Compact objects in modified gravity

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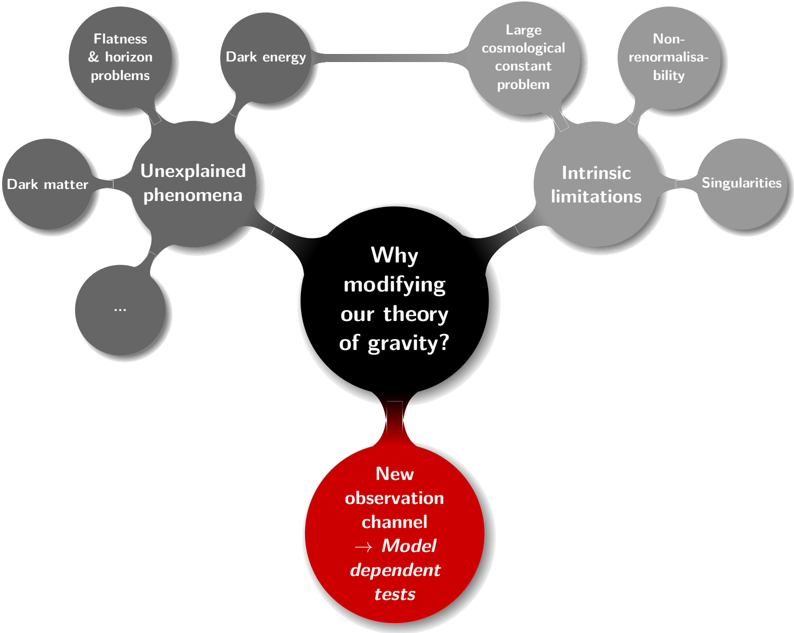
Institut Denis Poisson, 24 March 2022

My past experience

- 2014: **Agrégation** of physics and chemistry
- 2015-2018: **PhD, LPT**, Orsay *(C. Charmousis)*
- 2018-2020: **Postdoc, Nottingham U.** *(A. Padilla, T. Sotiriou)*
- 2020-2022: **Postdoc, IST**, Lisbon *(V. Cardoso)*

My research interests

- Alternative theories of gravity (scalar-tensor, extra dimensions)
- Black hole and neutron star physics
- Theory of dark energy


Why modifying our theory of gravity?
Unexplained phenomena
Flatness & horizon problems
Dark energy
Dark matter
...
Intrinsic limitations
Large cosmological constant problem
Non-renormalisa-bility
Singularities
New observation channel → *Model dependent tests*

How to modify our theory of gravity?

Starting point: general relativity

Spatiotemporal intervals measured by

$$\mathrm{d}s^2 = \boldsymbol{g}_{\boldsymbol{\mu\nu}}(x^\rho)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Dynamical and minimally coupled metric $\boldsymbol{g}_{\boldsymbol{\mu\nu}}$:

$$S = S_{\text{gravity}}[\boldsymbol{g}, \boldsymbol{\partial g}] + S_{\text{matter}}[\boldsymbol{g}, \psi, \partial\psi]$$

$$S_{\text{gravity}} = \int \mathrm{d}^4x\sqrt{-g}\,\frac{c^4}{16\pi G}(R - 2\Lambda)$$

Einstein equations [Einstein 1915]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Anything else with just a metric?

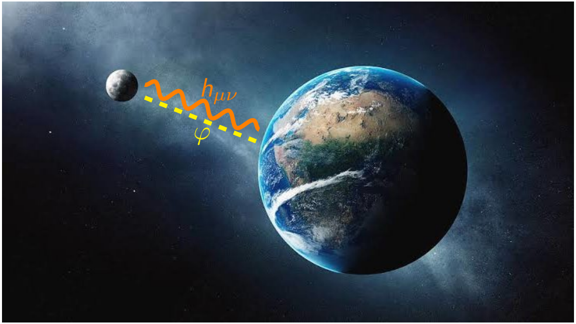
No [Lovelock '71]

In 4D, assuming a single metric degree of freedom, second-order field equations that are symmetric and divergence-less, the field equations can contain **only $G_{\mu\nu}$ and $g_{\mu\nu}$**.

What can we change then?

- New fields
- Number of dimensions
- Massive gravity
- Connection ($\neq$ Levi-Civita)
- Break Lorentz invariance
- ...

Adding a scalar degree of freedom: why?



Promoting Newton's constant to a field [Brans & Dicke '61]

$$\int \mathrm{d}^4x\sqrt{-g}\,\frac{c^4}{16\pi G}R \underset{G(x^\mu)}{\longrightarrow} \int \mathrm{d}^4x\sqrt{-g}\left[\varphi R - (\nabla\varphi)^2\right]$$

Adding a scalar degree of freedom: why?

Adding a scalar field φ

- **Simple and successful** *(inflation, quintessence, axion...)*
- **Many** alternative gravity models **related** *(massive gravity, extra dimensions, Horava gravity...)*

On the Extra Mode and Inconsistency of Hořava Gravity

D. Blas,[a] O. Pujolàs,[b] S. Sibiryakov,[a,c]

The aim of the present paper is to clarify this issue. We show that Hořava gravity does possess an additional light scalar mode. For a general background the equation of motion of

Massive gravity

From Wikipedia, the free encyclopedia

We may understand the smaller light bending as follows. The Fierz–Pauli massive graviton, due to the broken diffeomorphism invariance, propagates three extra degrees of freedom compared to the massless graviton of linearized general relativity. These three degrees of freedom package themselves into a vector field, which is irrelevant for our purposes, and a scalar field. This scalar mode exerts an extra attraction in the massive case compared to the massless case. Hence, if one wants

4D Gravity on a Brane in 5D Minkowski Space

Gia Dvali, Gregory Gabadadze, Massimo Porrati

freedom 3 of which couple to a conserved energy-momentum tensor. Thus, having the propagator as in (17) is equivalent of having a tensor-scalar gravity from 4D point of view. This extra scalar polarization degree of freedom yields additional

Kaluza–Klein theory

From Wikipedia, the free encyclopedia

his results to Einstein in 1919,[2] and published them in 1921.[3] Kaluza presented a purely classical extension of general relativity to 5D, with a metric tensor of 15 components. 10 components are identified with the 4D spacetime metric, four components with the electromagnetic vector potential, and one component with an unidentified scalar field sometimes called the "radion" or the "dilaton". Correspondingly, the 5D Einstein equations yield the 4D Einstein field equations, the

Adding a scalar degree of freedom: how?

Horndeski theory [Horndeski '74]

$$S_{\mathrm{H}} = \int \mathrm{d}^4 x \mathcal{L}_{\mathrm{H}}(g_{\mu\nu}, g_{\mu\nu,i_1}, \dots, g_{\mu\nu,i_1\dots i_p};\ \varphi, \varphi_{,i_1}, \dots, \varphi_{,i_1\dots i_q})$$

$$\Downarrow \quad \frac{1}{\sqrt{-g}} \frac{\delta\ \cdot}{\delta g_{\mu\nu}}$$

$$\mathcal{E}^{\mu\nu}(g_{\mu\nu}, g_{\mu\nu,i_1}, g_{\mu\nu,i_1,i_2};\ \varphi, \varphi_{,i_1}, \varphi_{,i_1,i_2}) = 0$$

Generically, higher than 2^{nd} order field equations ⇒ Ostrogradski ghost

Horndeski theory (aka generalized Galileons) [Horndeski '74]

$$S_{\mathrm{H}} = \int \sqrt{-g}\, d^4x\, (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)$$

$$\mathcal{L}_2 = \boldsymbol{G_2}(\varphi, X)$$

$$\mathcal{L}_3 = -\boldsymbol{G_3}(\varphi, X)\Box\varphi$$

$$\mathcal{L}_4 = \boldsymbol{G_4}(\varphi, X)R + \boldsymbol{G_{4X}}\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2\right]$$

$$\mathcal{L}_5 = \boldsymbol{G_5}(\varphi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi - \frac{\boldsymbol{G_{5X}}}{6}\left[(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3\right]$$

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$V(\varphi), (\nabla\varphi)^2, \Lambda \subset \quad \mathcal{L}_2 = \boldsymbol{G_2}(\varphi, X)$

DGP term $\Box\varphi(\nabla\varphi)^2 \subset \quad \mathcal{L}_3 = -\boldsymbol{G_3}(\varphi, X)\Box\varphi$

Ricci scalar $\subset \quad \mathcal{L}_4 = \boldsymbol{G_4}(\varphi, X)R + \boldsymbol{G_{4X}}\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2\right]$

Dilaton-like term $\subset \quad \mathcal{L}_5 = \boldsymbol{G_5}(\varphi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi$

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$+$ **Beyond Horndeski/DHOST**: $S_{\rm H}[g_{\mu\nu}, \varphi] + S_{\rm m}[\boldsymbol{\tilde{g}_{\mu\nu}}, \psi]$ with
$\tilde{g}_{\mu\nu} = \boldsymbol{C}(\varphi, X)g_{\mu\nu} + \boldsymbol{D}(\varphi, X)\nabla_\mu\varphi\nabla_\nu\varphi$

No-hair theorems

Two kinds: **uniqueness** theorems & **no scalar hair** theorems

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Uniqueness theorems

[Israel, Carter, Robinson, Hawking... '67-'72]

- Metric $g_{\mu\nu}$ alone, no matter field
- Assumptions: asymptotic flatness, weak energy condition, stationarity
- **Only Kerr black holes**, parametrized by M and J

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- Metric $g_{\mu\nu}$ alone, no matter field
- Assumptions: asymptotic flatness, weak energy condition, stationarity
- **Only Kerr black holes**, parametrized by $\boldsymbol{M}$ and $\boldsymbol{J}$

No scalar hair

[Bekenstein, Hawking... '68-...]

- Add A^μ: Kerr-Newman, parametrized by electric charge Q
- Add φ: **nothing!**
- Assumptions: it depends

No-hair theorem for shift-symmetric Horndeski models

Assumptions

1. Spherical symmetry and staticity of φ and $g_{\mu\nu}$
2. Asymptotic flatness with constant φ at spatial infinity
3. Action possesses shift-symmetry $\phi \to \phi + C$
4. Finite norm of the associated Noether current J^2
5. Canonical kinetic term $X \subseteq G_2$, and analytic G_i functions around $\nabla_\mu \varphi = 0$

Theorem [Hui & Nicolis '13]

The solutions are identical to GR, with constant φ

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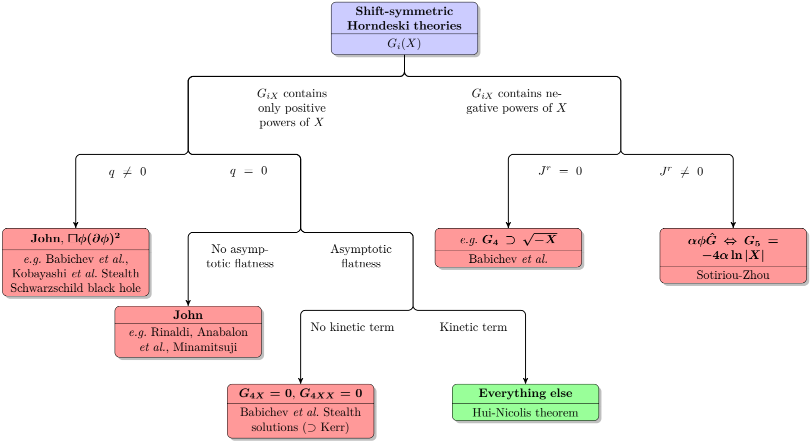
A classification of hairy solutions

[Babichev, Charmousis & AL '16]



1st example: Cosmological asymptotics

Metric and scalar field ansatz

$$\mathrm{d}s^2 = g_{tt}(r)\mathrm{d}t^2 + g_{rr}(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

$$\varphi(\boldsymbol{t},r) = \boldsymbol{qt+}\psi(r)$$

$g_{tt} \longleftrightarrow$  $g_{rr} \longleftrightarrow$ 

- Natural asymptotics if φ is a dark energy field:

$$\varphi(\tau) \underset{\tau\to\tau_0}{=} \varphi_0 + \dot{\varphi}_0(\tau-\tau_0) + \mathcal{O}[(\tau-\tau_0)^2]$$

- **Compulsory** hair

1st example: Cosmological asymptotics

Simplest quartic model [Babichev & Charmousis '14]

$$S_1 = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi}(R-2\Lambda) - (\nabla\varphi)^2 - \frac{1}{m^2}G_{\mu\nu}\nabla^\mu\varphi\,\nabla^\nu\varphi\right]$$

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- **Exact** Schwarzschild-de Sitter solutions:

$$-g_{tt} = g_{rr}^{-1} = 1 - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2$$

$$(\nabla\varphi)^2 = q^2 = \frac{\boldsymbol{m^2} - \Lambda}{16\pi}$$

de Sitter: static vs flat slicing

$$-\left(1-\frac{\Lambda r^2}{3}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{\Lambda r^2}{3}} + r^2\mathrm{d}\Omega^2 = -\mathrm{d}\tau^2 + e^{\sqrt{3\Lambda}\tau}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2\right)$$

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- **Exact** Schwarzschild-de Sitter solutions
- Pass **local** geometric **tests**
- **Self-tuning**: $\Lambda_{\text{eff}} = m^2$ independent of Λ
- Invoked to alleviate the large cosmological constant problem
- **Resists phase transitions**

Caveats

- Speed of gravitational waves (can be fixed)
- Not a true solution to large CC problem ($\leftrightarrow$ stability)

2^{nd} example: 4D Einstein-Gauss-Bonnet gravity

Starting point: higher dimensional Lovelock theory

$$S_D = \int \mathrm{d}^D x \sqrt{-g^{(D)}} \left(R^{(D)} - \hat{\alpha} \mathcal{G}^{(D)} \right)$$

with $\mathcal{G}^{(D)} = R^{(D)}_{\mu\nu\rho\sigma} R^{(D)\,\mu\nu\rho\sigma} - 4 R^{(D)}_{\mu\nu} R^{(D)\,\mu\nu} + R^{(D)\,2}$

- Compactification on a $D-4$ maximally symmetric manifold
- $\hat{\alpha} = \alpha/(D-4)$ + limit $D \to 4$

4D: non-trivial scalar-tensor theory [Lu & Pang '20]

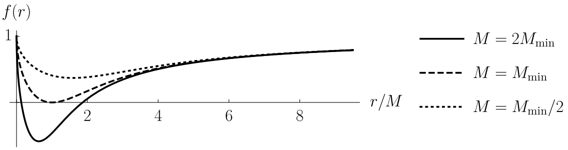
$$\int \mathrm{d}^4 x \sqrt{-g} \left\{ R + \alpha \left[\phi \mathcal{G} + 4 G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - 4 (\nabla\phi)^2 \Box\phi + 2 (\nabla\phi)^4 \right] \right\}$$

2^{nd} example: 4DEGB black holes

Exact vacuum solution

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$-g_{tt} = g_{rr}^{-1} = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M}{r^3}}\right), \quad \phi = \int \mathrm{d}r \frac{1 - \sqrt{g_{rr}}}{r}$$

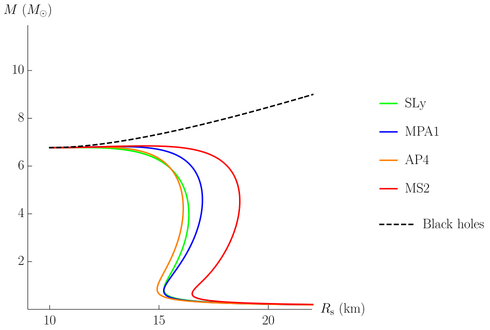


- Solar system tests: $|\alpha| < \mathbf{10^{10}\ m^2}$
  [Clifton, Carrilho, Fernandes & Mulryne '20]
- No horizon screening of small bodies: $\alpha > \mathbf{-10^{-30}\ m^2}$
  [Charmousis, AL, Smyrniotis & Stergioulas '21]

2^{nd} example: 4DEGB neutron stars

Observational constraints

- Reproduce heaviest pulsar and lightest black hole
- $\mathbf{0 \leq \alpha \lesssim 40}$ **km^2**



[Charmousis, AL, Smyrniotis & Stergioulas '21]

Remarkable property

- **Universal** point of convergence (independent of the EOS)
- Identical to the extremal black hole

Instability?

Self-tuned Schwarzschild-de Sitter solutions

$$S_1 = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{16\pi}(R - 2\Lambda) - (\partial\varphi)^2 - \frac{1}{m^2} G_{\mu\nu} \nabla^\mu \varphi \nabla^\nu \varphi \right]$$

$$-g_{tt} = g_{rr}^{-1} = 1 - \frac{2M}{r} - \frac{\Lambda_{\mathrm{eff}}}{3} r^2, \quad \varphi = qt + \psi(r)$$

Instability claim [Ogawa, Kobayashi & Suyama '16]

Arbitrarily negative Hamiltonian density

The conclusion is in fact more subtle

[Babichev, Charmousis, Esposito-Farèse & AL '18]

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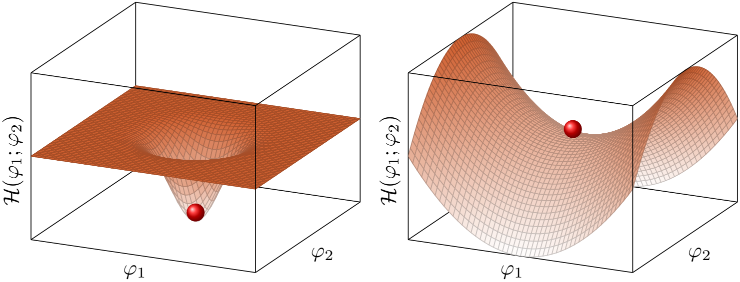
Hamiltonian and stability

Momentum p and Hamiltonian $\mathcal{H}$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \qquad \mathcal{H} = p\dot{\varphi} - \mathcal{L}$$



- Bounded $\mathcal{H}$ and energy conservation $\Rightarrow$ stability
- Unbounded $\mathcal{H}$ $\Rightarrow$ instability?

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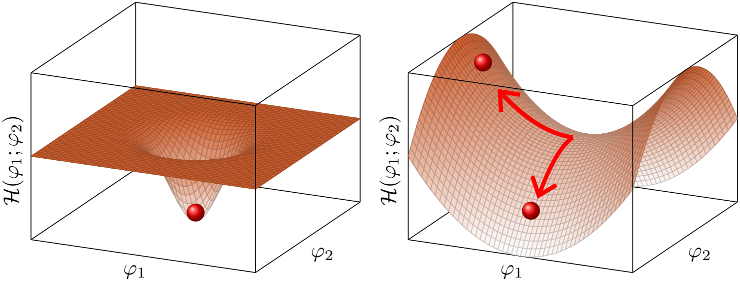

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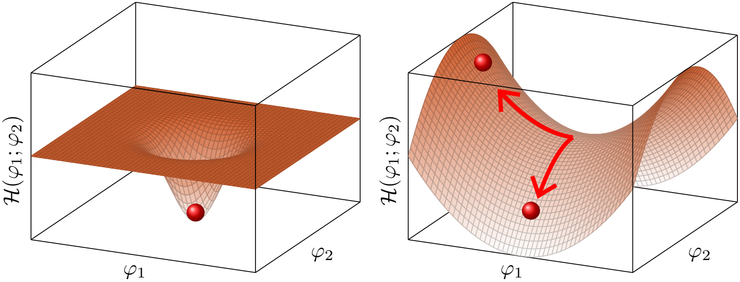

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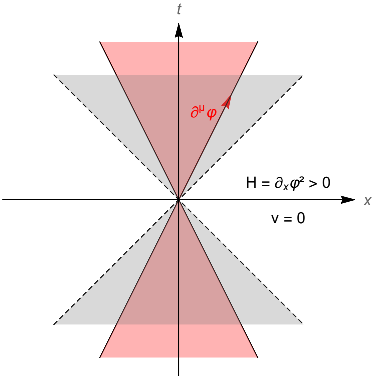
Hamiltonian and stability

Simple example

$$\mathcal{L} = -\frac{1}{2}\mathcal{S}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \text{ with } \mathcal{S}^{\mu\nu} = \begin{bmatrix} -1/c_s^2 & 0 \\ 0 & 1 \end{bmatrix}$$



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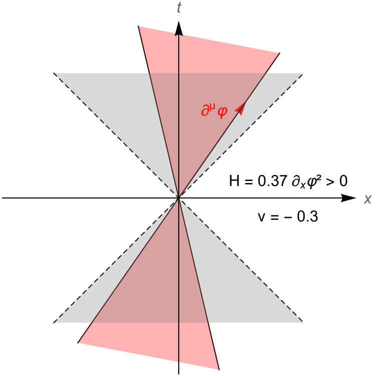

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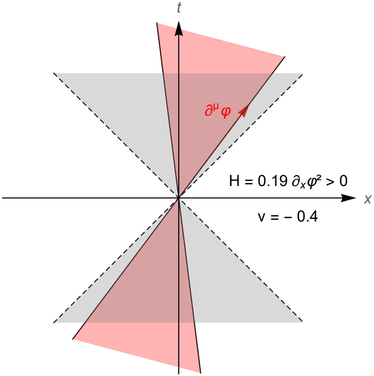

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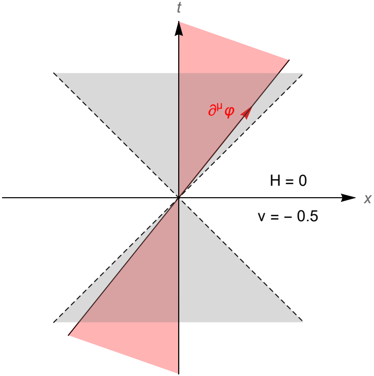

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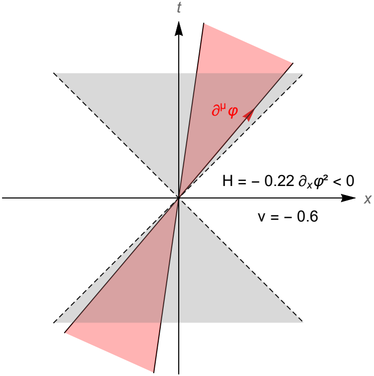
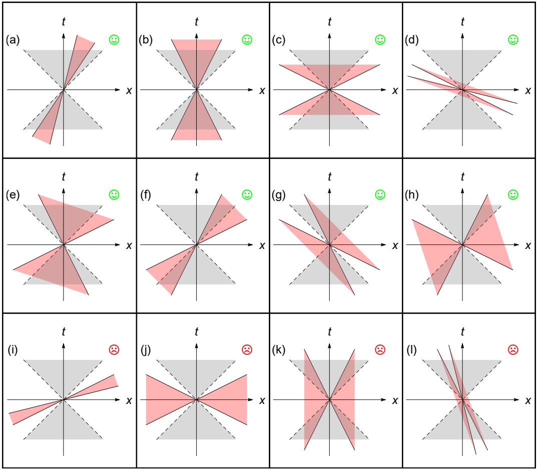


Correct stability criterion for causal cones

1. Common timelike direction
2. Common spacelike hypersurface



Application to scalar-tensor theories

Reminder of investigated solution

$$S_1 = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi}(R-2\Lambda) - (\partial\varphi)^2 - \frac{1}{m^2}G_{\mu\nu}\nabla^\mu\varphi\,\nabla^\nu\varphi\right]$$

$$-g_{tt} = g_{rr}^{-1} = 1 - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2, \quad \varphi = qt + \psi(r)$$

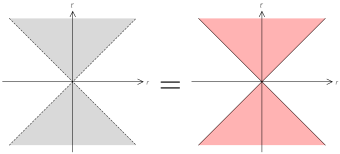
A priori, three different causal cones:

1. **Matter** causal cone (light)
2. **Gravitational** waves
3. **Scalar** waves

Mixing of space and time ⟶ similar to boosts

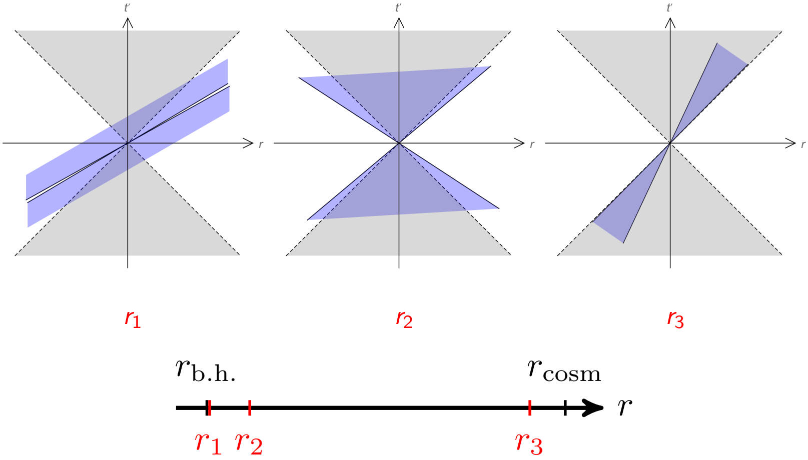
Application to scalar-tensor theories

Couple matter to $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + \frac{16\pi}{m^2 + 16\pi X} \nabla_\mu \varphi \nabla_\nu \varphi$ instead of $g_{\mu\nu}$



$c_{\text{grav}} = c_{\text{light}}$

Application to scalar-tensor theories



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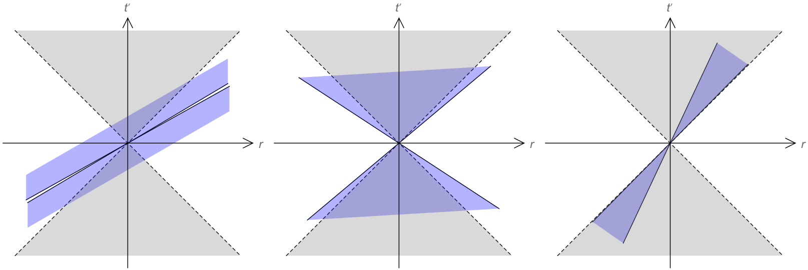
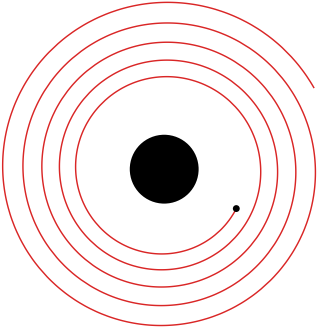


Stability window

- **Stable solutions** if $\Lambda/3 < \Lambda_{\text{eff}} < \Lambda$
- $\boldsymbol{c_{\text{grav}} = c_{\text{light}}}$ **even close to a black hole**
- $\boldsymbol{\Lambda_{\text{eff}} \simeq \Lambda}$: no resolution of large CC problem



Simple question

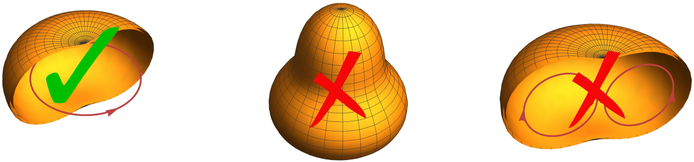
If a particle loses energy, **does it always fall** towards the center?
In **any** theory of gravity?

[AL & Cardoso '22]

Metricity ($\nabla_\rho g_{\mu\nu} = 0$) and no torsion ($\nabla_{[\mu}\nabla_{\nu]} f = 0$)

Symmetries of the spacetime

- **Stationary** & **axisymmetric** spacetime: Killing vectors ξ^μ, ψ^μ
- **Circular** spacetime $\xi^\mu R_\mu^{[\nu}\xi^\rho\psi^{\sigma]} = \psi^\mu R_\mu^{[\nu}\xi^\rho\psi^{\sigma]} = 0$
- **Equatorial symmetry**



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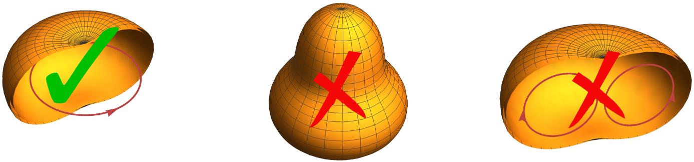


$$\mathrm{d}s^2 = g_{tt}(r,\theta)\mathrm{d}t^2 + 2g_{t\varphi}(r,\theta)\mathrm{d}t\mathrm{d}\varphi + g_{\varphi\varphi}(r,\theta)\mathrm{d}\varphi^2 + g_{rr}(r,\theta)\mathrm{d}r^2 + g_{\theta\theta}(r,\theta)\mathrm{d}\theta^2$$

Geodesics

- Timelike geodesics: $g_{\mu\nu}u^\mu u^\nu = -1$
- Conserved quantity n°1: $E = -g_{\mu\nu}u^\mu \xi^\nu$
- Conserved quantity n°2: $L = g_{\mu\nu}u^\mu \psi^\nu$

We will focus on **equatorial and circular** geodesics

$$g_{rr}\dot{r}^2 = \frac{g_{\varphi\varphi}E^2 + 2g_{t\varphi}EL + g_{tt}L^2}{g_{t\varphi}^2 - g_{\varphi\varphi}g_{tt}} - 1 \equiv -V(r, E, L)$$

Stable circular orbits

$$V(r, E, L) = 0,\ V'(r, E, L) = 0,\ V''(r, E, L) > 0$$

Newtonian case

$$E(r) = -M/(2r), \; L_\pm(r) = \pm\sqrt{Mr}$$

Generic case

$$E_\pm(r) = -\frac{g_{tt} + g_{t\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}$$

$$L_\pm(r) = \frac{g_{t\varphi} + g_{\varphi\varphi}\Omega_\pm}{\sqrt{\beta_\pm}}$$

$$\beta_\pm(r) \equiv -g_{tt} - 2g_{t\varphi}\Omega_\pm - g_{\varphi\varphi}\Omega_\pm^2$$

$$\Omega_\pm(r) \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{-g'_{t\varphi} \pm \sqrt{C}}{g'_{\varphi\varphi}}$$

Necessary conditions for existence of orbits:

$$\beta_\pm > 0, \qquad C \equiv g'^2_{t\varphi} - g'_{tt}g'_{\varphi\varphi} > 0$$

- Particle losing energy ($\boldsymbol{\delta E < 0}$) on sequence of circular orbits
- $\delta E = E'\delta r$: $\boldsymbol{E'}$ **determines whether orbits shrink or grow**

Newtonian case

$E' = M/(2r^2) > 0 \;\Rightarrow$ Orbits always shrink

Generic case

$$\boldsymbol{E' = \frac{B}{2\sqrt{C\beta_\pm}}\frac{(-g'_{tt})V''}{(\sqrt{C} \pm g'_{t\varphi})}}$$

- $B \equiv g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} > 0$ outside of a horizon
- $\sqrt{C} \pm g'_{t\varphi}$ can only change sign if g'_{tt} does

$$E' = \frac{B}{2\sqrt{C\beta_\pm}} \frac{\boldsymbol{(-g'_{tt})V''}}{(\sqrt{C} \pm g'_{t\varphi})}$$

When approaching the center,

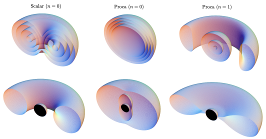
- **either** $V'' = 0$: orbits become unstable
- **or** $g'_{tt} = 0$; coincides with $\Omega_- = 0$: **static rings**

[Collodel, Kleihaus & Kunz '18]

Conclusion

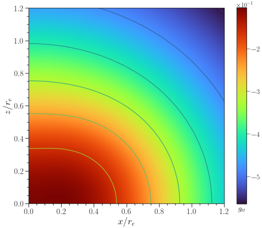
For a theory of gravity that respects the weak equivalence principle, a particle losing energy on a sequence of circular orbits will **either plunge** towards the center **or settle down at minima of the generalized Newtonian potential**

Stationary spacetimes with a local maximum of g_{tt}?



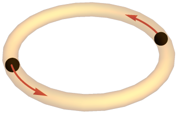
Kerr black holes with scalar/Proca hair

Fig. from [Santos, Benone, Crispino, Herdeiro & Radu '20] [Collodel, Doneva & Yazadjiev '21]



Rapidly rotating neutron stars

Plot by courtesy of Panagiotis Iosif



Time-averaged binaries?

Conclusions

- No-hair theorems can be bypassed in consistent settings
- Stable solutions
- Theorem predicting generic behavior of geodesics

Outlook

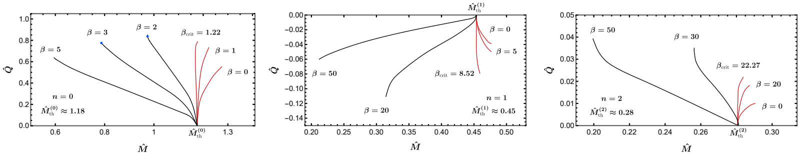
- Promising future: tests through gravitational waves (quasi-normal modes, EMRIs...)
- Dynamical regime (collapse, Cauchy problem)

Thank you for your attention!

Primary hair

New field → Solutions described by one more free parameter

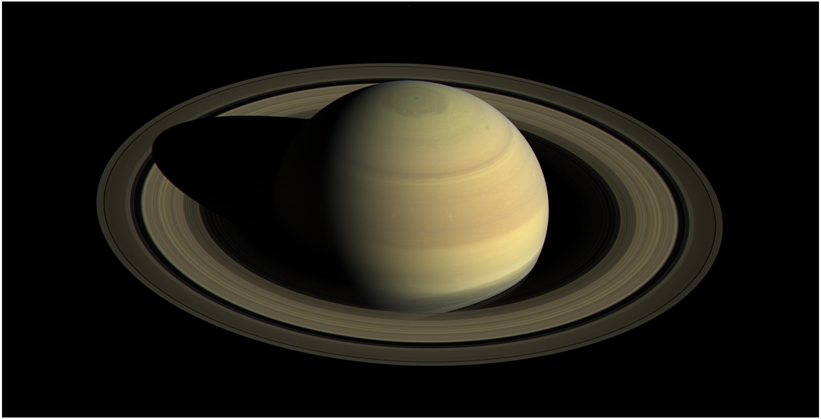
- First example: Einstein-Yang-Mills, new (discrete) parameter, counting number of nodes [Volkov & Gal'tsov '89]
- Similar type of hair for scalarization models: [Antoniou, AL, Sotiriou & Ventagli '21]



Secondary hair

New field → Solutions $\neq$ Kerr-Newman, but no new parameter

4D Einstein-Gauss-Bonnet black holes for instance



Circularizing effects

- Collisions
- Tidal heating
- Radiative effects

Very specific gravitational wave signal

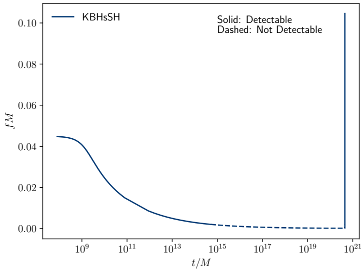
- Energy loss is due to GW emission
- Test body + quadrupolar approximation:

$$m\dot{E}_\pm = \frac{1}{5}\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}_{ij} = \frac{32}{5}m^2 r^4 \Omega_\pm^6$$

- On a static ring, $\Omega_- \to 0$ and test body will freeze

Object around a supermassive Kerr black hole with scalar hair **close to forming a static ring**

Fig. from [Collodel, Doneva & Yazadjiev '21]



What about energy injected to the test body ($\delta E > 0$)?

Internal energy injection

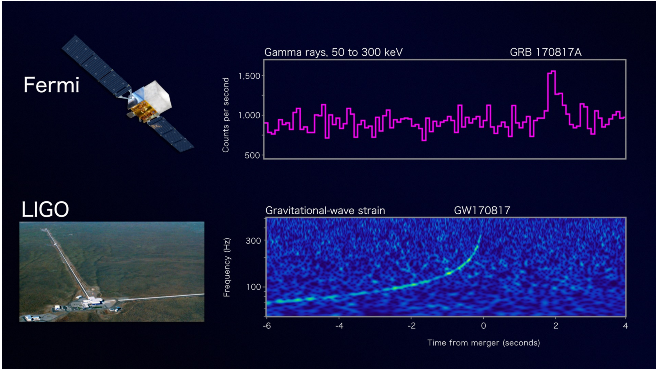
- **Tidal acceleration**: rotational energy of the central body transferred to gravitational energy of the test body
- Moon moves away from the Earth at 3.8 cm/year
- Condition: $\Omega < \Omega_{\text{central body}}$ ($\simeq$ superradiance)

External energy injection

- **GW cosmic background** [Blas & Jenkins '21]
- **Triple systems** [Bonga, Yang & Hughes '21]

Previous calculation remains predictive upon knowledge of detailed energy balance

GW170817: light and gravity have identical speeds



A further important point is that Eq. (23), a distinctive feature of two-metric theories, suggests that a search for time delays between simultaneously emitted gravitational and electromagnetic bursts could prove a valuable experimental tool. An experimental limit of $\lesssim 10^{-8}$ for $|c_g - c_{em}|/$

[Eardley, Lee & Lightman ’73]

GW170817: light and gravity have identical speeds

- $|c_{\rm grav}/c_{\rm light} - 1| < 10^{-15}$
- Caveat 1: Validity of EFT of dark energy at $f \simeq 100$ Hz?
- Caveat 2: Problem only if φ is non-trivial at cosmological level

Easy solution: disformal transformation

Couple matter to $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + \dfrac{16\pi}{m^2 + 16\pi X}\,\partial_\mu\varphi\,\partial_\nu\varphi$ instead of $g_{\mu\nu}$

Well-posed or ill-posed?

System described by

- Differential equations(s) determining the evolution
- Initial data

Well-posed Cauchy problem [Hadamard 1902]

The solution

1. exists (at least locally)
2. is unique
3. depends continuously on the initial data

Model with well-posed Cauchy problem

- Most complicated (and also most interesting) Horndeski models are ill-posed in a very broad class of gauges [Papallo & Reall, PRD '17]
- Finally, well-posed in a certain gauge [Kovacs & Reall, PRL '20]
- **Important limitation**: weak coupling (GR + ϵ)

Idea : Transfer goods properties of general relativity

$$S = \int d^4x \sqrt{-\tilde{g}} \left[M_{\rm Pl}^2 \tilde{R} - (\tilde{\nabla\phi})^2 \right]$$

$$\downarrow \quad g_{\mu\nu} = \tilde{g}_{\mu\nu} - D\partial_\mu\phi\partial_\nu\phi$$

$$G_2(X) = \frac{X}{\sqrt{1-2DX}}, \quad G_4(X) = \sqrt{1-2DX}$$