Scattering of Test Fields in the Interior of Black Holes

Mokdad (Rémi) Mokdad
Institut de Mathématiques de Bourgogne
Univeristé de Bourgogne

Collaborators:
D. Häfner (IF – UGA), J.-P. Nicolas (LMBA – UBO), R. Nasser (ETH – Zürich)

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- Spacetime $(\mathcal{M}, g)$: a 4-dimensional Lorentzian manifold.
  - Timelike: $g(X, X) > 0$,
  - Null: $g(X, X) = 0$,
  - Spacelike: $g(X, X) < 0$.

Reissner-Nordström-(Anti-)de Sitter (RN(A)dS)

charged black hole solutions to Einstein-Maxwell System

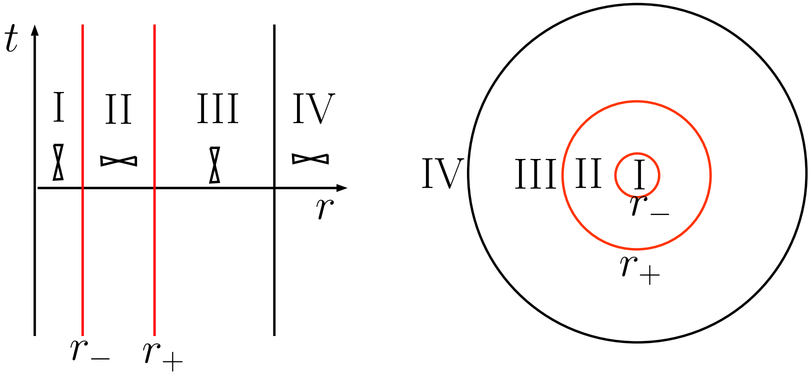
- Spacetime $(\mathcal{M}, g)$: a 4-dimensional Lorentzian manifold.
- Spherically symmetric solutions to Einstein-Maxwell system: RN(A)dS **charged** black hole spacetime,

$$\mathcal{M} = \mathbb{R}^4 \backslash \{0\} = \mathbb{R}_t \times]0, +\infty[_r \times \mathrm{S}^2_{\theta,\varphi} \ ,$$

$$g = f(r)\mathrm{d}t^2 - \frac{1}{f(r)}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2 \ ,$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2 \ .$$

Black hole spacetime



Black hole interior: $\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_x \times \mathrm{S}^2_\omega$

$$g = -f(r)\left(\mathrm{d}t^2 - \mathrm{d}x^2\right) - r^2\mathrm{d}\omega^2$$

$r = r(t)$ defined by:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = f(r)$$

Renaming old t as x.

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(1) $f \in \mathcal{C}^\infty\left([r_-, r_+]\right)$,

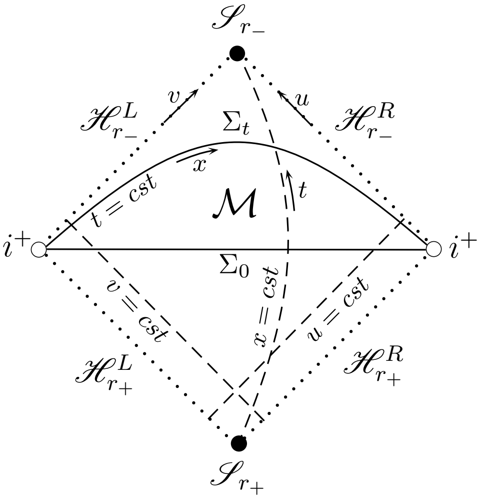
(2) $f < 0$ on (r_-, r_+),

(3) $f(r_\pm) = 0 \neq f'(r_\pm)$.

Horizons as boundaries

We define $u = t - x\,,\ v = t + x$, and we add to $\mathcal{M}$:

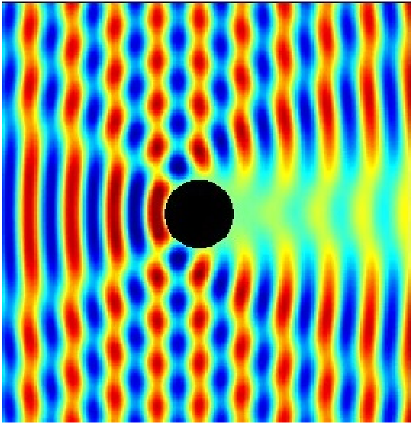
$$\begin{aligned}
\mathscr{H}_{r_-}^{L} &:= \{r = r_-\} \times \mathbb{R}_v \times \mathrm{S}^2_\omega\,,\\
\mathscr{H}_{r_-}^{R} &:= \{r = r_-\} \times \mathbb{R}_u \times \mathrm{S}^2_\omega\,,\\
\mathscr{H}_{r_+}^{L} &:= \{r = r_+\} \times \mathbb{R}_u \times \mathrm{S}^2_\omega\,,\\
\mathscr{H}_{r_+}^{R} &:= \{r = r_+\} \times \mathbb{R}_v \times \mathrm{S}^2_\omega\,.
\end{aligned}$$



The interior between the Cauchy and the event horizons.

General Idea:

$$\text{Past Profile} \xleftrightarrow{\text{Scattering Operator}} \text{Future Profile}$$



1^{st} Approach: *Via the transmission and reflection coefficients.*

$$\text{Dynamic in time} \xrightarrow{\text{Fourier}} \text{Stationary: fixed frequency}$$

Scattering Matrix S

$$S\phi_{in} = \phi_{out}$$

2^{nd} Approach: *Via the wave operators.*

$\forall \phi \in \mathcal{H}$, $\exists \tilde{\phi} \in \mathcal{H}$, and vice–versa, such that:

$$\left\| U_0(t,0)\tilde{\phi} - U(t,0)\phi \right\|_{\mathcal{H}} \xrightarrow[t\to\pm\infty]{} 0.$$

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$$\Updownarrow$$

$$W^{\pm} = s - \lim_{t \to \pm\infty} U(0,t) U_0(t,0) \quad ; \quad \Omega^{\pm} = s - \lim_{t \to \pm\infty} U_0(0,t) U(t,0)\,.$$

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Scattering Operator $S = \Omega^{+}W^{-}$**.**

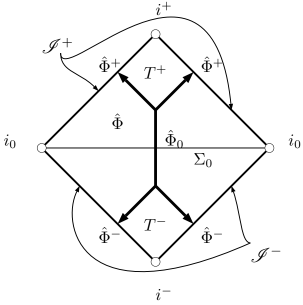
3^{rd} Approach: *Via the trace operators.*
Rescale and compactify (if necessary), then take "traces".

Trace operators

$$T^{\pm}(\hat{\Phi}_0) = \hat{\Phi}|_{\mathscr{I}^{\pm}}$$

Scattering operator

$$S = T^{+}(T^{-})^{-1}$$



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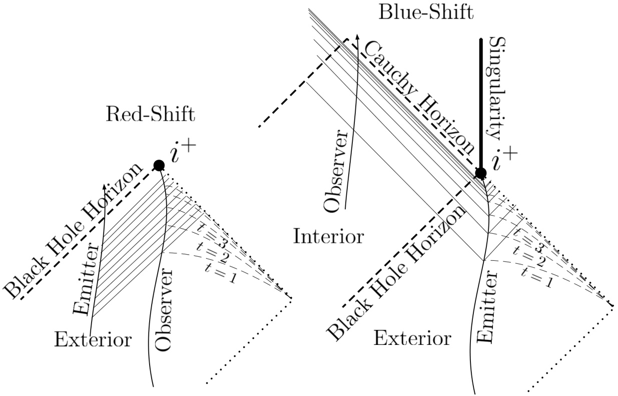

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Charged and massive Dirac equations:

$$\begin{cases} \left(\nabla^{AA'} - iqA^{AA'}\right)\phi_A & = \frac{m}{\sqrt{2}}\chi^{A'} \,, \\ \left(\nabla_{AA'} - iqA_{AA'}\right)\chi^{A'} & = -\frac{m}{\sqrt{2}}\phi_A \,, \end{cases}$$

$$\Psi = {}^{\mathrm{t}}\left(\phi_0, \phi_1, \chi^{0'}, \chi^{1'}\right) : \mathcal{M} \to \mathbb{C}^4$$

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Always possesses a conserved current defining an L^2-norm.

Hamiltonian Formulation

- The Schrödinger form of Dirac's equation: $\partial_t \Psi(t) = iH(t)\Psi(t)$

$$\text{in} \qquad \mathcal{H} = L^2(\Sigma = \mathbb{R} \times \mathcal{S}^2\,;\ \mathbb{C}^4)\,,\ \|\Psi\|^2_{\mathcal{H}} = \int_{\Sigma} |\Psi|^2 \mathrm{d}x \mathrm{d}\omega$$

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- Comparison dynamics at each horizon: $H_0^\pm$ " $=$ " $\lim_{t\to\pm\infty} H(t)$.

Scattering Theory for Dirac Fields

Theorem (D.Häfner , J.-P. Nicolas , M.M.)

$W^{\pm}$ and $\Omega^{\pm}$ are well-defined on $\mathcal{H}$ as:

$$W^{\pm} = s - \lim_{t \to \pm\infty} \mathcal{U}(0,t) e^{itH_0^{\pm}},$$

$$\Omega^{\pm} = s - \lim_{t \to \pm\infty} e^{-itH_0^{\pm}} \mathcal{U}(t,0),$$

are unitary on $\mathcal{H}$.

$$W^{\pm}\Omega^{\pm} = \Omega^{\pm}W^{\pm} = \mathrm{Id}_{\mathcal{H}}.$$

Scattering operator $S = \Omega^{+}W^{-}$ is unitary on $\mathcal{H}$.

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Theorem (M.M.)

The trace and scattering operators are isometries: $S = T^{+}(T^{-})^{-1}$,

$$T^{\mp} : \mathcal{H}_t \simeq L^2(\Sigma_t; \mathbb{C}^4) \longrightarrow L^2(\mathscr{H}^L_{r_\pm}; \mathbb{C}^2) \oplus L^2(\mathscr{H}^R_{r_\pm}; \mathbb{C}^2)$$

The Wave Equation

The geometric wave equation:

$$\Box_g \phi = 0.$$

In (t, x, θ, φ) coordinates:

$$\Box_g = \nabla^a \nabla_a = \frac{1}{f}(\partial_x^2 - \partial_t^2) - \frac{2}{r}\partial_t - \frac{1}{r^2}\Delta_{\mathcal{S}^2}$$

Energy-momentum tensor

The energy-momentum tensor

$$\mathbf{T}_{ab} := \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \ .$$

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- Divergence-free:

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- Dominant Energy Cond. : for X and Y causal

$$\mathbf{T}_{ab}X^aY^b \geq 0$$

.

Energy

For X a vector field , S a hypersurface:
The geometric "energy" flux:
let $J^a = \mathbf{T}^{ab} X_b$,

$$\mathcal{E}_X[\phi](S) := \int_S i_J \mathrm{dVol}_{\mathbf{g}}.$$

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- If X is Killing, $\mathcal{E}$ is conserved (by Stokes' theorem).

However, inside the black hole there is no timelike Killing vector field!
Therefore, no energy norm is conserved.

Energies

Choose X to be $T := \partial_t$.

$$\begin{aligned}\mathcal{E}[\phi](t) :=& \mathcal{E}_T[\phi](\Sigma_t) = \int_{\Sigma_t} \mathbf{T}_{00} r^2 \mathrm{d}x \wedge \mathrm{d}\omega^2 \\ =& \frac{1}{2} \int_{\mathbb{R}_x \times \{t\} \times \mathcal{S}^2_\omega} \left((\partial_t \phi)^2 + (\partial_x \phi)^2 - \frac{f}{r^2} |\nabla_{\mathcal{S}^2} \phi|^2 \right) r^2 \mathrm{d}x \mathrm{d}^2\omega\end{aligned}$$

Energies

Choose X to be $T := \partial_t$.

$$\mathcal{E}[\phi](t) := \frac{1}{2}\int_{\mathbb{R}_x\times\{t\}\times\mathcal{S}^2_\omega}\left((\partial_t\phi)^2+(\partial_x\phi)^2-\frac{f}{r^2}|\nabla_{\mathcal{S}^2}\phi|^2\right)r^2\mathrm{d}x\mathrm{d}^2\omega$$

T extends smoothly and becomes normal to the horizons:

$$\mathcal{E}_T[\phi](\mathscr{H}^R_{r_-}) = \int_{\mathbb{R}_u\times\{r_-\}\times\mathcal{S}^2}(\partial_u\phi)^2 r_-^2\mathrm{d}u\mathrm{d}^2\omega,$$

$$\mathcal{E}_T[\phi](\mathscr{H}^L_{r_-}) = \int_{\mathbb{R}_v\times\{r_-\}\times\mathcal{S}^2}(\partial_v\phi)^2 r_-^2\mathrm{d}v\mathrm{d}^2\omega.$$

Finite Energy Spaces

$\mathcal{C}_c^\infty(t) := \mathcal{C}_c^\infty(\Sigma_t) \times \mathcal{C}_c^\infty(\Sigma_t)$ with the energy norm $\|(\psi_0, \psi_1)\|_{\mathcal{E}(t)}$

$$\|(\phi(t), \partial_t\phi(t))\|^2_{\mathcal{E}(t)} := \mathcal{E}[\phi](t)$$

$\mathcal{H}(t)$ the finite energy space: completion of $\mathcal{C}_c^\infty(t)$.

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$\mathcal{H}^+$ on the Cauchy horizon $\mathscr{H}_{r_-}$ with norm

$$\|(\xi, \zeta)\|_{\mathcal{H}^+} = \left(\int_{\mathbb{R}_u \times \mathcal{S}^2} (\partial_u \xi)^2 r_-^2 \mathrm{d}u \mathrm{d}^2\omega + \int_{\mathbb{R}_v \times \mathcal{S}^2} (\partial_v \zeta)^2 r_-^2 \mathrm{d}v \mathrm{d}^2\omega \right)^{\frac{1}{2}}.$$

$\mathcal{H}^-$ analogously on the event horizon $\mathscr{H}_{r_+}$.

Direct scattering between the two horizons

Theorem (C. Kehle, Y. Shlapentokh-Rothman)

In the interior of a ***Reissner–Nordström*** *black hole ($\Lambda = 0$), the scattering map $S : \mathcal{H}^- \to \mathcal{H}^+$ is a Hilbert space isomorphism.*

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C^1-blowup at the Cauchy horizon.

Breakdown result

Due to C. Kehle and Y. Shlapentokh-Rothman

Theorem (C. Kehle, Y. Shlapentokh-Rothman)

Breakdown of scattering for generic Klein-Gordon and cosmological settings ($\Lambda \neq 0$):
$\exists (\phi_n)_n$ *with* $\mathcal{E}_T[\phi_n](\mathscr{H}_{r_+}) = 1 \;\; \forall n$, *but* $\lim_{n\to\infty} \mathcal{E}_T[\phi_n](\mathscr{H}_{r_-}) = \infty$.

Breakdown of Scattering between the horizons and Σ_t

Theorem (R. Nasser , M.M.)

The trace mappings $T^{\pm} : \mathcal{H}(0) \to \mathcal{H}^{\pm}$, defined by:

$$T^{\mp}(\Phi_0, \Psi_0) = (\phi|_{\mathscr{H}^L_{r_\pm}}, \phi|_{\mathscr{H}^R_{r_\pm}}), \qquad (\Phi_0, \Psi_0) \in \mathcal{C}^\infty_c(0)$$

*are linear bounded maps but they do **not** have bounded inverses.*

There exist "*decaying*" sequences $(\phi_n^{\pm})_n$ of solutions:

$$\phi_n^{\pm}|_{\Sigma_0} \in \mathcal{C}_c^{\infty}(\Sigma_0) \quad \textit{and} \quad \mathcal{E}[\phi_n^{\pm}](0) = 1 \qquad \forall n,$$

$$\textit{and}$$

$$\lim_{n\to\infty} \lim_{t\to\pm\infty} \mathcal{E}[\phi_n^{\pm}](t) = 0.$$

Note that

$$\lim_{t\to\pm\infty} \mathcal{E}[\phi_n^{\pm}](t) = \|T^{\pm}(\phi_n(0), \partial_t\phi_n(0))\|_{\mathcal{H}^{\pm}}.$$

As we shall see, the breakdown of scattering is a direct consequence of the behavior of solutions at high angular momenta (ℓ) and small spatial frequencies (ω).

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However,

$$\int_{\Sigma_t} |\nabla_{\mathcal{S}^2}\phi|^2 \mathrm{d}x\mathrm{d}^2\omega \leq D \int_{\Sigma_t} |\partial_x \phi|^2 \mathrm{d}x\mathrm{d}^2\omega, \quad \forall t \geq 0, \qquad (Cond.)$$

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yields a "scattering theory"!

One way to impose $(Cond.)$: $|\omega| \geq \omega_0 > 0$ and $\ell \leq \ell_0$.

Reduction to 1+1-dimensions and rescaling

- Simplify $\Box\phi = 0$ by rescaling $u := r\phi$.

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- Simplify $\Box\phi = 0$ by rescaling $u := r\phi$.
- Decompose on spherical harmonics $u = \sum_\ell u_\ell(t,x) Y_\ell(\theta,\varphi)$:

$$\partial_t^2 u_\ell - \partial_x^2 u_\ell + V_\ell(t) u_\ell = 0 \ , \qquad (\star_\ell)$$

$$V_\ell = -\frac{f}{r^2}\Big(\ell(\ell+1) + rf'\Big)$$

Note that $V_\ell > 0$ only for $\ell \geq \ell_0 > 0$.

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- Auxiliary "energy":

$$E_\ell[u](t) = \int_{\mathbb{R}_x} (\partial_t u_\ell)^2 + (\partial_x u_\ell)^2 + V_\ell u_\ell^2 \mathrm{d}x.$$

Note that $E_\ell[u] \eqsim \mathcal{E}_\ell[\phi]$ for all $\ell \geq \ell_0$.

A general potential

A general form of Equation $(\star_\ell)$:

$$\partial_t^2 u - \partial_x^2 u + V(t)u = 0 \ , \quad (t,x) \in \mathbb{R}_t^+ \times \mathbb{R}_x \qquad (\star)$$

with

$$\begin{cases} 0 < V \in \mathcal{C}^\infty(\mathbb{R}^+) \\ V' < 0 \quad \text{and} \quad V \eqsim e^{-\lambda t}, \quad \forall t > t_{large} \geq 0 \quad \text{with} \quad \lambda > 0. \end{cases} \qquad (GC)$$

Note that V_ℓ satisfies (GC) for $\ell \geq \ell_0$ on both $t = \pm\infty$.

Proposition

Consider ($\star$) with V satisfying (GC) and

$$E[u](t) = \int_{\mathbb{R}_x} (\partial_t u)^2 + (\partial_x u)^2 + V u^2 \mathrm{d}x.$$

$\exists (u_n)_n$ *of solutions to ($\star$) such that* $E[u_n](0) = 1$ *and* $u_n(0,x) \in \mathcal{C}_c^\infty(\mathbb{R}_x)$ *for all* n, *and*

$$\lim_{n\to\infty} \lim_{t\to+\infty} E[u_n](t) = 0.$$

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Black hole case follows as a corollary.

Consider the case $V(t) = e^{-\lambda t}$, $\forall t \in \mathbb{R}^+$.

- Take Fourier transform of $(\star)$ in x:

$$\hat{u}''_\omega(t) + (\omega^2 + e^{-\lambda t})\hat{u}_\omega(t) = 0.$$

Explicit solution using Bessel functions.

but first a toy-model...

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  Explicit solution using Bessel functions.

- Carefully chosen $(\hat{u}_\omega(0), \hat{u}_\omega'(0))$:

$$E_\omega[\hat{u}_\omega](t) = |\hat{u}_\omega'(t)|^2 + (\omega^2 + V(t))|\hat{u}_\omega(t)|^2 \xrightarrow[t \to +\infty]{\omega \to 0} 0.$$

For $\omega = 0$:

$$(\hat{u}_0(0), \hat{u}_0'(0)) = \left(2J_0\left(\frac{2}{\lambda}\right), 2J_1\left(\frac{2}{\lambda}\right)\right)$$

$$\hat{u}_0(t) = J_0\left(\frac{2}{\lambda}e^{-\frac{1}{2}\lambda t}\right)$$

and indeed

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Toy-model: decaying sequence

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Decaying sequence: $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_x)$ with $\|\varphi\|_{L^2} = 1$,

$$(u_n(0,x), \partial_t u_n(0,x)) = \frac{2\varphi(\frac{x}{n})}{\sqrt{n}}\left(J_0\left(\frac{2}{\lambda}\right), J_1\left(\frac{2}{\lambda}\right)\right)$$

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$$\hat{u}''(t) + (w^2 + V(t))\hat{u}(t) = 0.$$

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  Then $(u_n)_n$ is the decaying sequence we are looking for.

Analytic scattering: wave operators

Scattering breakdown for $(\star)$:

- Full dynamics: $\mathbb{H}(t) = \big(H^1(\mathbb{R}) \times L^2(\mathbb{R}), \|\cdot\|_{E(t)}\big)$

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Analgous theorem for black hole interior.

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