, the atoms consist of a positively charged nucleus surrounded by a system of electrons kept together by attractive forces from the nucleus; the total negative charge of the electrons is equal to the positive charge of the nucleus. Further, the nucleus is assumed to be the seat of the essential part of the mass of the atom, and to have linear dimensions exceedingly small compared with the linear dimensions of the whole atom. The number of electrons in an atom is deduced to be approximately equal to half the atomic weight. Great interest is to be attributed to this atom-model; for, as Rutherford has shown, the assumption of the existence of nuclei, as those in question, seems to be necessary in order to account for the results of the experiments on large angle scattering of the α rays‡.

In an attempt to explain some of the properties of matter on the basis of this atom-model we meet, however, with difficulties of a serious nature arising from the apparent

* Communicated by Prof. E. Rutherford, F.R.S.
† E. Rutherford, Phil. Mag. xxi. p. 669 (1911).
‡ See also Geiger and Marsden, Phil. Mag. April 1913.

Phil. Mag. S. 6. Vol. 26. No. 151. *July* 1913. B

1990's: Quantum Jumps Observed

- First (clean) observations of quantum jumps
  - Fluorescence monitoring and photon counting

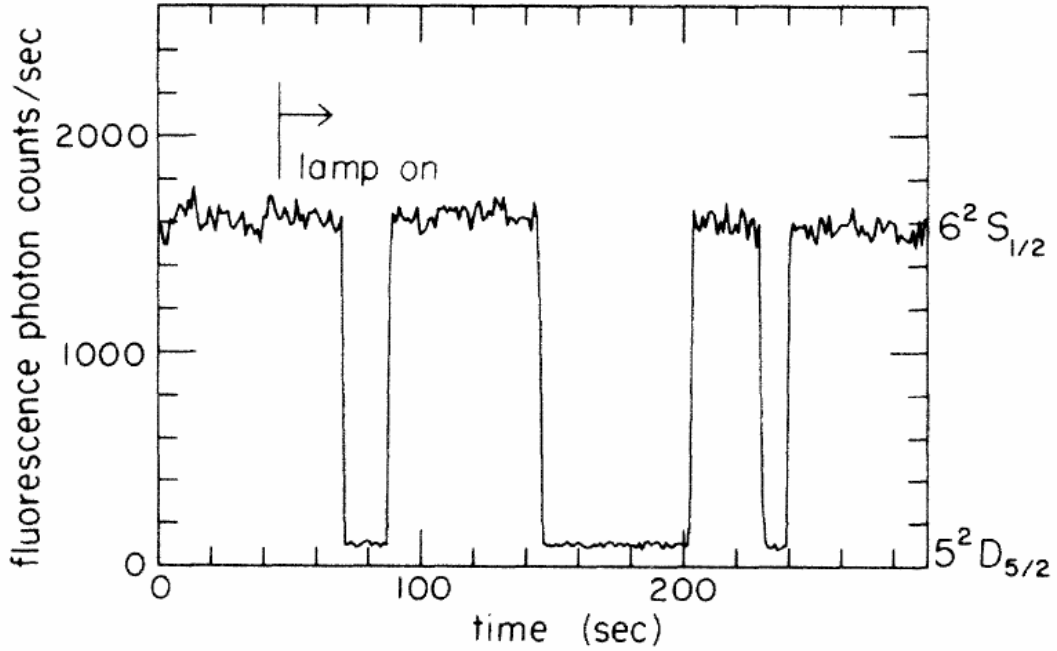
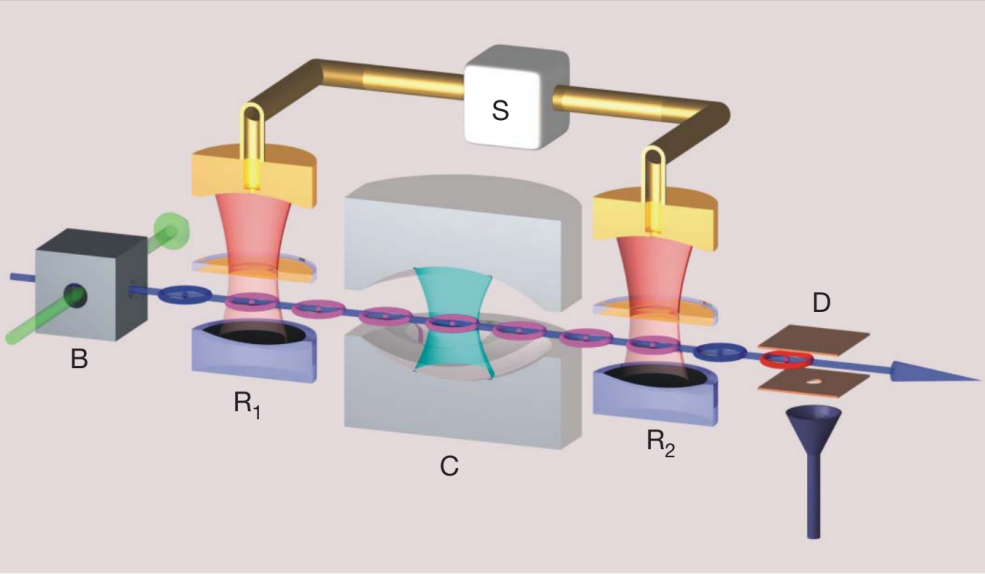


FIG. 2. A typical trace of the 493-nm fluorescence from the $6^2P_{1/2}$ level showing the quantum jumps after the hollow cathode lamp is turned on. The atom is definitely known to be in the shelf level during the low fluorescence periods.

2000's: Quantum Jumps and Repeated Measurements

- Birth and death of a photon in a cavity
  - S. Gleyzes et al (including S. Haroche), **446** (2007) 297-300, Quantum jumps of light recording ...



  - A model of thermalization observed by a non-demolition measurement

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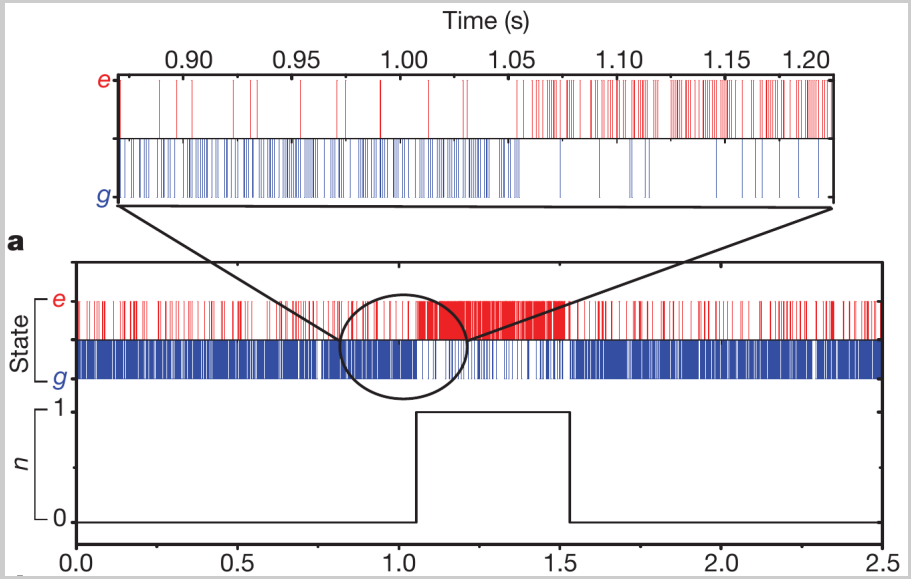


  - A model of thermalization observed by a non-demolition measurement

Aims and Miscellanies

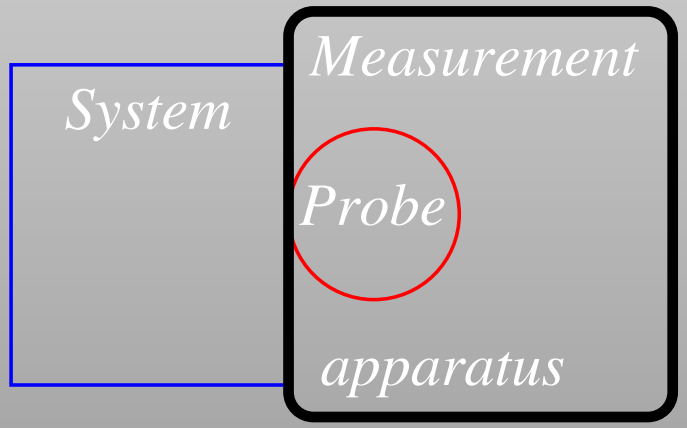
- Aims:
  - Describe some of the tools involved in the manipulation of simple quantum systems
    - In particular the observation of quantum jumps
  - Describe the mathematical framework in discrete and in continuous time
    - In particular how jumps appear and are characterized
- Results alluded to in this presentation:
  - All obtained in collaboration with Tristan Benoist, Denis Bernard, and Antoine Tilloy

Markovian Open Quantum Systems

Indirect Measurements

Indirect measurements (1)

- For a compound system $C = A \cup B$ with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$
  - Learn something on A...
  - ... by measuring on B.
- In this context, A is called the system and B is called the probe.

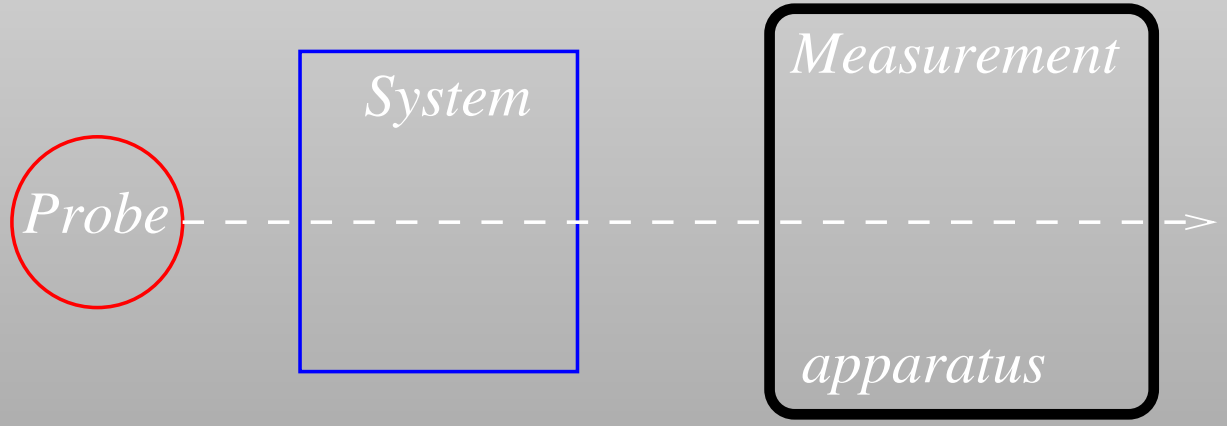


Indirect measurements(2)

- Thus measure an observable $\Lambda = \mathrm{Id}_A \otimes \Lambda_B$ that does nothing on A.
  - If $|\phi\rangle_C = |\varphi\rangle_A \otimes |\psi\rangle_B$ (resp. $\rho_C = \rho_A \otimes \rho_B$) is a pure tensor product just before the measurement...
  - ... then it is $|\varphi\rangle_A \otimes |\psi'\rangle_B$ (resp. $\rho_A \otimes \rho'_B$) just after the measurement.
- But
  - If $|\Phi\rangle_C$ (resp. ρ_C) is not a pure tensor product just before the measurement...
  - ...something really happens to A in the measurement process.
- This is due to entanglement.

Indirect measurements(3)

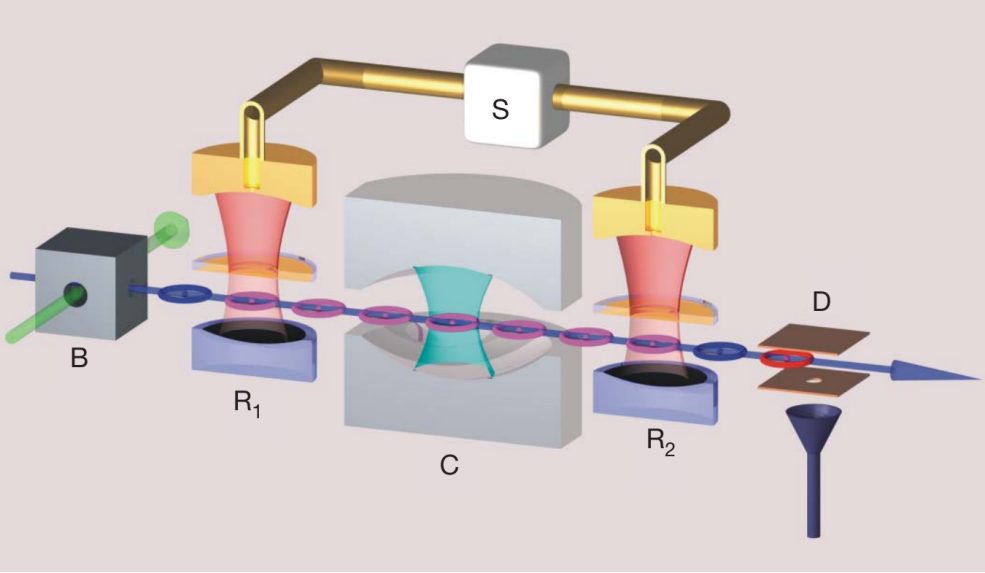
- Common protocol:
  - Start from tensor product state $|\Phi\rangle_C = |\varphi\rangle_A \otimes |\psi\rangle_B$ (resp. $\rho_C = \rho_A \otimes \rho_B$),
  - System-probe interaction goes on for a while, $|\Phi\rangle_C \to U|\Phi\rangle_C$ (resp. $\rho_C \to U\rho_C U^{-1}$)
    - The state is not a pure tensor product anymore.
  - Measure the probe.



- For an indirect measurement, it is natural to view the system-probe interaction as part of the measurement process.

Indirect measurements(4)

- This is the abstract setting of the real experiment.

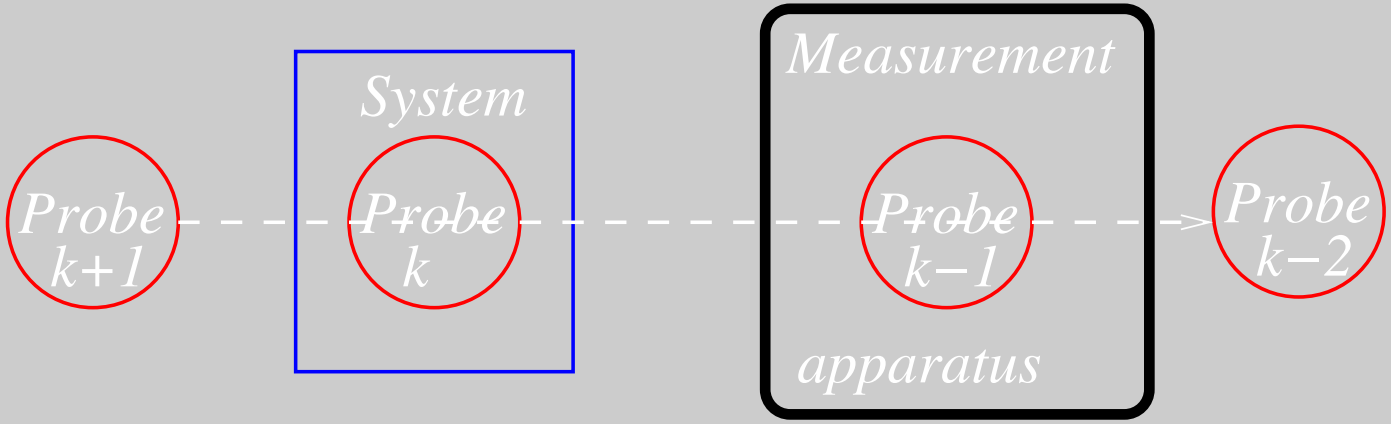


- Rydberg atoms are sent one after the other to interact with the photons in the cavity

Repeated indirect measurements

Generic protocol

- Make several indirect measurements in a row, for a given system but for different probes.



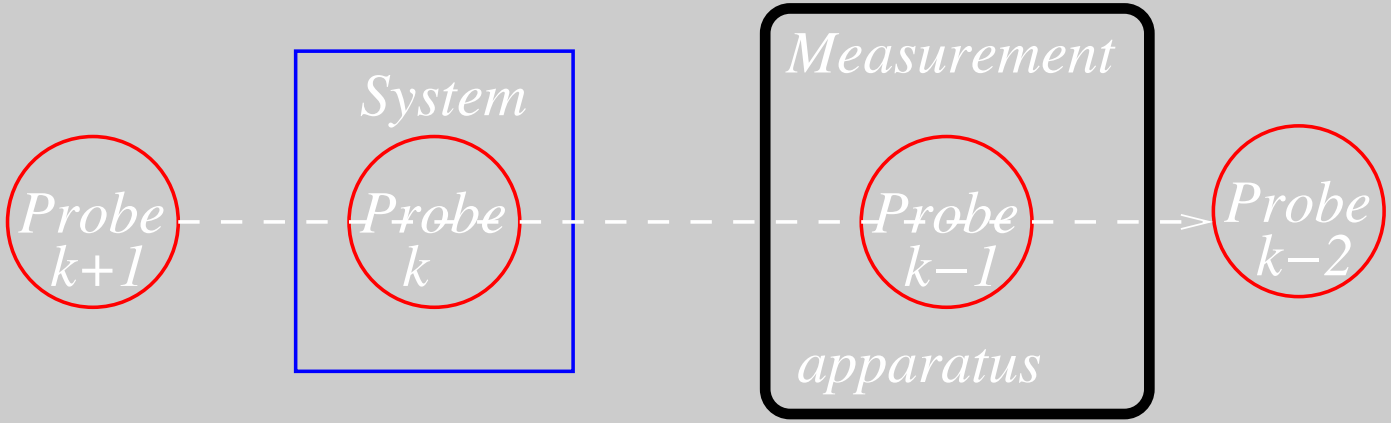
Idealizations

- Assume the probes do not interact with each other.
- Assume the system-probe interaction time is finite.

Repeated indirect measurements

Mathematical setting

- Thus $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_{p^1} \otimes \cdots \otimes \mathcal{H}_{p^n}$.
- Time evolution during the interaction with probe k is given by U_k, acting as U on $\mathcal{H}_s \otimes \mathcal{H}_{p^k}$ and doing nothing to the other probes.
- There is a sequence $i_1, \cdots, i_n$ of outcomes of the probe measurements (assuming that $\Lambda_p = \sum_{i \in E} \lambda_i | \, i \, \rangle\langle \, i \, |$, all λ_i distinct).



Dynamical Equations

Discrete Time Equations(1)

- Iteration of the random dynamical system

$$\rho \to \rho' := \frac{\sum_{i \in I_r} A_i \rho A_i^\dagger}{\pi_r} \text{ with proba } \pi_r := \sum_{i \in I_r} \mathrm{Tr}_{\mathcal{H}_s} A_i \rho A_i^\dagger$$

  - ρ is the density matrix of the system
  - A_i, $i \in I$ are such that $\sum_i A_i^\dagger A_i = \mathrm{Id}_{\mathcal{H}_s}$
  - $I = \cup_r I_r$ is a partition

- Any family A_i can be realized as

$$A_i := \langle\, i \,|\, U \,|\, \psi \,\rangle$$

for some appropriate probe Hilbert space $\mathcal{H}_p$, some unitary evolution U on $\mathcal{H}_s \otimes \mathcal{H}_p$, some orthonormal basis $|\, i \,\rangle$ in $\mathcal{H}_p$ and some fixed state $|\, \psi \,\rangle$in $\mathcal{H}_p$.

Discrete Time Equations (2)

Extreme cases

- Ideal indirect measurement: (of a non-degenerate observable on $\mathcal{H}_p$ projecting on the basis $|\, i \,\rangle$)

$$\rho \to \rho' := \frac{A_i \rho A_i^\dagger}{p_i} \text{ with proba } \pi_i := \mathrm{Tr}_{\mathcal{H}_s} A_i \rho A_i^\dagger$$

- No reading at all of the measurement outcome:

$$\rho \to \rho' := \sum_{i \in I} A_i \rho A_i^\dagger$$

  - Describes also the Markovian limit of interaction with an environment (partial trace on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_p$)

$$\sum_{i \in I} A_i \rho A_i^\dagger = \mathrm{Tr}_{\mathcal{H}_p} U \rho_{\mathcal{H}} U^\dagger$$

Continuous Time Limit Equations: Barchielli, Belavkin ... Pellegrini

$$d\rho_t = (-i[H,\rho_t] + \sum_a \mathcal{L}_{B_a}(\rho_t) + \sum_b \mathcal{L}_{N_b}(\rho_t))\, dt + \sum_b \mathcal{Q}_{N_b}(\rho_t)\, dW_t^{(b)}$$

- dt : general Lindbladian, H is a self-adjoint operator (Hamiltonian), the B_as and N_bs are arbitrary operators on $\mathcal{H}_s$
  - $\mathcal{L}_O(\rho) := O\rho O^\dagger - \frac{1}{2}(O^\dagger O\rho + \rho O^\dagger O)$
- $dW_t^{(b)}$: stochastic innovation term, the W_t^bs are centered continuous Gaussian Markov processes with independent increments and covariance
  - $dW_t^{(b)} dW_t^{(b')} = dt(\delta^{b,b'} - \sqrt{p_b p_{b'}})$ $\quad (\sum_b p_b = 1)$
  - $\mathcal{Q}_O(\rho) := O\rho + \rho O^\dagger - \rho \mathrm{Tr}_{\mathcal{H}_s}(O\rho + \rho O^\dagger)$ (non-linear term)

Non Demolition Experiments

Non Demolition Experiments (1)

Definition

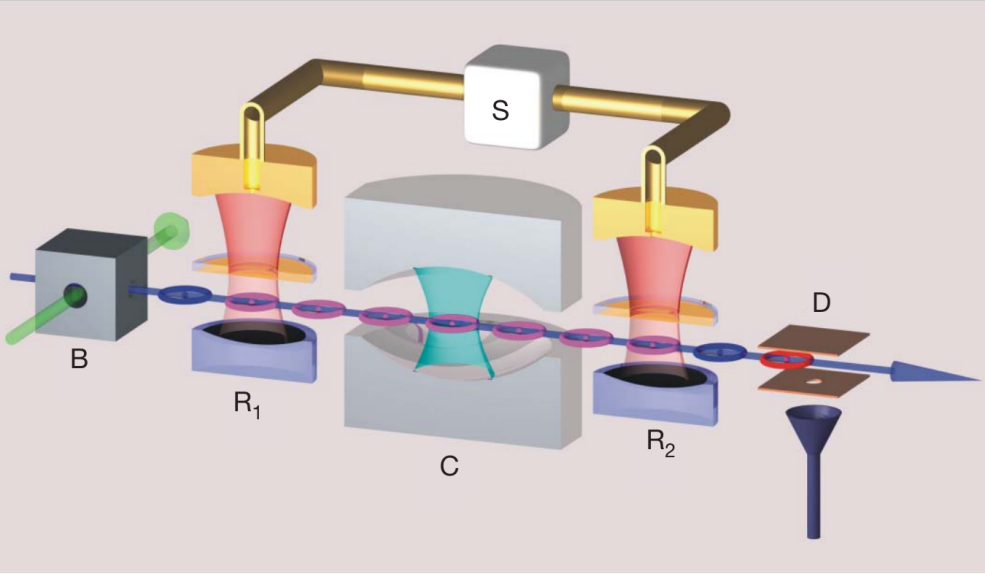
- Time evolution during the interaction with probe k is given by U_k, acting as U on $\mathcal{H}_s \otimes \mathcal{H}_{p^k}$ and doing nothing to the other probes.
- A Non Demolition experiment is when the U_k's commute.

Consequence

- There is an orthonormal basis $|\alpha\rangle$ (pointer states) in $\mathcal{H}_s$ such that
$$U = \sum_\alpha |\alpha\rangle\langle\alpha| \otimes U_\alpha.$$
- In the pointer basis, the operators A_i are diagonal.
$$(A_i)_{\alpha\beta} = \delta_{\alpha\beta} c(i|\alpha)$$

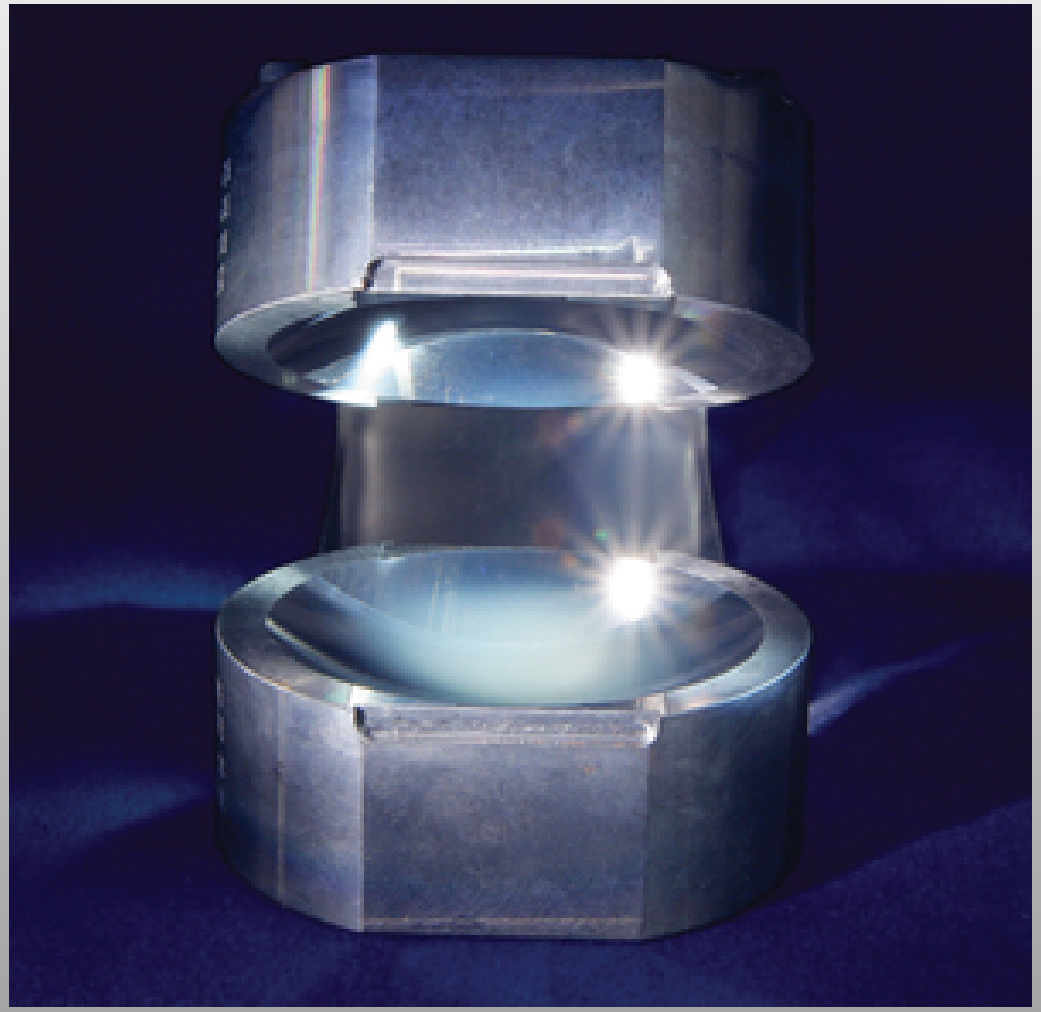
Non Demolition Experiments (3)

A real experiment



- Each probe (a Rydberg atom) behaves as a two-level system
- The preferred basis is that of photon number $n = 0, 1, \cdots$
- In an appropriate basis, $U_n = e^{i\theta n \sigma_z}$ where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - Due to the value of θ, U_n is periodic modulo 8.
- The probe observable Λ is a Pauli matrix along some axis perpendicular to the z axis.

The Cavity



Non Demolition Experiments (2)

Non demolition dynamics

- In the pointer basis, iteration of :

$$\rho'_{\alpha\beta} = \rho_{\alpha\beta} \frac{c(i|\alpha)\overline{c(i|\beta)}}{\sum_\gamma \rho_{\gamma\gamma}|c(i|\gamma)|^2} \text{ with proba } \sum_\gamma \rho_{\gamma\gamma}|c(i|\gamma)|^2$$

- For each α, $p(i|\alpha) := |c(i|\alpha)|^2$ is a probability measure on probe measurement outcomes.
  - The measurement is called non-degenerate if the $p(\cdot|\alpha)$ are distinct for different αs, i.e. if measurements discriminate the different αs (assumed in what follows)

Consequence

- The (non)diagonal elements of ρ are (super)martingales

Non Demolition Experiments (3)

Von Neumann equivalence

- At large times (i.e. after many iterates) ρ_n converges to a projector on some pointer $|\Gamma\rangle\langle\Gamma|$
  - Beware that Γ is random (i.e. depends on the experiment)
  - Convergence is exponential, rates given by relative entropies
- The probability that ρ_n ends in $|\gamma\rangle\langle\gamma|$ is $\mathcal{P}(\Gamma = \gamma) = \langle\gamma|\rho_0|\gamma\rangle$.

Reading the outcome

- The asymptotic frequency of outcome i in a given experiment is $p(i|\Gamma)$

Holography

- As the sequence of probe outcomes $i_1, i_2, \cdots$ is exchangeable (the nondemolition condition) any (infinite, very large) subsequence $i_{n_1}, i_{n_2}, \cdots$ allows to recover Γ.

Outlook

- Iterated non demolition measurements are a subtle tool to implement standard measurements on a fragile quantum system
- Aims in what follows:
  - Use probes coupled with a non demolition interaction to a system whose intrinsic time evolution does not preserve pointer states
  - Study the strong measurement regime, when time between probes is small with respect to the time scales of the system (in this regime, asymptotic holography holds)
  - Make contact with some real experiments

Quantum Jumps and Spikes

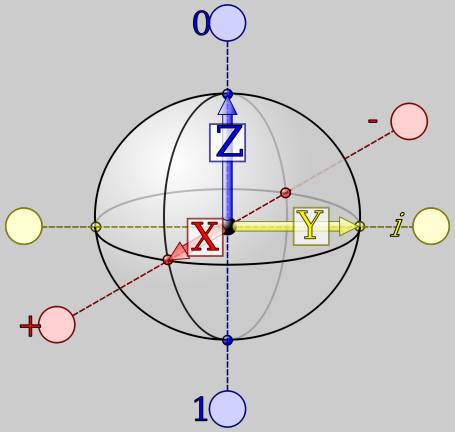
Two-levels Systems: $\dim \mathcal{H}_s = 2$

- The general 2 by 2 density matrix is

$$\rho = \frac{1}{2}\begin{pmatrix} 1+Z & X-iY \\ X+iY & 1-Z \end{pmatrix}$$

$$X^2 + Y^2 + Z^2 \leq 1$$

  - The Bloch sphere



- Our illustrations involve real 2 by 2 density matrices

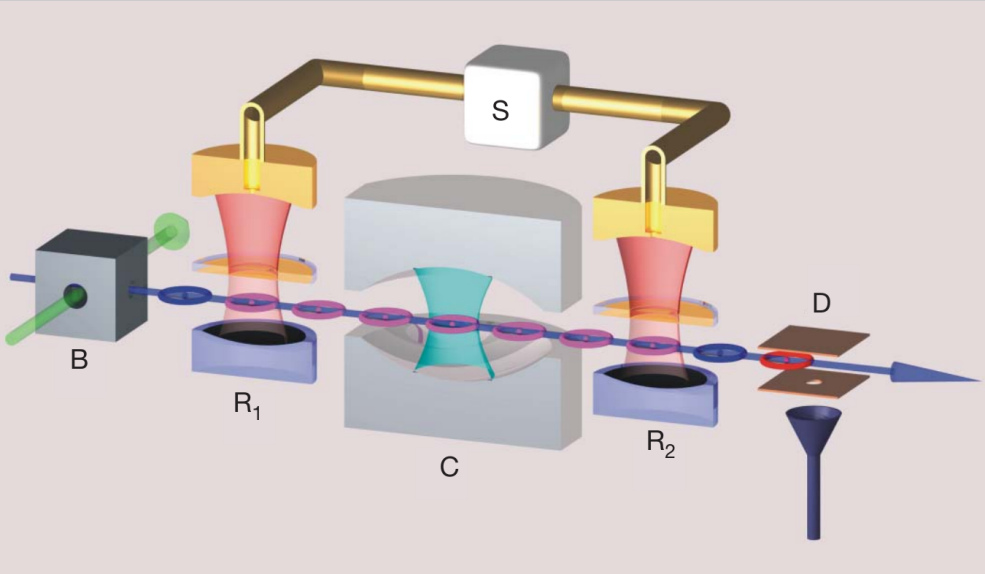
$$\rho = \frac{1}{2}\begin{pmatrix} 1+Z & X \\ X & 1-Z \end{pmatrix} \qquad X^2 + Z^2 \leq 1$$

- The Bloch disk $Z^2 + X^2 \leq 1$ bounded by the Bloch circle
- Set $Q =: (1+Z)/2$

Thermal Noise plus Measurement

The Experiment

- Birth and death of a photon in a cavity
  - S. Gleyzes et al (including S. Haroche), **446** (2007) 297-300, Quantum jumps of light recording ...



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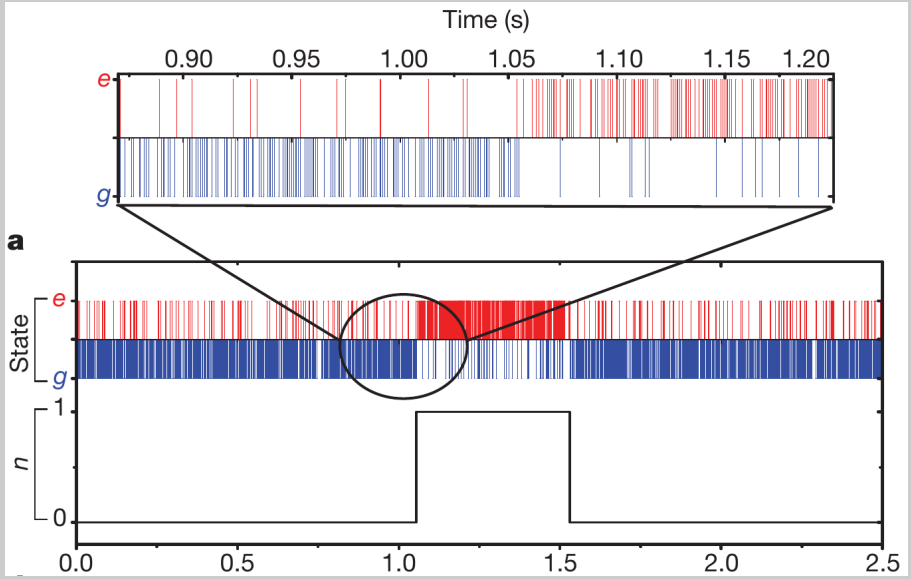

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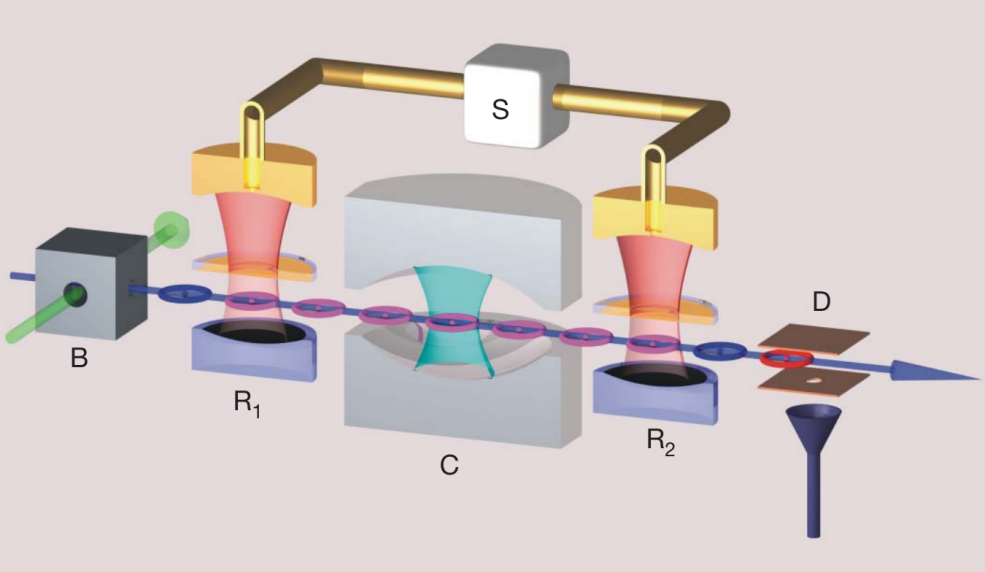
Physical setting



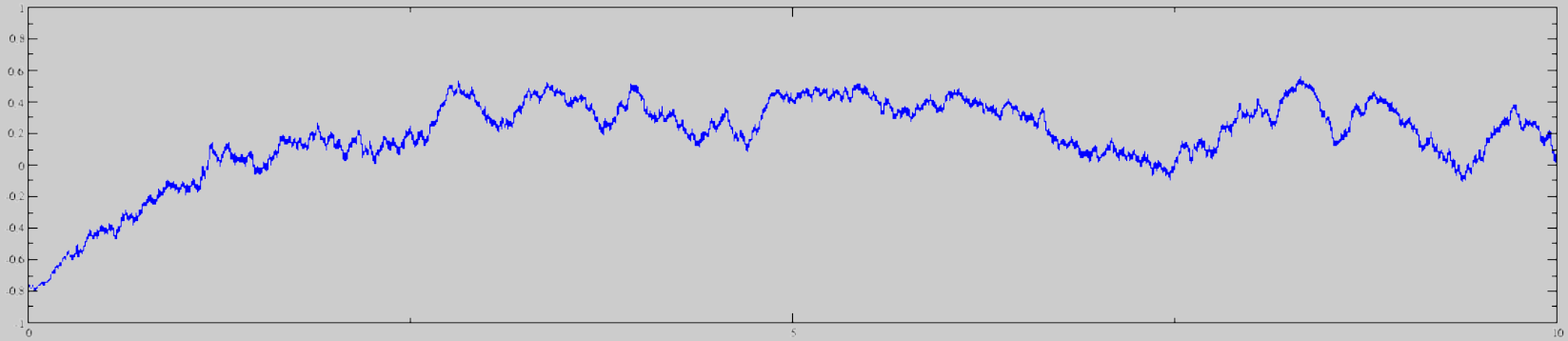
- The setting is very much the same as before but :
  - Due to the value of θ, U_n is periodic modulo 2
  - The cavity is modeled by a two-level system, containing 0 photon (i.e. an even number of photons) or 1 photon (i.e. an odd number of photons)
  - Thermal noise may induce transition between 0 ($Q = 1$, $Z = 1$) and 1 ($Q = 0$, $Z = -1$) photon states

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

A plot of Q_t, small γ



- Fluctuations around the stationary limit

Mathematical Model

Thermal Noise

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

- Markovian approximation
  - No memory effects
- Diagonal elementary processes
  - In the photon number basis, the elementary processes trigger only transition from 0 to 1 photon and from 1 to 0 photon.
- Equilibrium at temperature β:
  - Energy $H := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$ and $\rho_{eq} \propto e^{-\beta H} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta\epsilon} \end{pmatrix}$

Mathematical Model

Measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

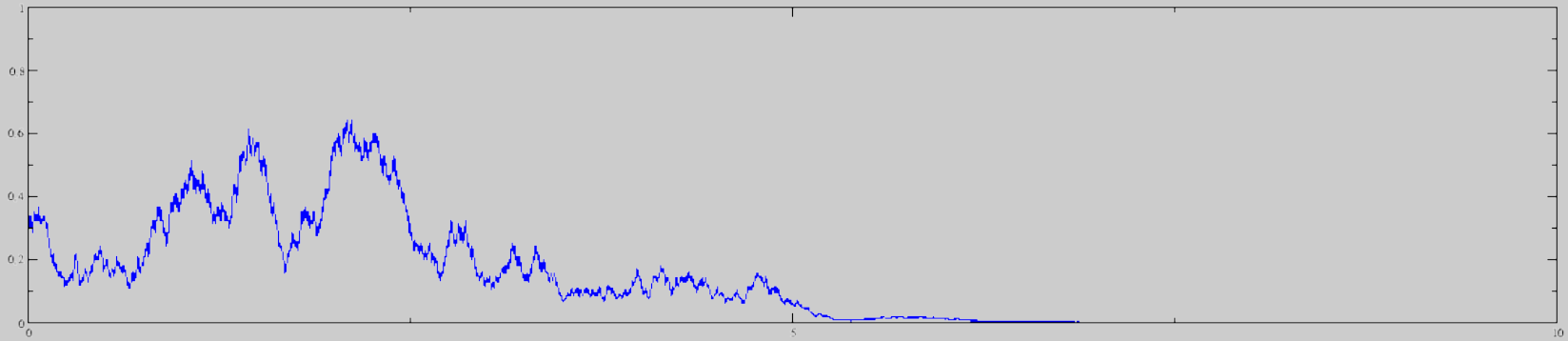
- Continuous time approximation
  - Each probe measurement has a small effect on the cavity
  - Time resolution large compared to lapse between two probes
- Measurement is responsible for non-linearities
- Each probe measurement can have two outcomes
  - Measurement statistics is a random walk correlated to the cavity
  - In continuous time, leads to a diffusion
- The probes couple to the photon number in the cavity

Mathematical Model

Measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

Convergence of Q_t to 0



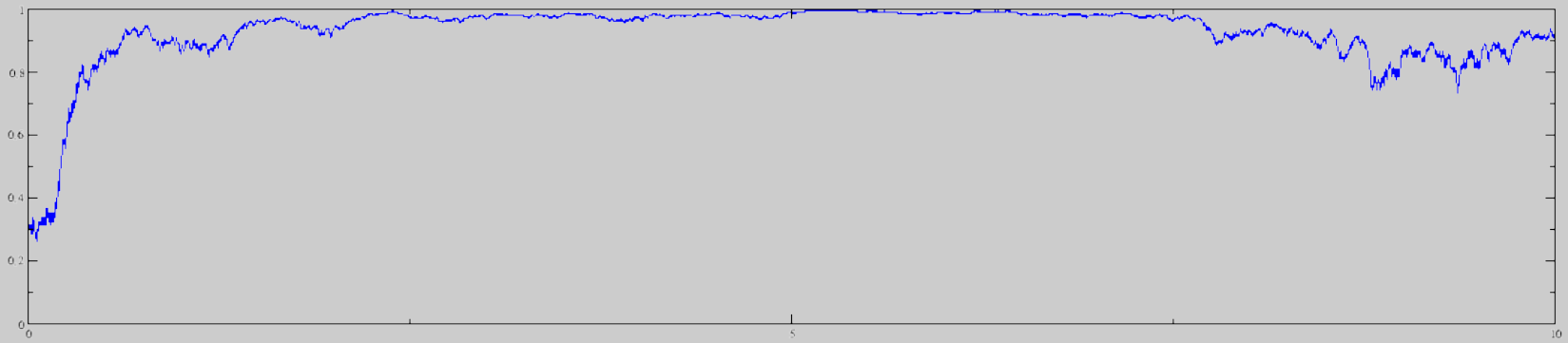
- Rapid convergence to 0

Mathematical Model

Measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

Convergence of Q_t to ?



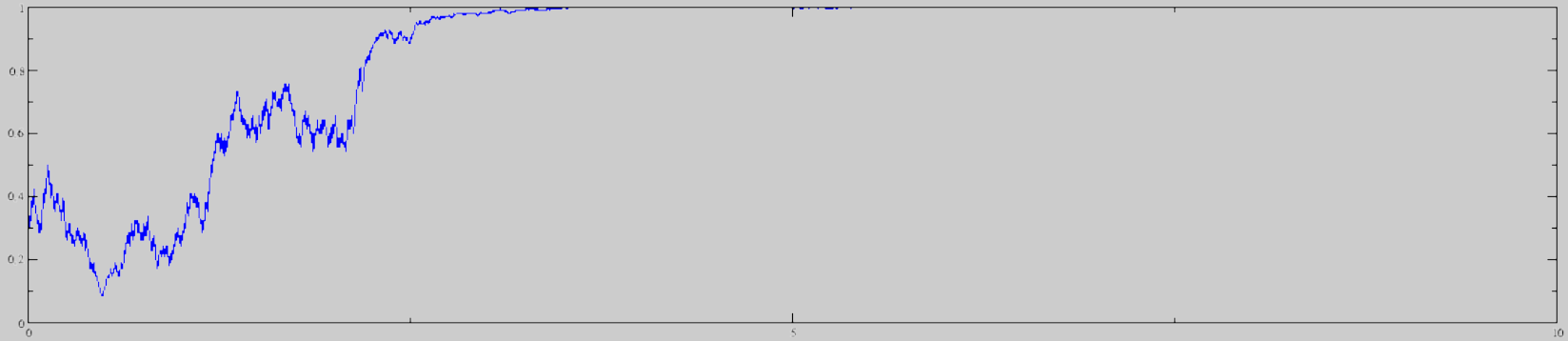
- Hesitations

Mathematical Model

Measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

Convergence of Q_t to 1



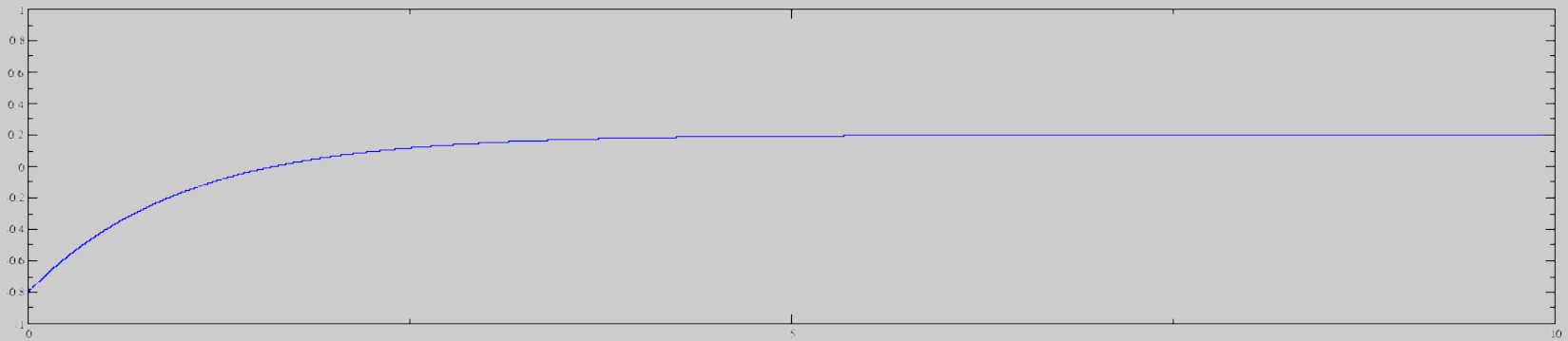
- Rapid convergence to 1

Mathematical Model

Competition between thermal fluctuations and measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

A plot of Q_t, $\gamma = 0$



- Convergence to the stationary limit

Mathematical Model

Competition between thermal fluctuations and measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
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\end{aligned}
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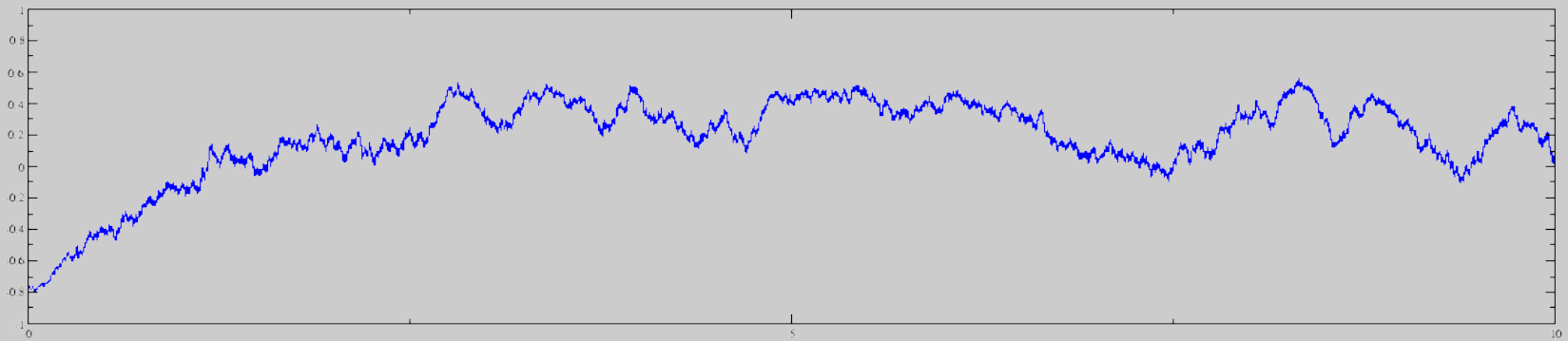
A plot of Q_t, small γ



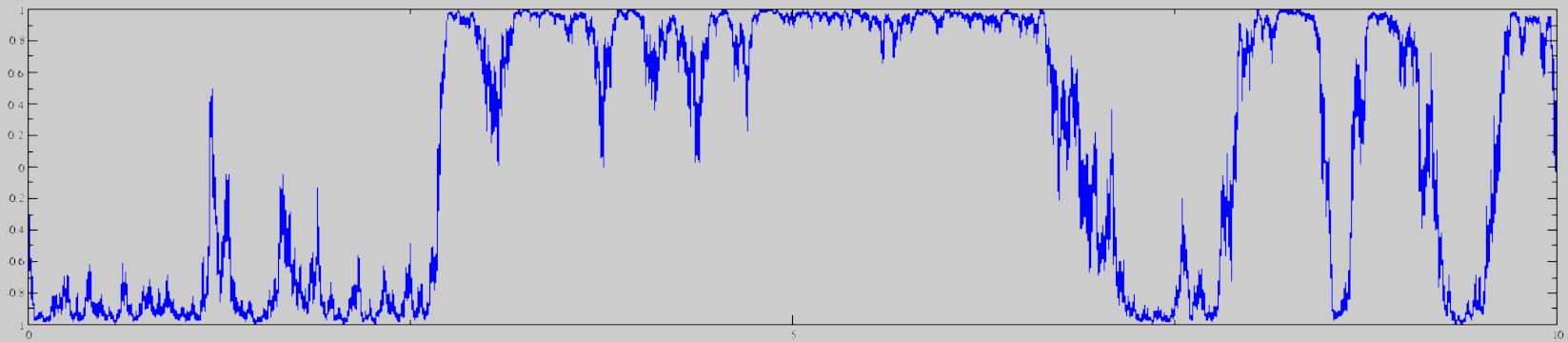
- Fluctuations around the stationary limit

Mathematical Model

Competition between thermal fluctuations and measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

A plot of Q_t, moderate γ



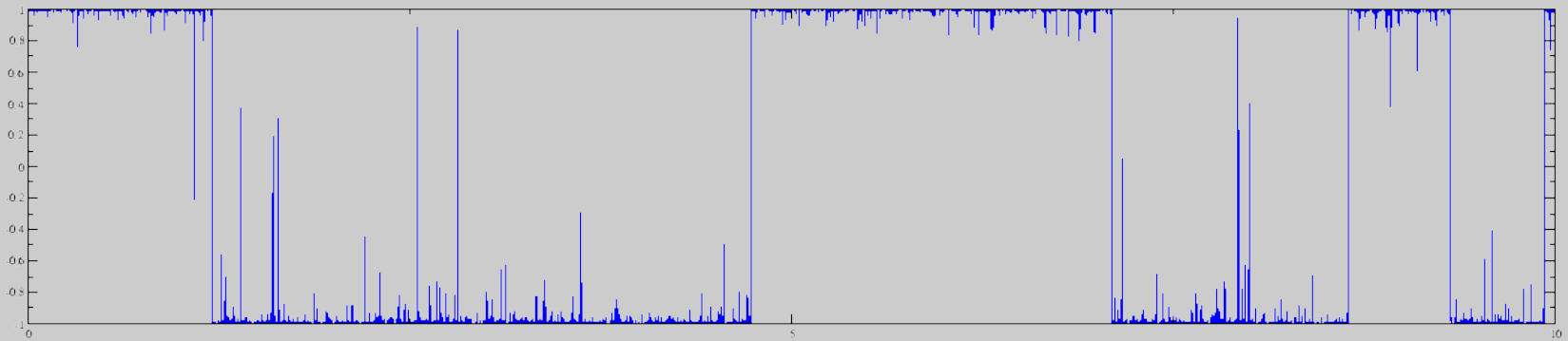
- Progressive deformation of the shape

Mathematical Model

Competition between thermal fluctuations and measurement

$$
\begin{aligned}
dZ_t &= \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

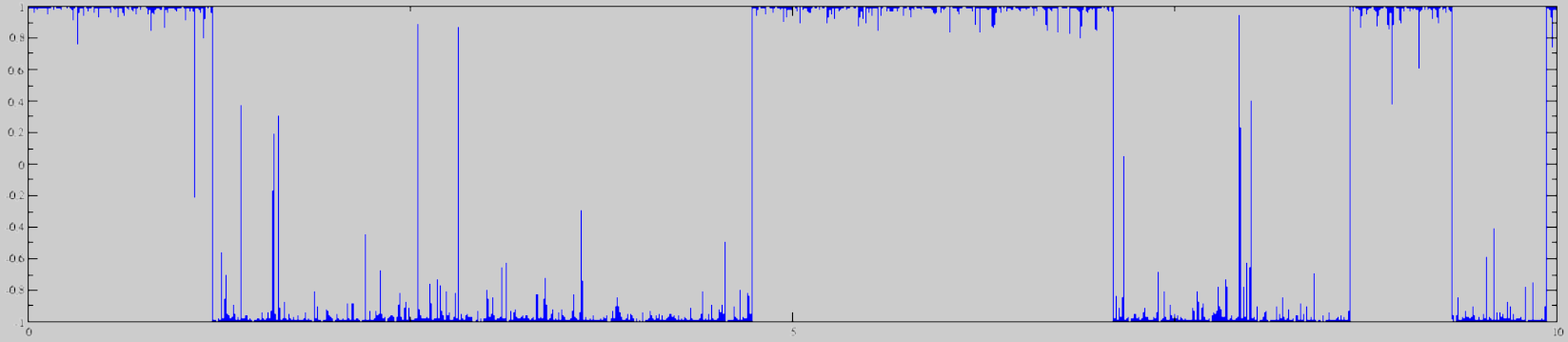
A plot of Q_t, large γ



- Emergence of jumps and spikes

Goals

- The equation seems to account for jumps (at large γ, i.e. in the *many probes per unit time* limit)
  - Describe the limiting jump process
- The equations exhibits also unexpected spikes
  - Are the spikes mathematically and/or physically real ?
  - If so, describe the limiting spike process



Jumps

Strategy

- The Markov kernel of the measurement part $dZ_t = -\gamma(1 - Z_t^2)\, dB_t$ can be computed explicitly
- At large γ treat the thermal noise part as a perturbation

Theorem

- In the limit $\gamma \to \infty$, the finite dimensional distributions of $Q_t = \frac{1}{2}(1 + Z_t)$ converge weakly (i.e. in law) towards those of a finite state Markov process with states 0 ($Q \simeq 1$) and 1 ($Q \simeq 0$) with Markov generator

$$\frac{\lambda}{2}\begin{pmatrix} -1 + \tanh\frac{\beta\epsilon}{2} & 1 + \tanh\frac{\beta\epsilon}{2} \\ 1 - \tanh\frac{\beta\epsilon}{2} & -1 - \tanh\frac{\beta\epsilon}{2} \end{pmatrix}$$

- The Markov matrix is already apparent in the thermal noise part (i.e. master equation) $d\mathbb{E}(Z_t) = \lambda(\tanh\beta\epsilon - \mathbb{E}(Z_t))\, dt$

Spikes

Strategy

- Let $\tau(q_i, q_f)$ be the random time it takes to go to q_f starting from q_i
- Describe the original process in terms of $\tau(q_i, q_f)$
- Limiting law of $\tau(q_i, q_f)$ for $\gamma \to \infty$ can be computed

$$\frac{q_i}{q_f}\delta(t)dt + \left(1 - \frac{q_i}{q_f}\right)\frac{p\lambda}{q_f}e^{-\frac{p\lambda}{q_f}t}dt \qquad p := \frac{1}{2}\left(1 + \tanh\frac{\beta\epsilon}{2}\right)$$

Theorem

- One can reconstruct the process in the limit $\gamma \to \infty$ and in law from two time-homogeneous space-time Poisson point processes $\mathbb{P}\mathrm{ois}_0$ and $\mathbb{P}\mathrm{ois}_1$ on $[0,1] \times [0,+\infty]$.
  - For instance, the density of $\mathbb{P}\mathrm{ois}_0$ is:

$$d\nu_0 := \left(\delta(1-q)dq + \frac{dq}{q^2}\right)p\lambda dt$$

Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

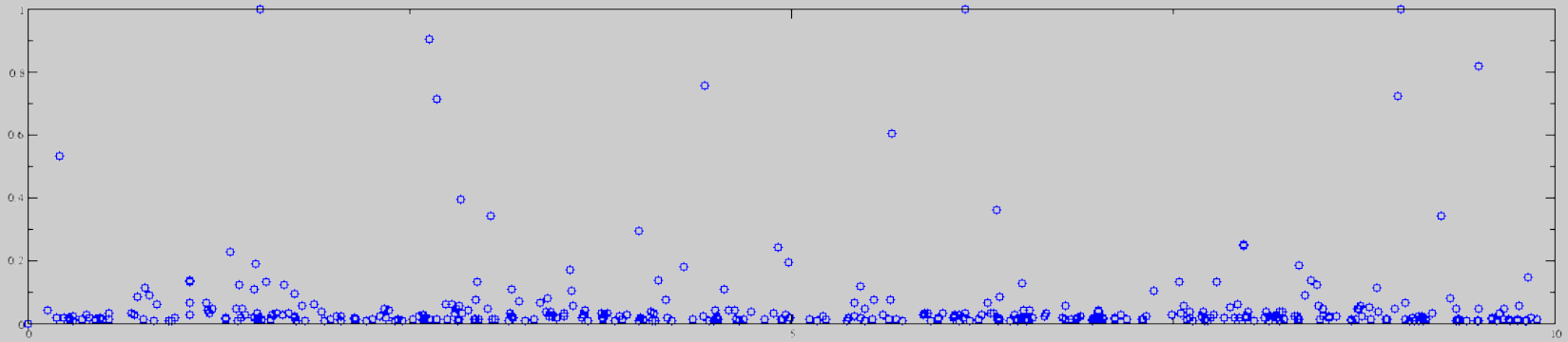
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- $\mathbb{P}\mathrm{ois}_0$



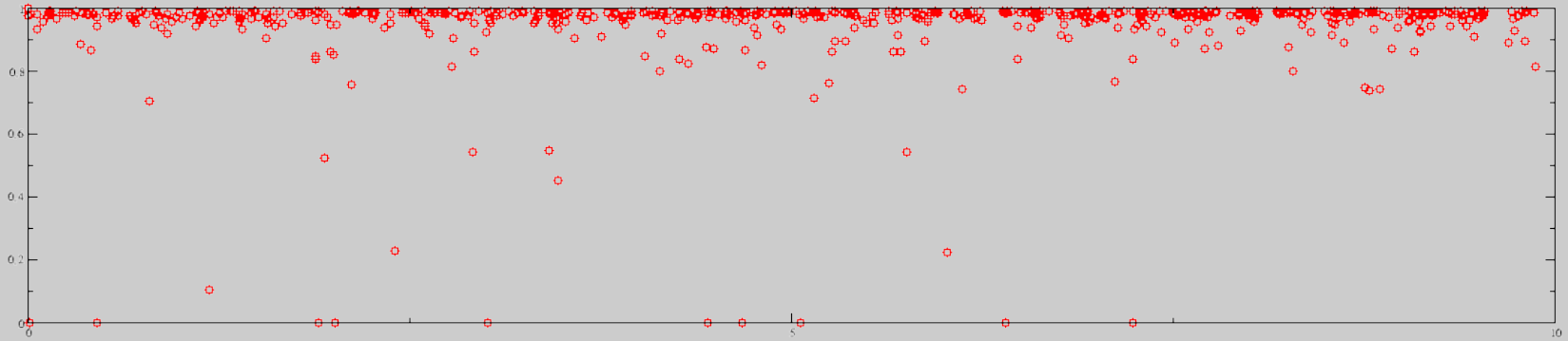
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- $\mathbb{P}\mathrm{ois}_1$



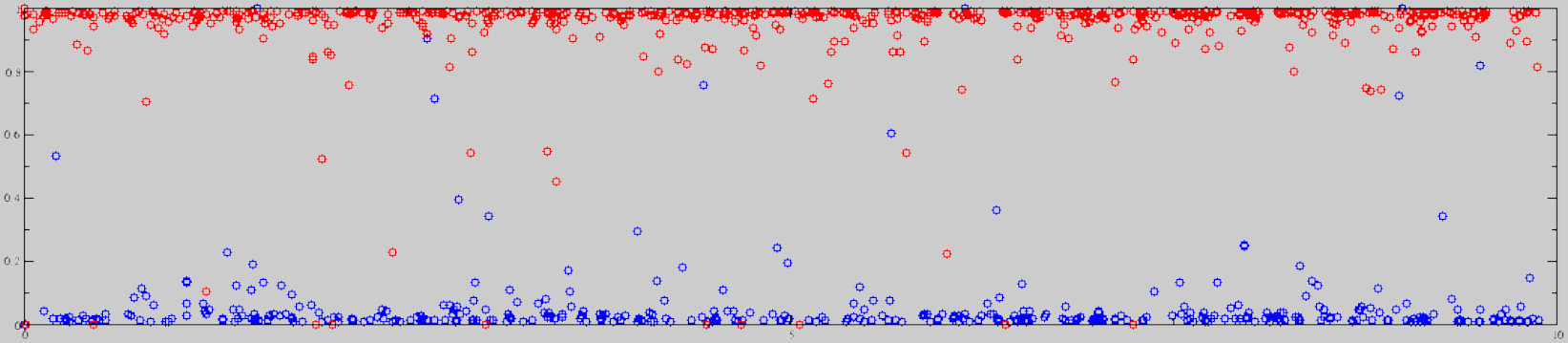
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- $\mathbb{P}\mathrm{ois}_1$ and $\mathbb{P}\mathrm{ois}_1$



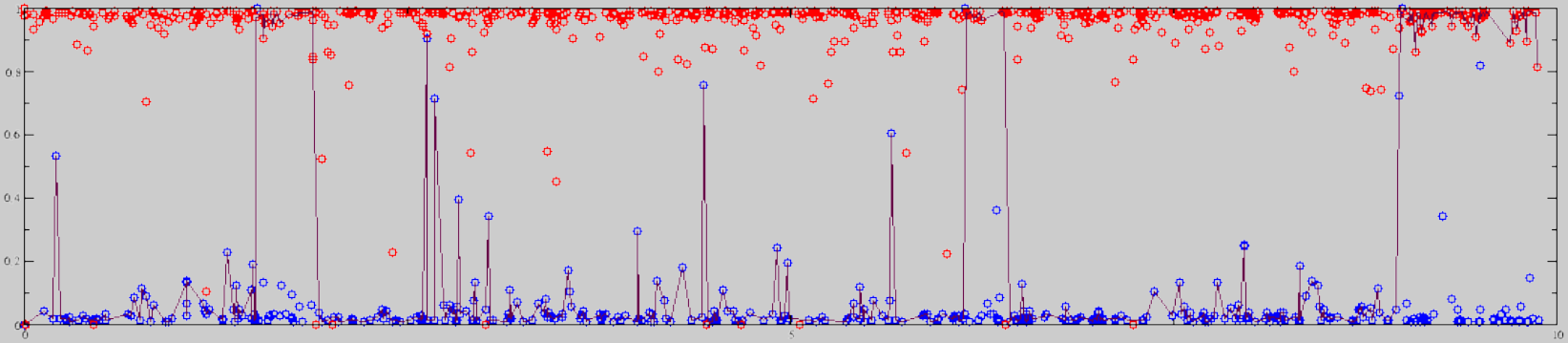
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- Spike process from $\mathbb{P}\mathrm{ois}_1$ and $\mathbb{P}\mathrm{ois}_1$



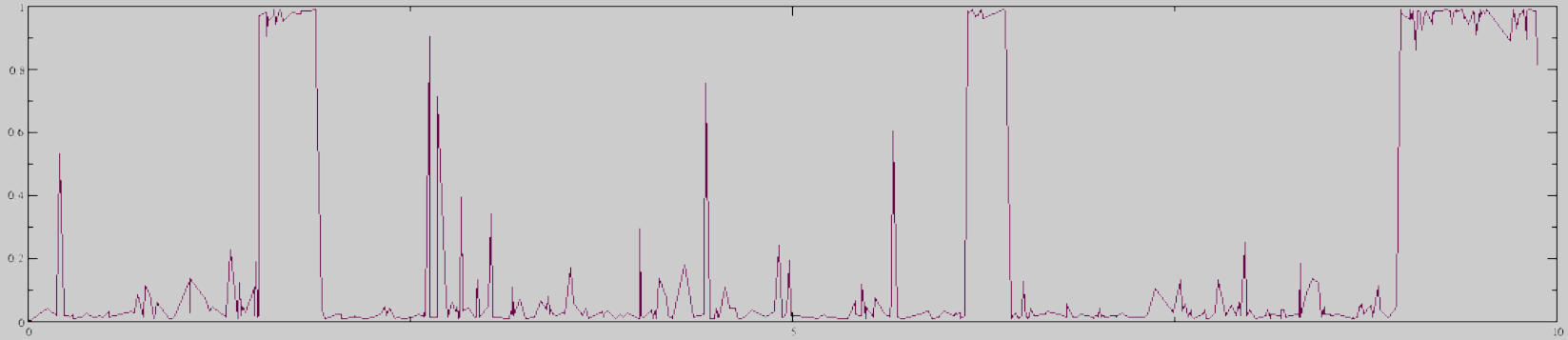
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- Spike process



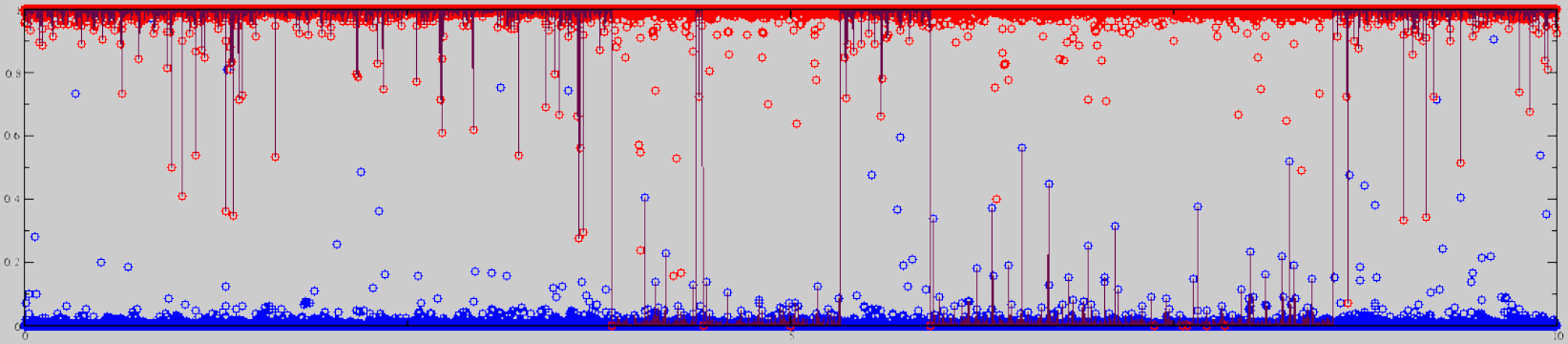
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- More points



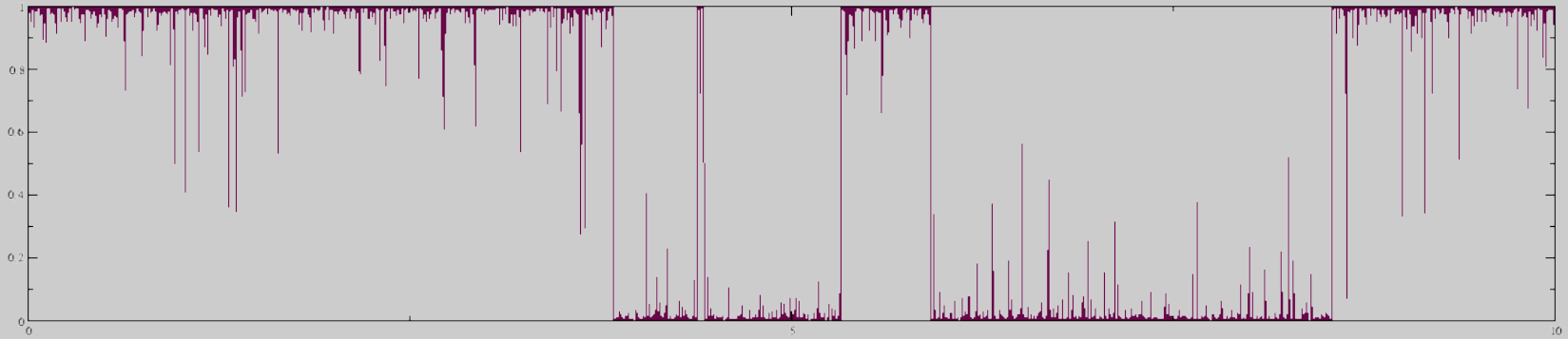
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

From Poisson to Spikes

- More points



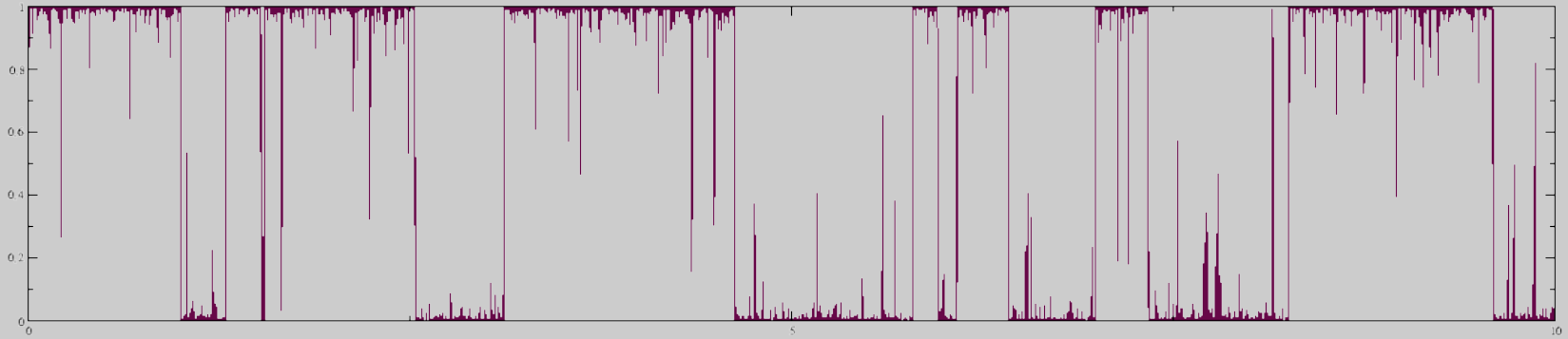
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter Q_0

Original process at large γ

- More points



The Status of Spikes

Mathematical status

- Spikes are predicted by both the discrete and the continuous time model

Physical status

- Experiments are not yet precise enough to see spikes (but they should be there)
- Spikes do not have an unavoidable quantum origin ...
  - The equation

$$dZ_t = \lambda(\tanh \beta\epsilon - Z_t)\, dt - \gamma(1 - Z_t^2)\, dB_t$$

    also describes a cavity jumping between the 0 and the 1 photon states according to a thermal Markovian law, as observed by a fuzzy but purely classical (no disturbance of the cavity) repeated measurement

... and possibly no physical reality

To Summarize

- The mathematical model for thermal fluctuations observed by repeated non-demolition measurements
  - Accounts for jumps
  - Predicts (unexpected?) spikes
- Jumps are described by a finite state Markov process whose Markov matrix can be read on the averaged equations of motion
- Spikes are described by Poisson point processes, and are aborted jumps
  - Spikes are scale invariant (dq/q^2 at small q)

- What about other systems ?
  - A simple possibility is to replace the thermal noise by a Hamiltonian evolution

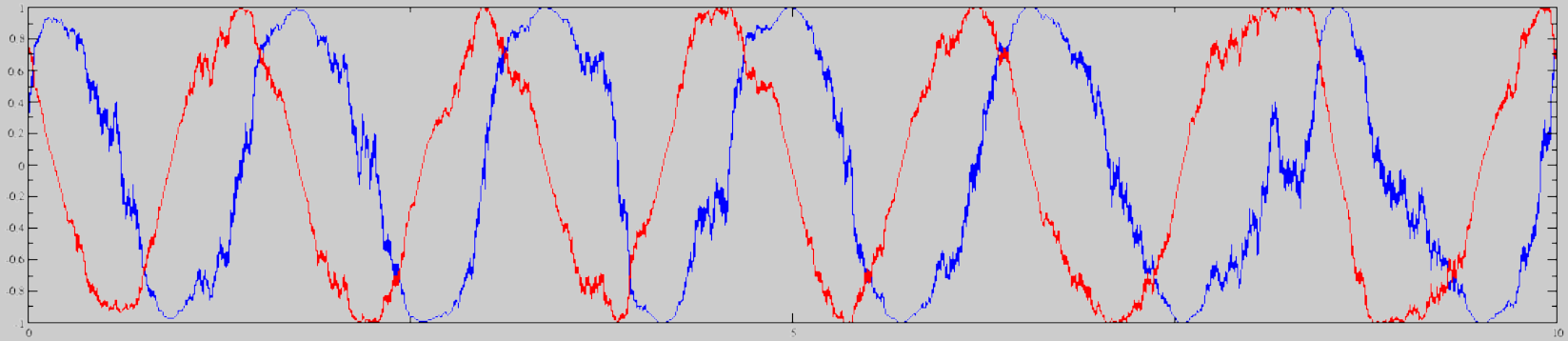
Rabi Oscillations plus Measurement

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= UX_t\,dt - \gamma(1 - Z_t^2)\,dB_t \\
dX_t &= -UZ_t\,dt - \frac{\gamma^2}{2}X_t\,dt + \gamma X_t Z_t\,dB_t
\end{aligned}
$$

A plot of Z_t and X_t, small γ



- Small deformation of Rabi oscillations, purification

Mathematical Model

Measurement

$$
\begin{aligned}
dZ_t &= UX_t\,dt - \gamma(1 - Z_t^2)\,dB_t \\
dX_t &= -UZ_t\,dt - \frac{\gamma^2}{2}X_t\,dt + \gamma X_t Z_t\,dB_t
\end{aligned}
$$

- No difference with the previous system

Mathematical Model

Rabi oscillations

$$
\begin{aligned}
dZ_t &= UX_t - \gamma(1 - Z_t^2)\, dB_t \\
dX_t &= -UZ_t\, dt - \frac{\gamma^2}{2} X_t\, dt + \gamma X_t Z_t\, dB_t
\end{aligned}
$$

- Of the form $d\rho_t = -i[H, \rho_t]\, dt$ with $H := \begin{pmatrix} 0 & -iU \\ iU & 0 \end{pmatrix}$

Mathematical Model

Quantum Zeno Effect

$$
\begin{aligned}
dZ_t &= UX_t\,dt - \gamma(1 - Z_t^2)\,dB_t \\
dX_t &= -UZ_t\,dt - \frac{\gamma^2}{2}X_t\,dt + \gamma X_t Z_t\,dB_t
\end{aligned}
$$

- In the large γ limit, complete freezing of the dynamics
  - Seen in explicit divergences of transition times
- Need to rescale U with γ to get a limit

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

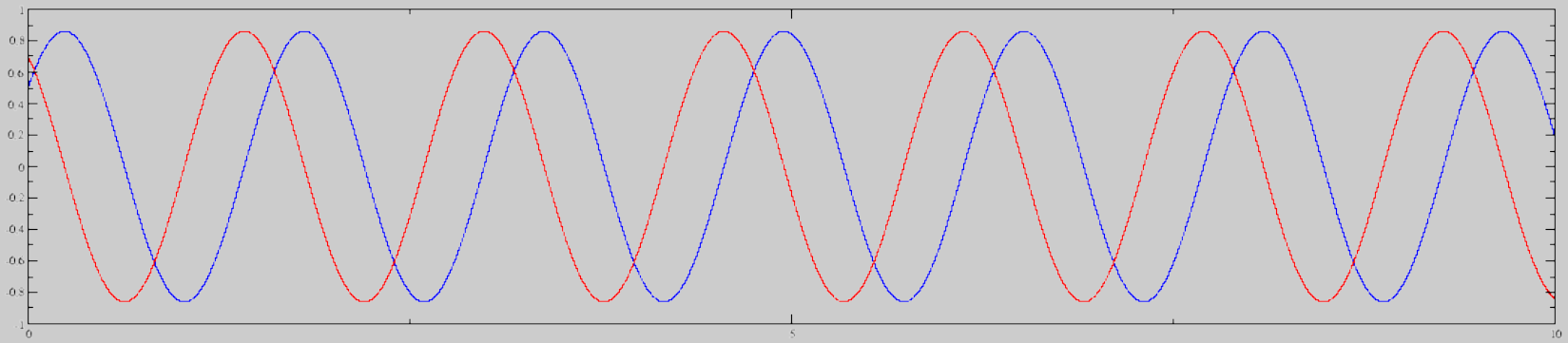
- In the large γ limit, complete freezing of the dynamics
- Need to rescale U with γ to get a limit
  - Set $U := u\gamma$, $u > 0$ fixed as $\gamma \to \infty$

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

A plot of Z_t and X_t, $\gamma = 0$ U finite



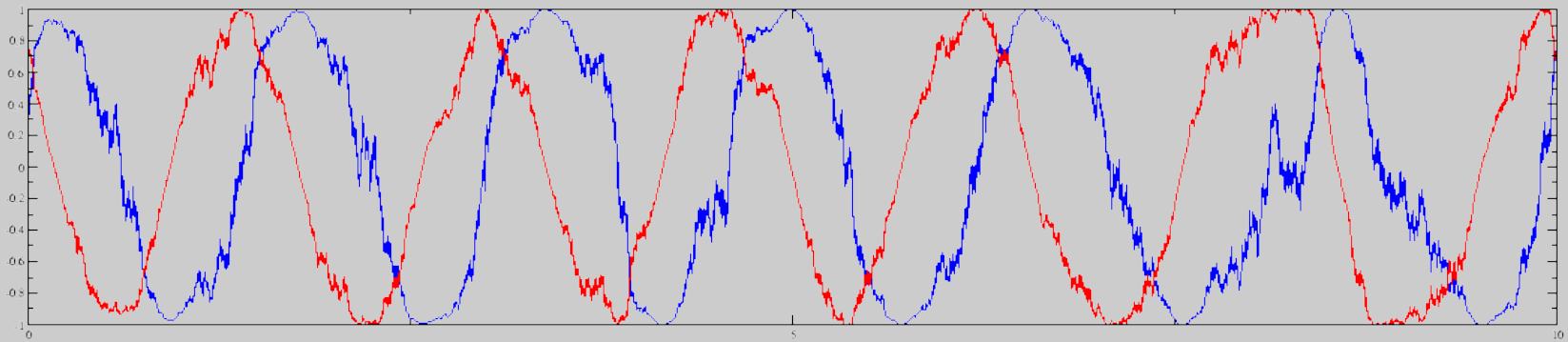
- Rabi oscillations

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

A plot of Z_t and X_t, small γ, u fixed



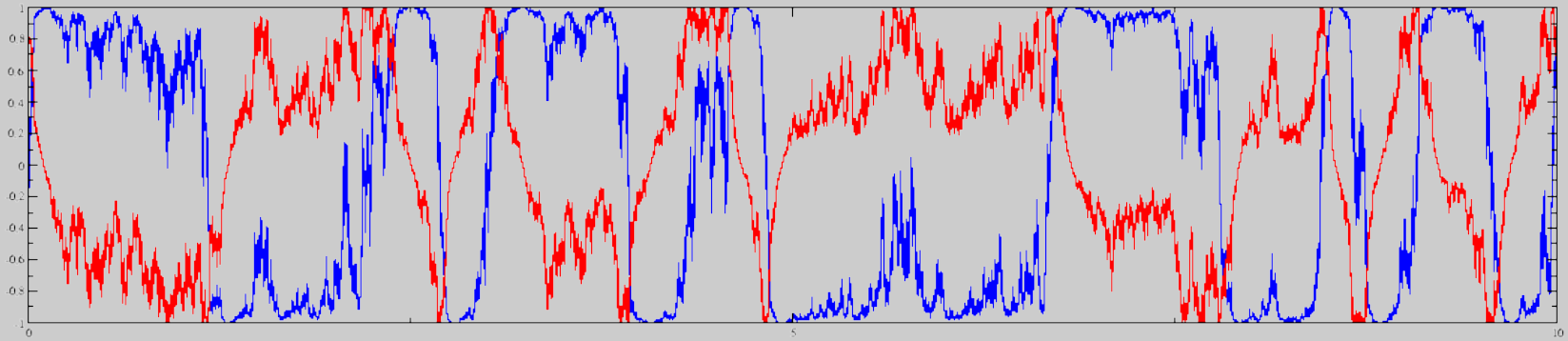
- Small deformation of Rabi oscillations

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

A plot of Z_t and X_t, larger γ, u fixed



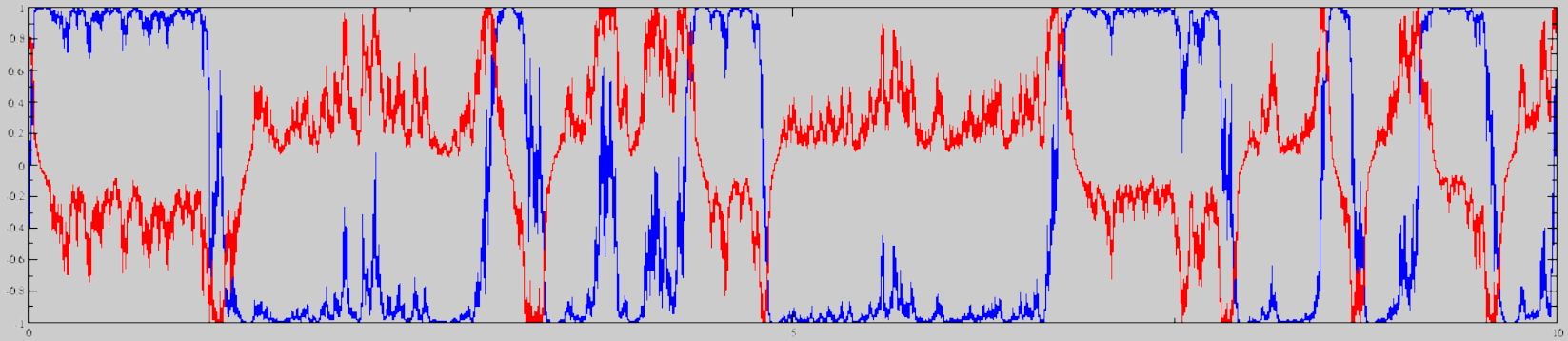
- Rabi oscillations fade away

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

A plot of Z_t and X_t, still larger γ, u fixed



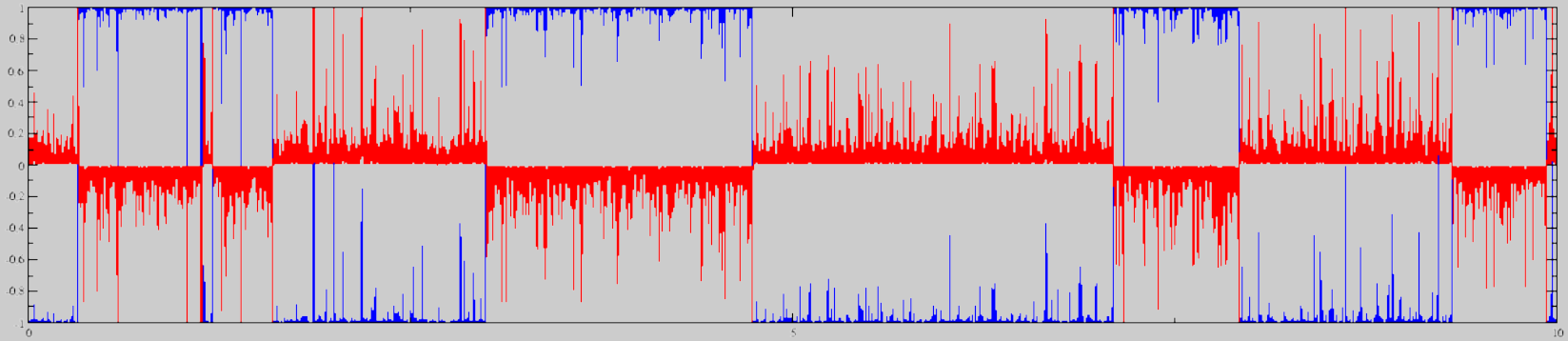
- Rabi oscillations fade away

Mathematical Model

Basic SDE

$$
\begin{aligned}
dZ_t &= \gamma u X_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{aligned}
$$

A plot of Z_t and X_t, large γ, u fixed



- Jumps and spikes again

Jumps

Strategy

- The Markov kernel of the measurement part $dZ_t = -\gamma(1 - Z_t^2)\, dB_t$ can be computed explicitly
- At large γ treat the thermal noise part as a perturbation
- Additional difficulty : X_t cannot be left aside

Theorem

- In the limit $\gamma \to \infty$, the finite dimensional distributions of $Q_t = \frac{1}{2}(1 + Z_t)$ converge weakly (i.e. in law) towards those of a finite state Markov process with states 0 ($Q \simeq 1$) and 1 ($Q \simeq 0$) with Markov generator

$$\begin{pmatrix} -u^2 & u^2 \\ u^2 & -u^2 \end{pmatrix}$$

The Markov Matrix, Quick and Dirty

- Rescale $X_t := K_t/\gamma$

$$\begin{aligned} dZ_t &= uK_t\,dt - \gamma(1 - Z_t^2)\,dB_t \\ dK_t &= -\gamma^2\left(uZ_t + \frac{1}{2}K_t\right)\,dt + \gamma K_t Z_t\,dB_t \end{aligned}$$

- Take expectations (remember $Q_t = \frac{1}{2}(1 + Z_t)$)

$$\begin{aligned} d\mathbb{E}\left(Q_t\right) &= \frac{u}{2}\mathbb{E}\left(K_t\right)\,dt \\ d\mathbb{E}\left(K_t\right) &= -\gamma^2\left(u(2\mathbb{E}\left(Q_t\right) - 1) + \frac{1}{2}\mathbb{E}\left(K_t\right)\right)\,dt \end{aligned}$$

- For large γ set $u(2\mathbb{E}\left(Q_t\right) - 1) + \frac{1}{2}\mathbb{E}\left(K_t\right) = 0$ to get correct master equation

$$d\mathbb{E}\left(Q_t\right) = u^2(1 - 2\mathbb{E}\left(Q_t\right))\,dt$$

Large γ Behavior for K_t

- Depending whether $Z_t = \pm 1$, K_t is distributed according to the stationary measure of
$$dK_t = -\gamma^2(\frac{1}{2}K_t \pm u)\,dt \pm \gamma K_t\,dB_t$$

- Using $s = \gamma^2 t$ as time, $W_s := \gamma dB_t$ is a standard Brownian, and
$$dK_s = -(\frac{1}{2}K_t \pm u)\,ds \pm K_s\,dW_s$$

- Brownian representation
$$K_\infty \overset{Law}{=} \mp u \int_0^{+\infty} ds\, e^{\pm \overline{B}_s - s}$$

- Law has large tail, density:
$$\mu_\pm(k) := \mp 4u^2\; e^{\pm u/k}\; \frac{1}{k^3}\; 1_{\mp k > 0}$$

Spikes

Strategy

- The strategy remains the same

Theorem

- One can reconstruct the process in the limit $\gamma \to \infty$ and in law from two time-homogeneous space-time Poisson point processes $\mathbb{P}\mathrm{ois}_0$ and $\mathbb{P}\mathrm{ois}_1$ on $[0,1] \times [0,+\infty]$.
  - For instance, the density of $\mathbb{P}\mathrm{ois}_0$ is:

$$d\nu_0 := \left(\delta(1-q)dq + \frac{dq}{q^2}\right) u^2 dt$$

Remarks

- The space factor is unchanged
- The time factor is dictated by the finite state Markov process jump rates

A Glance at the General Case

The General Case(1)

Starting point

$$d\rho_t = (-i[H,\rho_t] + \sum_a \mathcal{L}_{B_a}(\rho_t) + \sum_b \mathcal{L}_{N_b}(\rho_t))dt + \sum_b \mathcal{Q}_{N_b}(\rho_t)dW_t^{(b)}$$

- $\mathcal{L}_{N_b}$ quadratic and $\mathcal{Q}_{N_b}$ linear in N_b
- Non demolition : N_bs are diagonal in the pointer state basis

Strong measurement regime

- $N_b \to \gamma N_b$, large γ limit

The General Case(2)

Strong measurement regime

- Zeno freezing
  - The appropriate rescalings in H and some pieces of the B_as are understood
- Jumps
  - In the large γ limit, convergence of f.d.d. to a Markov process whose states are the pointer states
  - Explicit formula for the Markov transition kernel $M_{\alpha,\beta}$
- Spikes
  - Spikes are conjectured to occur, involving mixtures between two pointer states
  - Spikes from α to β are described by a Poisson process with measure

$$d\nu_{\alpha,\beta} := \left(\delta(1-q)dq + \frac{dq}{q^2}\right) M_{\alpha,\beta}\, dt$$

Conclusions

Conclusions

Iterated non demolition measurements

- Thorough understanding of asymptotic equivalence with standard Von Neumann measurements
- Standard mathematical tools (martingales, decomposition in extremal measures)
- Puzzling connections with De Finetti's theory and Sanov's large deviation theorem

Conclusions

Dynamics observed by iterated non demolition measurements

- Thorough understanding (jump, spikes) in two-levels systems
- For general systems:
  - Jumps are inherent to the strong continuous measurement regime
    - Well-understood and well-controlled finite state Markov processes
  - Spikes are present and conjectured to be described in terms of universal scale invariant Poisson processes
- Standard weak convergence theorems do not apply
  - Even the right space to formulate appropriate weak convergence is unknown

Conclusions

- Bohr and the other fathers of quantum mechanics would be astonished by today's experiments
  - Fast electronics and low temperature mastery allow to understand in detail
    - Simple quantum systems
    - Fundamental predictions of quantum mechanics
  - Jumps are observed daily in laboratories
    - Are quantitative aspects of spikes accessible to experiments ?
  - The hunt for quantum computing building blocks goes on