Detecting change points in dynamic networks

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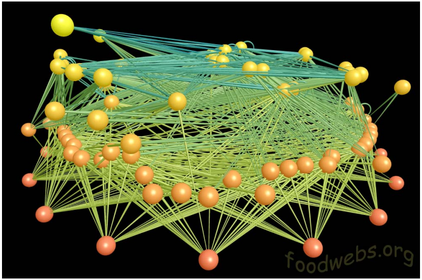
joint work with F. Enikeeva

Network model

Network analysis has become an important research field driven by applications in social sciences, computer sciences, statistical physics, biology,. . .



East-river trophic network [Yoon et al.(04)]

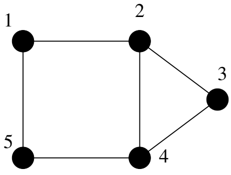
Approach

- The modeling of real networks as random graphs.
- Model-based statistical analysis.

Graph Notations

A (simple, undirected graph) $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ consists of

- a set of vertices $V = \{1, \ldots n\}$
- a set of edges $E \subset \{\{i,j\} : i,j \in V$ and $i \neq j\}$



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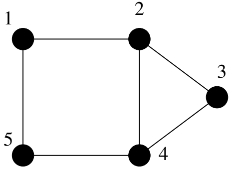

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The corresponding adjacency matrix is denoted $\mathbf{A} = (\mathbf{A}_{i,j}) \in \{0,1\}^{n \times n}$, where $\mathbf{A}_{i,j} = 1 \Leftrightarrow (i,j) \in E$

Network Model

- Our observations: the adjacency $\mathbf{A} = (\mathbf{A}_{ij})$
- $\mathbf{A}_{ij}$ independent Bernoulli random variables with **probability of connection**

$$\mathbf{\Theta}_{ij} = \mathbb{P}(\mathbf{A}_{ij} = 1), \quad 1 \leq j < i \leq n.$$

- $\mathbf{\Theta}$ $n \times n$ symmetric matrix with the coefficients $\mathbf{\Theta}_{ij}$ for $1 \leq j < i \leq n$ and zeros on the diagonal.

- **Inhomogeneous random graph model**

Dynamic networks

- Most of the real-life networks evolve over the time
- We observe a sequence of sparse graphs:

$$\mathbf{A}^1, \mathbf{A}^2, \ldots$$

- Each $\mathbf{A}^t$ = independent realization from an **unknown inhomogeneous random graph model**
- The underlying distribution of this sequence of graphs may change at some unknown time moment
- **Problem of detection of possible changes in a time sequence of networks**
  - intrusion detection
  - health care monitoring
  - fraud detection

Change Point Detection and Localization

Modeling dynamic networks

- "Signal + Noise" model of the observed data $\mathbf{A}^t$:

$$\mathbf{A}^t = \mathbf{\Theta}^t + \mathbf{W}^t \quad (1 \leq t \leq T),$$

- The connection probability matrix $\mathbf{\Theta}^t$ might change at an unknown time $1 \leq \tau \leq T-1$:

$$\mathbf{\Theta}^t = \mathbf{\Theta}^0 \mathbf{1}_{\{1 \leq t \leq \tau\}} + (\mathbf{\Theta}^0 + \Delta\mathbf{\Theta}^\tau)\mathbf{1}_{\{\tau+1 \leq t \leq T\}} \quad (t = 1, \ldots, T).$$

  - $\mathbf{\Theta}^0$ the connection probability matrix before the change
  - $\Delta\mathbf{\Theta}^\tau$ symmetric jump matrix of a change that occurs at time τ.

Change point Detection/Estimation in Dynamic Networks

- Detection:
  - **Chen et al (2019)**
  - **Wang et al (2021)**
  - ...
- Localization:
  - **Bhattacharjee et al (2018)**
  - **Wang et al (2021)**
  - ...
- Two-Sample Testing:
  - **Tang et al (2015)**
  - **Ghoshdastidar et al (2020)**
  - ...

Test problem

- We would like to test whether there is no change in $\mathbf{\Theta}^t$:

$$H_0 : \ \Delta\mathbf{\Theta}^\tau = 0 \quad \text{for all } \tau$$

against the alternative hypothesis of a change in $\mathbf{\Theta}^t$ at some point τ

$$H_1 : \|\Delta\mathbf{\Theta}^\tau\| \geq \rho_n \text{ for some } \tau.$$

- $\rho_n > 0$ is the minimal value of the jump that guarantees the change-point detection

Partial answer

Problem of the minimax distinguishability of hypotheses

- **Ghoshdastidar et al (2020)** (two sample test)
  - Spectral and Frobenius norm
  - Known τ
  - Additional $\log(n)$ factor
- **Wang et al (2021)** (localization)
  - Frobenius norm
  - up to a $\log$ factor
  - upper bound for a two-sample procedure

Sparsity

Main integral characteristics

- number of vertices n
- number of edges $|E|$

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$$\frac{n(n-1)}{2}$$

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Maximal number of edges

$$\frac{n(n-1)}{2}$$

- Dense graph $|E| \sim n^2$
- Real world networks are sparse : $|E| = o(n^2)$
  - more difficult to handle

Modeling sparsity

- $\mathbf{\Theta}_{ij}^t \to 0$ as $n \to \infty$ for some (or all) (i,j)
- Usual assumption: $r_n = \max_t \|\mathbf{\Theta}^t\|_\infty$ and $r_n \to 0$ as $n \to \infty$
- **Sparsity:** $\kappa_n = \max_t \|\mathbf{\Theta}^t\|_{1,\infty} = \max_{t,j} \sum_i \mathbf{\Theta}_{ij}$:
  - $\kappa_n / n \to 0$ as $n \to \infty$
  - $\kappa_n \leq r_n n$
  - more flexible
  - can be easily estimated!

Energy of the change point

- Difficulty of assessing the existence of a change-point: *energy*

$$\mathcal{E}(\tau) = \mathbf{q}(\tau/\mathbf{T}) \|\boldsymbol{\Delta\Theta}^{\tau}\|_{\mathbf{2\to 2}}$$

- $q(t) = \sqrt{t(1-t)}$: quantifies the impact of change-point location on the difficulty of detecting the change
- Problem of testing whether the energy is zero or is at least $\rho_n > 0$:

$$H_0 : \ \mathcal{E}(\tau) = 0 \text{ for all } \tau$$

against the alternative hypothesis of a change in Θ^t:

$$H_1 : \ \mathcal{E}(\tau) \geq \rho_n > 0 \text{ for some } \tau$$

Minimax detection rate

- *Energy*: $\mathcal{E}(\tau) = q(\tau/T)\|\Delta\Theta^{\tau}\|_{2\to 2}$
- $H_0: \ \mathcal{E}(\tau) = 0$ for all τ against the alternative hypothesis

  $H_1: \ \mathcal{E}(\tau) \geq \rho_n > 0$ for some τ

Goal

Find the minimal amount of energy that guarantees the change-point detection

Minimax detection rate

- *Energy*: $\mathcal{E}(\tau) = q(\tau/T)\|\Delta\Theta^{\tau}\|_{2\to 2}$
- $H_0 : \ \mathcal{E}(\tau) = 0$ for all τ vs $H_1 : \ \mathcal{E}(\tau) \geq \rho_n > 0$ for some τ

Detection threshold (up to log factors)

$$\left(\frac{\kappa_n}{T}\right)^{1/2}$$

- $\kappa_n = \max_t \|\mathbf{\Theta}^t\|_{1,\infty}$: sparsity
- T time points
- Lower and upper bound:
  - if the energy of the change point is smaller $\Rightarrow$ we can not distinguish between these two hypothesis.
  - a test which satisfies the upper detection condition

Matrix CUSUM statistic

$$Z_T(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^{t} \mathbf{A}^s - \frac{1}{T-t} \sum_{s=t+1}^{T} \mathbf{A}^s \right)$$

- Measures of the difference between the average number of connections before and after the point t
- A change of the parameter matrix Θ^t at time $\tau \Rightarrow$ the value of the process Z_T will be maximal in the neighborhood of τ
- If $\|Z_T(t)\|$ is sufficiently large at some point $t \Rightarrow$ there is a change in the connection matrix $\mathbf{\Theta}^t$ of the network
- **In what norm?**

Matrix CUSUM statistic

$$Z_T(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^{t} \mathbf{A}^s - \frac{1}{T-t} \sum_{s=t+1}^{T} \mathbf{A}^s \right)$$

- "signal +noise" model

$$Z_T(t) = -\mu_T(t)\Delta\mathbf{\Theta}^\tau + \xi(t), \quad t = 1, \dots, T-1,$$

  - $\mu_T(t) = \sqrt{\frac{t(T-t)}{T}} \left((\tau/t)\mathbf{1}_{\{\tau+1 \le t \le T\}} + (T-\tau)/(T-t)\mathbf{1}_{\{1 \le t \le \tau\}} \right)$ (max at the true change-point $t = \tau$)
  - centered noise

$$\xi(t) = \sqrt{\frac{t(T-t)}{T}} \left(\frac{1}{t} \sum_{s=1}^{t} W^s - \frac{1}{T-t} \sum_{s=t+1}^{T} W^s \right)$$

In what norm?

$$Z_T(t) = -\mu_T(t)\Delta\boldsymbol{\Theta}^\tau + \xi(t), \quad t = 1, \dots, T-1,$$

- $\|\Delta\boldsymbol{\Theta}^\tau\|$ should be larger than $\|\xi(t)\|$
- We need to control $\|\xi(t)\|$
- **Matrix Hoeffding inequality:**

$$\mathbb{E}\|\xi(t)\|_{2\to 2} \leq 3\sqrt{2}\left(\sqrt{1-t/T} + \sqrt{t/T}\right)(\kappa_n)^{1/2}$$

- Can not work with Frobenius norm as in sparse regime: assuming $\Theta^t_{ij} \approx r_n \to 0$

$$\frac{\mathbb{E}\|\xi(t)\|_2^2}{n^2} \approx r_n \gg \frac{\|\Delta\Theta^\tau\|_2^2}{n^2} \approx r_n^2$$

Decision rule

- Matrix CUSUM statistics:

$$Z_T(t) = -\mu_T(t)\Delta\boldsymbol{\Theta}^\tau + \xi(t), \quad t = 1, \ldots, T-1,$$

- Decision rule:

$$\psi_{n,T}(Y) = \mathbb{1}_{\left\{\max\limits_{t\in\mathcal{T}} \|Z_T(t)\|_{2\to 2} > H_{\alpha,n,T}\right\}},$$

- $\mathcal{T} = \mathcal{T}^L \cup \mathcal{T}^R$: dyadic grid of $\{1, \ldots, T-1\}$ where

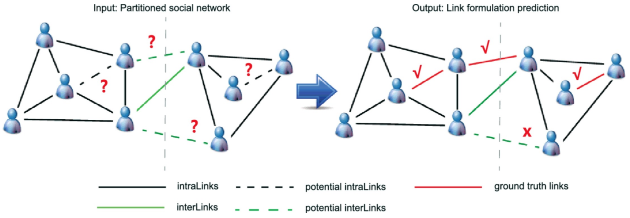
$$\mathcal{T}^L = \left\{2^k,\ k = 0, \ldots, \lfloor \log_2(T/2) \rfloor\right\}$$

$$\mathcal{T}^R = \left\{T - 2^k,\ k = 0, \ldots, \lfloor \log_2(T/2) \rfloor\right\}.$$

Missing Links

Real-life networks are only partially observed

- Exhaustive exploration of all interactions in a network is expensive
- Survey data: non-response or drop-out of participants
- Online social network data: sub-sample of the network



A balanced modularity maximization link prediction model in social networks [Wu et al.(2017)]

Missing links

- Imputation methods require observations from a homogeneous distribution
- Change-point detection and estimation methods are designed for the case of complete observations
- Can we detect the change point from partial observations of our networks?
- How will missing links affect the detection rate?

Change-point localization

- An estimator of τ, $\widehat{\tau}_n$, such that $|\widehat{\tau}_n - \tau| \leq \epsilon$ with high probability
- $\epsilon/T =$ *localization rate*
- an estimator is consistent if, as $T \to \infty$, its localization rate vanishes
- Using CUSUM statistics: $\widehat{\tau}_n \in \arg\max_{1 \leq t \leq T-1} \|Z_T(t)\|_{2\to 2}$

Theorem (Enikeeva and K., 2021)

Let $\gamma \in (0,1)$ *and* $x^* = \tau/T$. *Then, the estimated change-point* $\widehat{x} = \widehat{\tau}/T$ *satisfies*

$$|\widehat{x} - x^*| \leq \frac{3\, c_* \sqrt{\omega_n \log(nT/\gamma)}}{\mathcal{E}(\tau)\sqrt{T}}$$

with probability larger than $1-\gamma$.

Wang et al (2021): if $\mathcal{E}(\tau) \leq \dfrac{\sqrt{\kappa_n}}{\sqrt{33T}}$, then no consistent change-point estimator can exist.

Numerical experiments

- We applied three tests:
  - test $\psi^{\tau}_{n,T}$ at the given change-point τ
  - test $\psi_{n,T}$ over the dyadic grid $\mathcal{T}^d$
  - test $\psi^{full}_{n,T}(Y)$ based on the maximum over the whole set $\{1,\dots,T-1\}$
- Each test is calibrated at the significance level $\alpha = 0.05$
- The sparsity ρ_n is set to $n^{-1/2}$
- "Energy-to-noise ratio"

$$\mathrm{ENR} := \frac{\mathcal{E}(\tau)}{\sqrt{\kappa_n/T}}.$$

Adaptation to the unknown sparsity level

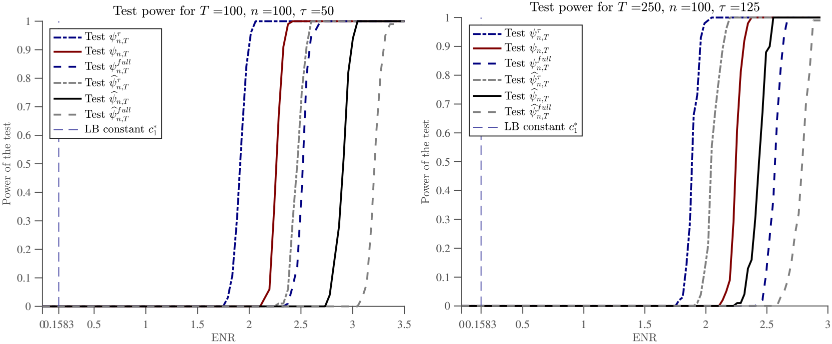


Figure: The test powers with known and estimated sparsity parameters for $n = 100$, $\tau/T = 0.5$. Left: SBM with two communities and change in connection probability between communities and $T = 100$. Right: SBM with three communities and change in connection probability between communities and $T = 250$.

Coping with missing links

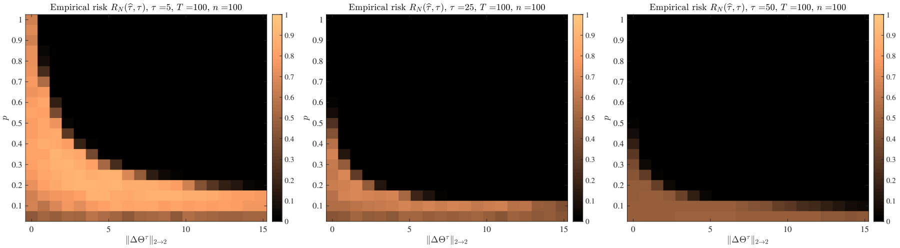


Figure: The risk of the change-point estimator for $T = 100$, $n = 100$ and $\tau = 5, 25, 50$ (left to right).

- We sample the links at the uniform rate p_n
- Average absolute error of $\hat{\tau}$ over $N = 100$ simulations: $R_N(\hat{\tau},\tau) = (NT)^{-1}\sum_{i=1}^{N}|\hat{\tau}_i - \tau|$
- dependence of the error on the sampling rate p_n and on the norm of the jump $\Delta\Theta^{\tau}$.

Transport for London (TfL) Open Data

- https://api.tfl.gov.uk
- Information about London Bicycle Sharing Network collected since 2012:
  - ID of each bicycle
  - ID and name of the origin and the destination trip stations
  - journey (rental) starting and ending time and date
  - ID and the duration of each trip
- Two-month period from June 24, 2012 to August 31, 2012
- Games of the XXX Olympiad

Transport for London (TfL) Open Data

- Dynamic network: a sequence of $T = 69$ daily observations
- Each observation: a graph with $n = 595$ vertices corresponding to the bike rental stations
- Two vertices are connected:
  - minimal trip duration is not less than 3 minutes
  - the number of trips is greater than a predefined threshold
    - 0.9975-level empirical quantile of the distribution of the total number of trips between every couple of stations
- Average sparsity $\bar{\kappa}_n = 43.2319$ (over $T = 69$ observations)
- The corresponding value of $\rho_n = \kappa_n / n = 0.0727 \asymp n^{-0.4}$

Transport for London (TfL) Open Data

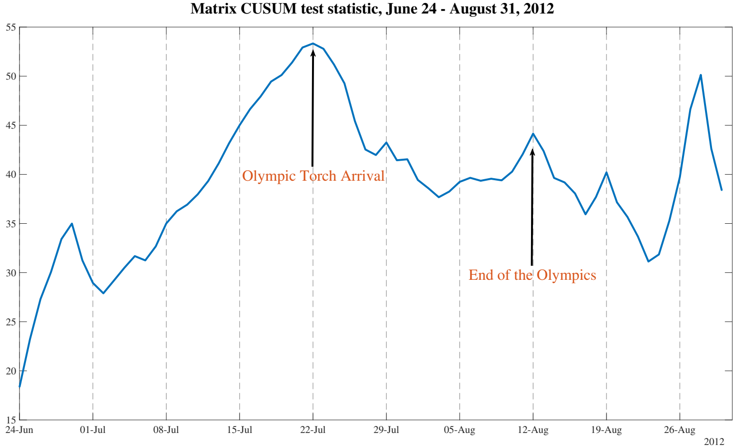


Figure: The value of the matrix CUSUM statistic calculated during the whole period of observations. The vertical grid lines correspond to Sundays.

Transport for London (TfL) Open Data

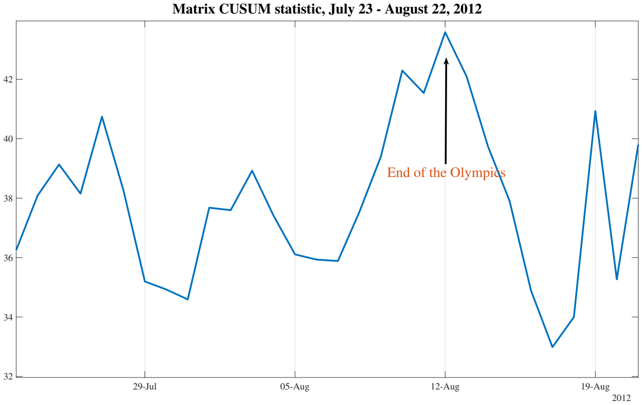


Figure: The value of the matrix CUSUM statistic calculated during 31 day from July 23 to August, 22.

Segmentation methods for multiple change-point localization

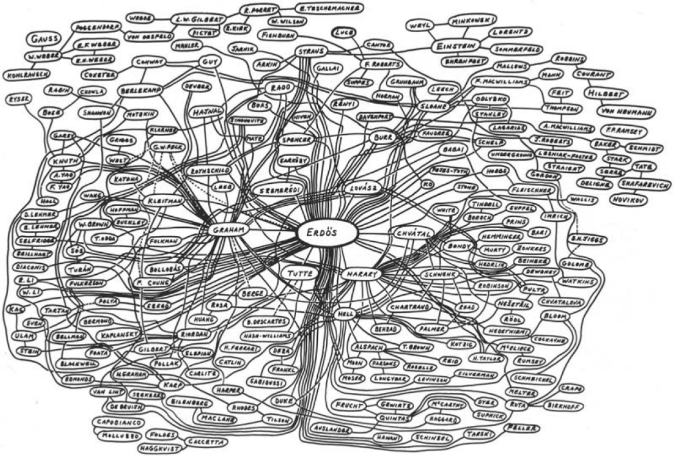
Summary

Problem of change point detection in dynamic network:

- **Minimax separation rate for the energy of the change point** $\mathcal{E}(\tau) = q(\tau/T)\|\Delta\Theta^{\tau}\|_{2\to 2}$

$$\sqrt{\kappa_n/T}$$

  - Sparsity: $\kappa_n = \max_t \|\boldsymbol{\Theta}^t\|_{1,\infty}$
  - Change points that are away from the end point may be detected at lower size of jump in the parameter matrix
- Test based on the spectral norm of the Matrix CUSUM statistics:
  - minimax optimal
  - robust to missing links
  - works for networks with changing size
- Localisation of the change point(s)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ZONKERS
BEINEKE
HEDETNIEMI
FLEISCHNER
STONE
TINDELL
WALLIS
GOLOMB
CHVATALOVA

Thank You !