Quantum Trajectories, Quantum Jumps, and Classical Probability

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Joint work with Denis Bernard and Antoine Tilloy

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Introduction
1913: Bohr on Quantum Jumps

On the constitution of atoms and molecules

• ... the emission lines correspond to a radiation emitted during the passing of the system between two different stationary states ...
1990’s: Quantum Jumps Observed

- First (clean) observations of quantum jumps
- Fluorescence monitoring and photon counting

![Graph showing fluorescence photon counts over time](image)

FIG. 2. A typical trace of the 493-nm fluorescence from the $6^2 P_{1/2}$ level showing the quantum jumps after the hollow cathode lamp is turned on. The atom is definitely known to be in the shelf level during the low fluorescence periods.
2000’s: Quantum Jumps and Repeated Measurements

- Birth and death of a photon in a cavity
  - S. Gleyzes et al (including S. Haroche), 446 (2007) 297-300, Quantum jumps of light recording ...

- A model of thermalization observed by a non-demolition measurement
2000’s: Quantum Jumps and Repeated Measurements

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- A model of thermalization observed by a non-demolition measurement
Aims and Miscellanies

• Aims:
  • Describe some of the tools involved in the manipulation of simple quantum systems
    • In particular the observation of quantum jumps
  • Describe the mathematical framework in discrete and in continuous time
    • In particular how jumps appear and are characterized

• Results alluded to in this presentation:
  • All obtained in collaboration with Tristan Benoist, Denis Bernard, and Antoine Tilloy
Markovian Open Quantum Systems
Indirect Measurements
Indirect measurements (1)

- For a compound system $C = A \cup B$ with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$
  - Learn something on $A$...
  - ... by measuring on $B$.
- In this context, $A$ is called the system and $B$ is called the probe.
Indirect measurements(2)

• Thus measure an observable $\Lambda = \text{Id}_A \otimes \Lambda_B$ that does nothing on $A$.
  • If $|\phi\rangle_C = |\varphi\rangle_A \otimes |\psi\rangle_B$ (resp. $\rho_C = \rho_A \otimes \rho_B$) is a pure tensor product just before the measurement...
  • ... then it is $|\varphi\rangle_A \otimes |\psi'\rangle_B$ (resp. $\rho_A \otimes \rho'_B$) just after the measurement.

• But
  • If $|\Phi\rangle_C$ (resp. $\rho_C$) is not a pure tensor product just before the measurement...
  • ...something really happens to $A$ in the measurement process.

• This is due to entanglement.
Indirect measurements (3)

- **Common protocol:**
  - Start from tensor product state $|\Phi\rangle_C = |\varphi\rangle_A \otimes |\psi\rangle_B$ (resp. $\rho_C = \rho_A \otimes \rho_B$).
  - **System-probe** interaction goes on for a while, $|\Phi\rangle_C \rightarrow U|\Phi\rangle_C$ (resp. $\rho_C \rightarrow U\rho_C U^{-1}$)
    - The state is not a pure tensor product anymore.
  - Measure the probe.

- For an indirect measurement, it is natural to view the **system-probe** interaction as part of the measurement process.
• This is the abstract setting of the real experiment.

• Rydberg atoms are sent one after the other to interact with the photons in the cavity.
Repeated indirect measurements

Generic protocol

- Make several indirect measurements in a row, for a given system but for different probes.

Idealizations

- Assume the probes do not interact with each other.
- Assume the system-probe interaction time is finite.
Repeated indirect measurements

Mathematical setting

- Thus $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_{p1} \otimes \cdots \otimes \mathcal{H}_{pn}$.
- Time evolution during the interaction with probe $k$ is given by $U_k$, acting as $U$ on $\mathcal{H}_s \otimes \mathcal{H}_{pk}$ and doing nothing to the other probes.
- There is a sequence $i_1, \cdots, i_n$ of outcomes of the probe measurements (assuming that $\Lambda_p = \sum_{i \in E} \lambda_i |i\rangle\langle i|$, all $\lambda_i$ distinct).
Dynamical Equations
Discrete Time Equations (1)

• Iteration of the random dynamical system

\[ \rho \rightarrow \rho' := \frac{\sum_{i \in I_r} A_i \rho A_i^\dagger}{\pi_r} \text{ with proba } \pi_r := \sum_{i \in I_r} \text{Tr}_{\mathcal{H}_s} A_i \rho A_i^\dagger \]

- \( \rho \) is the density matrix of the system
- \( A_i, i \in I \) are such that \( \sum_i A_i^\dagger A_i = \text{Id}_{\mathcal{H}_s} \)
- \( I = \bigcup_r I_r \) is a partition

• Any family \( A_i \) can be realized as

\[ A_i := \langle i | U | \psi \rangle \]

for some appropriate probe Hilbert space \( \mathcal{H}_p \), some unitary evolution \( U \) on \( \mathcal{H}_s \otimes \mathcal{H}_p \), some orthonormal basis \( | i \rangle \) in \( \mathcal{H}_p \) and some fixed state \( | \psi \rangle \) in \( \mathcal{H}_p \).
Discrete Time Equations (2)

Extreme cases

- Ideal indirect measurement: (of a non-degenerate observable on $\mathcal{H}_p$ projecting on the basis $|i\rangle$)

  $$\rho \rightarrow \rho' := \frac{A_i \rho A_i^\dagger}{p_i} \text{ with proba } \pi_i := \text{Tr}_{\mathcal{H}_s} A_i \rho A_i^\dagger$$

- No reading at all of the measurement outcome:

  $$\rho \rightarrow \rho' := \sum_{i \in I} A_i \rho A_i^\dagger$$

  - Describes also the Markovian limit of interaction with an environment (partial trace on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_p$)

    $$\sum_{i \in I} A_i \rho A_i^\dagger = \text{Tr}_{\mathcal{H}_p} U \rho U^\dagger$$
Continuous Time Limit Equations: Barchielli, Belavkin ... Pellegrini

\[
d\rho_t = (-i[H, \rho_t] + \sum_a \mathcal{L}_{B_a}(\rho_t) + \sum_b \mathcal{L}_{N_b}(\rho_t)) \, dt + \sum_b \mathcal{Q}_b(\rho_t) \, dW_t^{(b)}
\]

- \(dt\): general Lindbladian, \(H\) is a self-adjoint operator (Hamiltonian), the \(B_a\)s and \(N_b\)s are arbitrary operators on \(\mathcal{H}_s\)
  - \(\mathcal{L}_O(\rho) := O\rho O^\dagger - \frac{1}{2}(O^\dagger O\rho + \rho O^\dagger O)\)
- \(dW_t^{(b)}\): stochastic innovation term, the \(W_t^{(b)}\)s are centered continuous Gaussian Markov processes with independent increments and covariance
  - \(dW_t^{(b)} dW_t^{(b')} = dt(\delta^{b,b'} - \sqrt{p_b p_{b'}})\) \(\quad (\sum_b p_b = 1)\)
  - \(Q_O(\rho) := O\rho + \rho O^\dagger - \rho \text{Tr}_{\mathcal{H}_s}(O\rho + \rho O^\dagger)\) (non-linear term)

Quantum trajectories and quantum jumps
Non Demolition Experiments
Non Demolition Experiments (1)

**Definition**

- Time evolution during the interaction with probe $k$ is given by $U_k$, acting as $U$ on $\mathcal{H}_s \otimes \mathcal{H}_{p^k}$ and doing nothing to the other probes.
- A **Non Demolition** experiment is when the $U_k$’s commute.

**Consequence**

- There is an orthonormal basis $|\alpha\rangle$ (pointer states) in $\mathcal{H}_s$ such that
  \[
  U = \sum_{\alpha} |\alpha\rangle \langle \alpha | \otimes U_\alpha.
  \]
- In the pointer basis, the operators $A_i$ are diagonal.
  \[
  (A_i)_{\alpha\beta} = \delta_{\alpha\beta} c(i|\alpha)
  \]
Each probe (a Rydberg atom) behaves as a two-level system.

The preferred basis is that of photon number $n = 0, 1, \ldots$

In an appropriate basis, $U_n = e^{i\theta n \sigma_z}$ where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Due to the value of $\theta$, $U_n$ is periodic modulo 8.

The probe observable $\Lambda$ is a Pauli matrix along some axis perpendicular to the $z$ axis.
Non demolition dynamics

• In the pointer basis, iteration of:

\[ \rho'_\alpha \beta = \rho_{\alpha \beta} \frac{c(i|\alpha)c(i|\beta)}{\sum_\gamma \rho_{\gamma \gamma}|c(i|\gamma)|^2} \] with proba \[ \sum_\gamma \rho_{\gamma \gamma}|c(i|\gamma)|^2 \]

• For each \( \alpha \), \( p(i|\alpha) := |c(i|\alpha)|^2 \) is a probability measure on probe measurement outcomes.

  • The measurement is called non-degenerate if the \( p(\cdot|\alpha) \) are distinct for different \( \alpha \)s, i.e. if measurements discriminate the different \( \alpha \)s (assumed in what follows)

Consequence

• The (non)diagonal elements of \( \rho \) are (super)martingales
### Non Demolition Experiments (3)

#### Von Neumann equivalence
- At large times (i.e., after many iterates) $\rho_n$ converges to a projector on some pointer $|\Gamma\rangle\langle\Gamma|$
  - Beware that $\Gamma$ is random (i.e., depends on the experiment)
  - Convergence is exponential, rates given by relative entropies
- The probability that $\rho_n$ ends in $|\gamma\rangle\langle\gamma|$ is $P(\Gamma = \gamma) = \langle\gamma|\rho_0|\gamma\rangle$.

#### Reading the outcome
- The **asymptotic frequency** of outcome $i$ in a given experiment is $p(i|\Gamma)$

#### Holography
- As the sequence of probe outcomes $i_1, i_2, \cdots$ is exchangeable (the nondemolition condition) any (infinite, very large) subsequence $i_{n_1}, i_{n_2}, \cdots$ allows to recover $\Gamma$. 

**Quantum trajectories and quantum jumps**

Sec. 2: Markovian Open Quantum Systems
Outlook

- Iterated non demolition measurements are a subtle tool to implement standard measurements on a fragile quantum system.

- Aims in what follows:
  - Use probes coupled with a non demolition interaction to a system whose intrinsic time evolution does not preserve pointer states.
  - Study the strong measurement regime, when time between probes is small with respect to the time scales of the system (in this regime, asymptotic holography holds).
  - Make contact with some real experiments.
Quantum Jumps and Spikes
Two-levels Systems: $\dim \mathcal{H}_s = 2$

- The general 2 by 2 density matrix is
  \[
  \rho = \frac{1}{2} \begin{pmatrix} 1 + Z & X - iY \\ X + iY & 1 - Z \end{pmatrix}
  \]
  \[X^2 + Y^2 + Z^2 \leq 1\]

- The Bloch sphere

- Our illustrations involve real 2 by 2 density matrices
  \[
  \rho = \frac{1}{2} \begin{pmatrix} 1 + Z & X \\ X & 1 - Z \end{pmatrix}
  \]
  \[X^2 + Z^2 \leq 1\]

- The Bloch disk $Z^2 + X^2 \leq 1$ bounded by the Bloch circle

- Set $Q =: (1 + Z)/2$
Thermal Noise plus Measurement
The Experiment

• Birth and death of a photon in a cavity
  • S. Gleyzes et al (including S. Haroche), 446 (2007) 297-300, Quantum jumps of light recording ...

• A model of thermalization observed by a non-demolition measurement
The Experiment

- Birth and death of a photon in a cavity
  - S. Gleyzes et al (including S. Haroche), 446 (2007) 297-300,
    Quantum jumps of light recording ...

- A model of thermalization observed by a non-demolition measurement
The setting is very much the same as before but:

- Due to the value of $\theta$, $U_n$ is periodic modulo 2
- The cavity is modeled by a two-level system, containing 0 photon (i.e. an even number of photons) or 1 photon (i.e. an odd number of photons)
- Thermal noise may induce transition between 0 ($Q = 1$, $Z = 1$) and 1 ($Q = 0$, $Z = -1$) photon states
Mathematical Model

Basic SDE

\[
\begin{align*}
    dZ_t &= \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \\
    dX_t &= -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}
\]

A plot of $Q_t$, small $\gamma$

- Fluctuations around the stationary limit
Mathematical Model

Thermal Noise

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

- Markovian approximation
  - No memory effects
- Diagonal elementary processes
  - In the photon number basis, the elementary processes trigger only transition from 0 to 1 photon and from 1 to 0 photon.
- Equilibrium at temperature $\beta$:
  - Energy $H := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$ and $\rho_{eq} \propto e^{-\beta H} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta \epsilon} \end{pmatrix}$
Mathematical Model

Measurement

\begin{align*}
dZ_t &= \lambda(\tanh \beta \epsilon - Z_t) \, dt - \gamma(1 - Z_t^2) \, dB_t \\
dX_t &= -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}

- Continuous time approximation
  - Each probe measurement has a small effect on the cavity
  - Time resolution large compared to lapse between two probes
- Measurement is responsible for non-linearities
- Each probe measurement can have two outcomes
  - Measurement statistics is a random walk correlated to the cavity
  - In continuous time, leads to a diffusion
- The probes couple to the photon number in the cavity
Mathematical Model

**Measurement**

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]

\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

**Convergence of** \( Q_t \) **to** 0

- Rapid convergence to 0
Mathematical Model

**Measurement**

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]

\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

**Convergence of \( Q_t \) to ?**

- Hesitations
Mathematical Model

Measurement

\[\begin{align*}
    dZ_t &= \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \\
    dX_t &= -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}\]

Convergence of \(Q_t\) to 1

- Rapid convergence to 1
Mathematical Model

Competition between thermal fluctuations and measurement

\[dZ_t = \lambda(\tanh \beta \epsilon - Z_t) \, dt - \gamma(1 - Z_t^2) \, dB_t\]

\[dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t\]

A plot of \(Q_t, \gamma = 0\)

- Convergence to the stationary limit
Mathematical Model

Competition between thermal fluctuations and measurement

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]

\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Q_t \), small \( \gamma \)

- Fluctuations around the stationary limit
Mathematical Model

Competition between thermal fluctuations and measurement

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Q_t \), moderate \( \gamma \)

- Progressive deformation of the shape
Mathematical Model

Competition between thermal fluctuations and measurement

\[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]

\[ dX_t = -\frac{\lambda}{2} X_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Q_t \), large \( \gamma \)

- Emergence of jumps and spikes
Goals

- The equation seems to account for jumps (at large $\gamma$, i.e. in the many probes per unit time limit)
  - Describe the limiting jump process
- The equations exhibits also unexpected spikes
  - Are the spikes mathematically and/or physically real?
  - If so, describe the limiting spike process
Jumps

Strategy

• The Markov kernel of the measurement part
  \[ dZ_t = -\gamma(1 - Z_t^2) dB_t \]
  can be computed explicitly

• At large \( \gamma \) treat the thermal noise part as a perturbation

Theorem

• In the limit \( \gamma \to \infty \), the finite dimensional distributions of
  \( Q_t = \frac{1}{2}(1 + Z_t) \)
  converge weakly (i.e. in law) towards those
  of a finite state Markov process with states 0 (\( Q \simeq 1 \)) and 1
  (\( Q \simeq 0 \)) with Markov generator

\[
\begin{pmatrix}
\lambda/2 & 1 + \tanh\frac{\beta\epsilon}{2} \\
1 - \tanh\frac{\beta\epsilon}{2} & -1 - \tanh\frac{\beta\epsilon}{2}
\end{pmatrix}
\]

• The Markov matrix is already apparent in the thermal noise
  part (i.e. master equation)
  \[ d\mathbb{E}(Z_t) = \lambda(\tanh \beta \epsilon - \mathbb{E}(Z_t)) dt \]
Strategy

- Let $\tau(q_i, q_f)$ be the random time it takes to go to $q_f$ starting from $q_i$
- Describe the original process in terms of $\tau(q_i, q_f)$
- Limiting law of $\tau(q_i, q_f)$ for $\gamma \to \infty$ can be computed

$$
\frac{q_i}{q_f} \delta(t) dt + \left(1 - \frac{q_i}{q_f}\right) \frac{p \lambda}{q_f} e^{-\frac{p \lambda}{q_f} t} dt \quad p := \frac{1}{2} \left(1 + \tanh \frac{\beta \epsilon}{2}\right)
$$

Theorem

- One can reconstruct the process in the limit $\gamma \to \infty$ and in law from two time-homogeneous space-time Poisson point processes $\text{Pois}_0$ and $\text{Pois}_1$ on $[0, 1] \times [0, +\infty]$.
  - For instance, the density of $\text{Pois}_0$ is:

$$
d\nu_0 := \left(\delta(1 - q) dq + \frac{dq}{q^2}\right) p \lambda dt
$$
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter $Q_0$
Reconstructing Spikes (and Jumps)

Initial condition
- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes
- $\text{Pois}_0$
Reconstructing Spikes (and Jumps)

Initial condition
- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes
- $\text{Pois}_1$
Reconstructing Spikes (and Jumps)

**Initial condition**
- Bernoulli random variable with parameter $Q_0$

**From Poisson to Spikes**
- $\text{Pois}_1$ and $\text{Pois}_1$

Quantum trajectories and quantum jumps
Reconstructing Spikes (and Jumps)

Initial condition
- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes
- Spike process from $\text{Pois}_1$ and $\text{Pois}_1$
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes

- Spike process
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes

- More points
Reconstructing Spikes (and Jumps)

Initial condition

- Bernoulli random variable with parameter $Q_0$

From Poisson to Spikes

- More points
Reconstructing Spikes (and Jumps)

Initial condition
- Bernoulli random variable with parameter $Q_0$

Original process at large $\gamma$
- More points
The Status of Spikes

Mathematical status

- Spikes are predicted by both the discrete and the continuous time model

Physical status

- Experiments are not yet precise enough to see spikes (but they should be there)
- Spikes do not have an unavoidable quantum origin ...
  - The equation
    \[ dZ_t = \lambda (\tanh \beta \epsilon - Z_t) \, dt - \gamma (1 - Z_t^2) \, dB_t \]
    also describes a cavity jumping between the 0 and the 1 photon states according to a thermal Markovian law, as observed by a fuzzy but purely classical (no disturbance of the cavity) repeated measurement
    ... and possibly no physical reality
To Summarize

- The mathematical model for thermal fluctuations observed by repeated non-demolition measurements
  - Accounts for jumps
  - Predicts (unexpected?) spikes
- Jumps are described by a finite state Markov process whose Markov matrix can be read on the averaged equations of motion
- Spikes are described by Poisson point processes, and are aborted jumps
  - Spikes are scale invariant \( dq/q^2 \) at small \( q \)

What about other systems?
- A simple possibility is to replace the thermal noise by a Hamiltonian evolution
Rabi Oscillations plus Measurement
Mathematical Model

Basic SDE

\[ dZ_t = UX_t \, dt - \gamma(1 - Z_t^2) \, dB_t \]
\[ dX_t = -UZ_t \, dt - \frac{\gamma^2}{2}X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Z_t \) and \( X_t \), small \( \gamma \)

- Small deformation of Rabi oscillations, purification
Mathematical Model

\[ dZ_t = UX_t \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -UZ_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

- No difference with the previous system
Mathematical Model

Rabi oscillations

\[
\begin{align*}
\frac{dZ_t}{dt} &= UX_t - \gamma(1 - Z_t^2) \, dB_t \\
\frac{dX_t}{dt} &= -UZ_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}
\]

- Of the form \( d\rho_t = -i[H, \rho_t] \, dt \) with 
\[
H := \begin{pmatrix}
0 & -iU \\
iU & 0
\end{pmatrix}
\]
Mathematical Model

Quantum Zeno Effect

\[ \begin{align*}
\frac{dZ_t}{dt} &= UX_t dt - \gamma (1 - Z_t^2) dB_t \\
\frac{dX_t}{dt} &= -UZ_t dt - \frac{\gamma^2}{2} X_t dt + \gamma X_t Z_t dB_t
\end{align*} \]

- In the large \( \gamma \) limit, complete freezing of the dynamics
  - Seen in explicit divergences of transition times
- Need to rescale \( U \) with \( \gamma \) to get a limit
Mathematical Model

Basic SDE

\[ dZ_t = \gamma u X_t \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

- In the large \( \gamma \) limit, complete freezing of the dynamics
- Need to rescale \( U \) with \( \gamma \) to get a limit
  - Set \( U := u\gamma, \, u > 0 \) fixed as \( \gamma \to \infty \)
Mathematical Model

Basic SDE

\[ dZ_t = \gamma uX_t \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -\gamma uZ_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Z_t \) and \( X_t \), \( \gamma = 0 \) \( U \) finite

- Rabi oscillations
Mathematical Model

Basic SDE

\[
\begin{align*}
    dZ_t &= \gamma uX_t \, dt - \gamma (1 - Z_t^2) \, dB_t \\
    dX_t &= -\gamma uZ_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}
\]

A plot of \( Z_t \) and \( X_t \), small \( \gamma \), \( u \) fixed

- Small deformation of Rabi oscillations
Mathematical Model

**Basic SDE**

\[
\begin{align*}
    dZ_t & = \gamma uX_t \, dt - \gamma(1 - Z_t^2) \, dB_t \\
    dX_t & = -\gamma uZ_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t
\end{align*}
\]

A plot of $Z_t$ and $X_t$, larger $\gamma$, $u$ fixed

- Rabi oscillations fade away
Mathematical Model

Basic SDE

\[ dZ_t = \gamma u X_t \, dt - \gamma (1 - Z_t^2) \, dB_t \]

\[ dX_t = -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Z_t \) and \( X_t \), still larger \( \gamma, u \) fixed

- Rabi oscillations fade away
Mathematical Model

Basic SDE

\[ dZ_t = \gamma u X_t \, dt - \gamma (1 - Z_t^2) \, dB_t \]
\[ dX_t = -\gamma u Z_t \, dt - \frac{\gamma^2}{2} X_t \, dt + \gamma X_t Z_t \, dB_t \]

A plot of \( Z_t \) and \( X_t \), large \( \gamma \), \( u \) fixed

- Jumps and spikes again
Strategy

• The Markov kernel of the measurement part
  \[ dZ_t = -\gamma (1 - Z_t^2) dB_t \] can be computed explicitly
• At large \( \gamma \) treat the thermal noise part as a perturbation
• Additional difficulty: \( X_t \) cannot be left aside

Theorem

• In the limit \( \gamma \to \infty \), the finite dimensional distributions of
  \[ Q_t = \frac{1}{2} (1 + Z_t) \] converge weakly (i.e. in law) towards those
  of a finite state Markov process with states 0 (\( Q \simeq 1 \)) and 1
  (\( Q \simeq 0 \)) with Markov generator

\[
\begin{pmatrix}
  -u^2 & u^2 \\
  u^2 & -u^2
\end{pmatrix}
\]
The Markov Matrix, Quick and Dirty

- Rescale $X_t := K_t / \gamma$

  $$
  dZ_t = uK_t \, dt - \gamma (1 - Z_t^2) \, dB_t
  $$

  $$
  dK_t = -\gamma^2 \left( uZ_t + \frac{1}{2} K_t \right) \, dt + \gamma K_t Z_t \, dB_t
  $$

- Take expectations (remember $Q_t = \frac{1}{2} (1 + Z_t)$)

  $$
  d\mathbb{E} (Q_t) = \frac{u}{2} \mathbb{E} (K_t) \, dt
  $$

  $$
  d\mathbb{E} (K_t) = -\gamma^2 \left( u(2\mathbb{E} (Q_t) - 1) + \frac{1}{2} \mathbb{E} (K_t) \right) \, dt
  $$

- For large $\gamma$ set $u(2\mathbb{E} (Q_t) - 1) + \frac{1}{2} \mathbb{E} (K_t) = 0$ to get correct master equation

  $$
  d\mathbb{E} (Q_t) = u^2 (1 - 2\mathbb{E} (Q_t)) \, dt
  $$
Large $\gamma$ Behavior for $K_t$

- Depending whether $Z_t = \pm 1$, $K_t$ is distributed according to the stationary measure of
  $$dK_t = -\gamma^2\left(\frac{1}{2}K_t \pm u\right) dt \pm \gamma K_t dB_t$$

- Using $s = \gamma^2 t$ as time, $W_s := \gamma dB_t$ is a standard Brownian, and
  $$dK_s = -\left(\frac{1}{2}K_t \pm u\right) ds \pm K_s dW_s$$

- Brownian representation
  $$K_\infty \overset{Law}{=} \pm u \int_0^\infty ds e^{\pm B_s - s}$$

- Law has large tail, density:
  $$\mu_{\pm}(k) := \pm 4u^2 e^{\pm u/k} \frac{1}{k^3} 1_{\mp k > 0}$$
Spikes

**Strategy**

- The strategy remains the same

**Theorem**

- One can reconstruct the process in the limit $\gamma \to \infty$ and in law from two time-homogeneous space-time Poisson point processes $\mathcal{Pois}_0$ and $\mathcal{Pois}_1$ on $[0, 1] \times [0, +\infty]$.
  - For instance, the density of $\mathcal{Pois}_0$ is:
    \[
d\nu_0 := \left( \delta(1 - q)dq + \frac{dq}{q^2} \right) u^2 dt
    \]

**Remarks**

- The space factor is unchanged
- The time factor is dictated by the finite state Markov process jump rates
A Glance at the General Case
The General Case (1)

Starting point

\[ d\rho_t = (-i[H, \rho_t] + \sum_a \mathcal{L}_{B_a}(\rho_t) + \sum_b \mathcal{L}_{N_b}(\rho_t)) dt + \sum_b Q_{N_b}(\rho_t) dW_t^{(b)} \]

- \( \mathcal{L}_{N_b} \) quadratic and \( Q_{N_b} \) linear in \( N_b \)
- Non demolition: \( N_b \)s are diagonal in the pointer state basis

Strong measurement regime

- \( N_b \to \gamma N_b \), large \( \gamma \) limit
The General Case(2)

Strong measurement regime

- Zeno freezing
  - The appropriate rescalings in $H$ and some pieces of the $B_a$s are understood

- Jumps
  - In the large $\gamma$ limit, convergence of f.d.d. to a Markov process whose states are the pointer states
  - Explicit formula for the Markov transition kernel $M_{\alpha, \beta}$

- Spikes
  - Spikes are conjectured to occur, involving mixtures between two pointer states
  - Spikes from $\alpha$ to $\beta$ are described by a Poisson process with measure

$$d\nu_{\alpha, \beta} := \left( \delta(1 - q)dq + \frac{dq}{q^2} \right) M_{\alpha, \beta} dt$$
Conclusions
Conclusions

Iterated non demolition measurements

- Thorough understanding of asymptotic equivalence with standard Von Neumann measurements
- Standard mathematical tools (martingales, decomposition in extremal measures)
- Puzzling connections with De Finetti’s theory and Sanov’s large deviation theorem
### Conclusions

**Dynamics observed by iterated non demolition measurements**

- Thorough understanding (jump, spikes) in two-levels systems
- For general systems:
  - Jumps are inherent to the strong continuous measurement regime
    - Well-understood and well-controlled finite state Markov processes
  - Spikes are present and **conjectured to be described in terms of universal scale invariant Poisson processes**
- **Standard weak convergence theorems do not apply**
  - Even the right space to formulate appropriate weak convergence is unknown
Conclusions

- Bohr and the other fathers of quantum mechanics would be astonished by today’s experiments
  - Fast electronics and low temperature mastery allow to understand in detail
    - Simple quantum systems
    - Fundamental predictions of quantum mechanics
  - Jumps are observed daily in laboratories
    - Are quantitative aspects of spikes accessible to experiments?
  - The hunt for quantum computing building blocks goes on