Bogoliubov excitation spectrum of trapped Bose gases in the Gross–Pitaevskii regime

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Outline

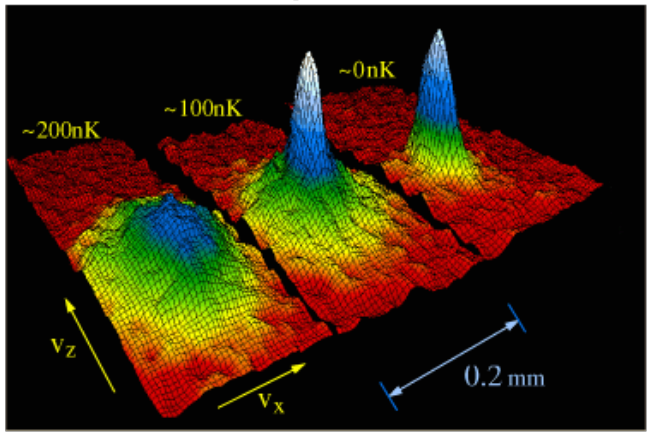
1. Physical motivation and model: BEC and Superfluidity

2. Mathematical context: the GP and the Thermodynamic Limit

3. Idea of proof: Bogoliubov's approximation

Bose-Einstein condensation and Superfluidity

- Prediction by Bose & Einstein (1924) for non-interacting particles
- 1st experimental realization (1995), 2001-Nobel Prize in Physics for Cornell–Wieman & Ketterle
- London (1938) links to superfluidity , Landau (1941) via the excitation spectrum, Bogoliubov (1947) microscopic derivation



Anderson–Ensher–Matthews–Wieman–Cornell (*Science*, 1995)
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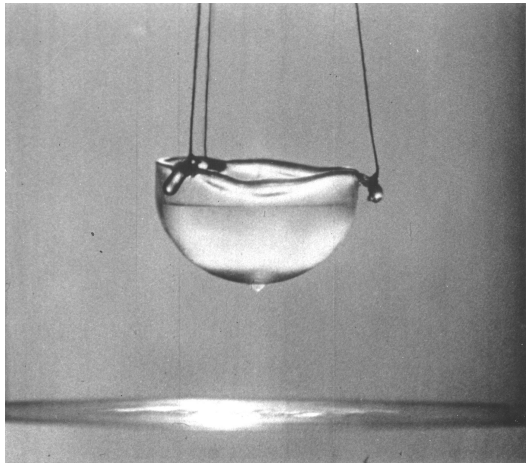

Alfred Leitner (*Liquid Helium*, *Superfluid*)

- Landau's argument: particle of momentum $\mathbf{v}$ in a superfluid
  ▷ conservation of the energy:

$$\frac{1}{2}\mathbf{v}^2 = \frac{1}{2}(\mathbf{v}-\mathbf{p})^2 + \varepsilon(\mathbf{p})$$

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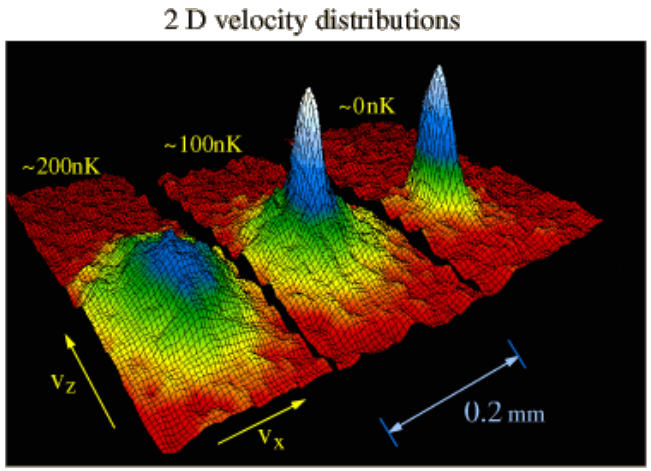

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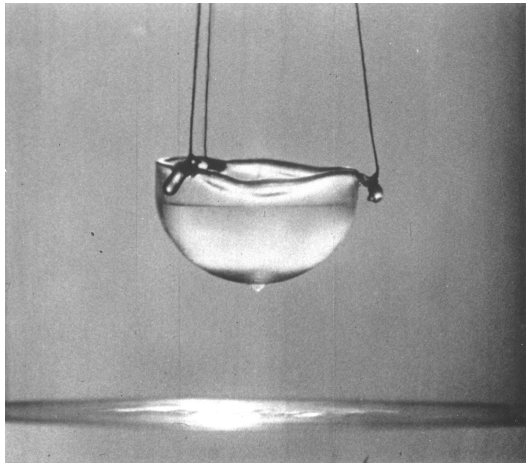

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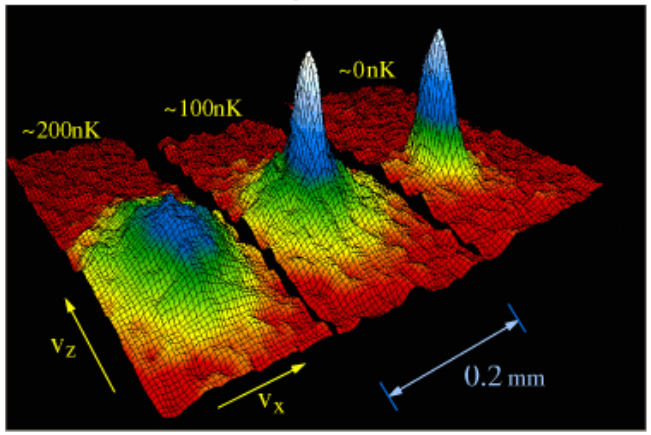

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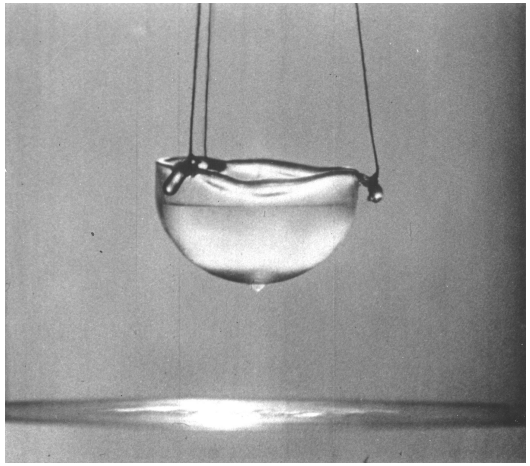

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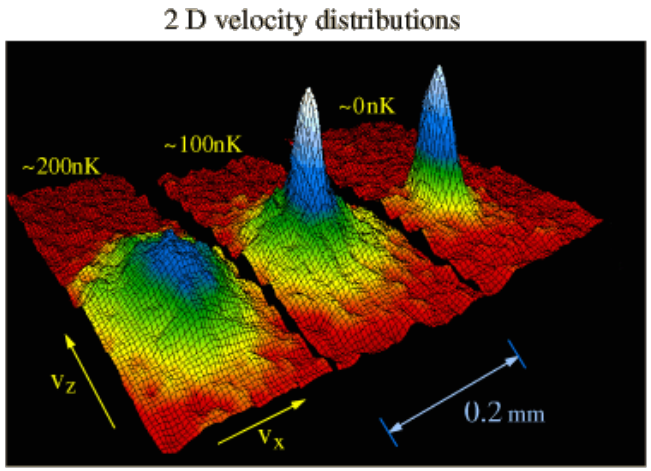

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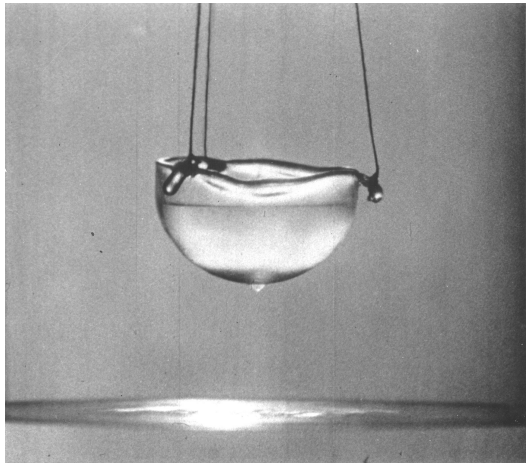

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- Landau's argument: particle of momentum $\mathbf{v}$ in a superfluid
  ▷ conservation of the energy:

$$\boxed{\mathbf{v}_{\mathbf{min}} = \min \frac{\varepsilon(\mathbf{p})}{|\mathbf{p}|}} > 0 \ ?$$

▷ intrinsically linked to interaction

Model and goal

Acting on $\bigotimes_s^N L^2(\mathbb{R}^3) = \left\{\Psi \in L^2(\mathbb{R}^{3N}) \middle| \Psi(x_{\sigma_1}, \dots, x_{\sigma_N}) = \Psi(x_1, \dots, x_N)\right\}$

$$H_N^{\mathrm{GP}} = \sum_{i=1}^{N} -\Delta_{x_i} + V_{\mathrm{ext}}(x_i) + \sum_{1 \le i < j \le N} N^2 V(N(x_i - x_j))$$

- Trapping potential: $0 \le V_{\mathrm{ext}} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$, $V_{\mathrm{ext}}(x) \to \infty$ as $|x| \to \infty$
- Interaction potential: $0 \le V \in L^1(\mathbb{R}^3)$, radial and compactly supported

$\longrightarrow$ **Goal**: understand the bottom of the spectrum (ultra cold gas)

$$\lambda_1(H_N^{\mathrm{GP}}) < \lambda_2(H_N) \le \lambda_3(H_N^{\mathrm{GP}}) \le \dots$$

- $\lambda_j(H_N^{\mathrm{GP}}) \simeq N$ but $\lambda_j(H_N^{\mathrm{GP}}) - \lambda_1(H_N^{\mathrm{GP}}) \simeq 1$

$\longrightarrow$ **Difficult regime**: mean-field approximation non valid

$$\Psi \not\simeq u(x_1) \dots u(x_N).$$

The Gross–Pitaevskii regime **vs** Thermodynamic limit

Thermodynamic limit:

$$H_N^{\mathrm{TL}} = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1\le i<j\le N} V(x_i - x_j) \quad \text{on} \quad \bigotimes_{s}^{N} L^2([-L/2, L/2]^3)$$

Energy in the **infinite volume limit** : $e(\rho, a) := \lim_{\substack{N\to\infty \\ N/L^3\to\rho}} \inf_\Psi \frac{\langle \Psi, H_N^{\mathrm{TL}}\Psi\rangle}{\|\Psi\|^2}$

$$e(\rho, a) = 4\pi a\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + 8\left(\frac{4\pi}{3} - \sqrt{3}\right)\rho a^3 \log\left(\rho a^3\right) + \mathcal{O}\left(\rho a^3\right)\right),$$

as $\rho a^3 \to 0$. The **scattering length**

$$4\pi \mathfrak{a}_0 = \inf_{f(x) \xrightarrow[x\to\infty]{} 1} \left\{ \int_{\mathbb{R}^3} |\nabla f|^2 + \int_{\mathbb{R}^3} V|f|^2 \right\}$$

$\longrightarrow$ [Lee-Huang-Yang '57], [Wu '59], [Dyson '57], [Lieb-Yngvason '99], [Yau-Yin 09'] (also [Basti-Cenatiempo-Schlein '21]), [Fournais-Solovej '21]

A simpler setting: mean-field bosons

Weakly interacting bosons

$$H_N^{\mathrm{MF}} = \sum_{i=1}^{N} -\Delta_{x_i} + V_{\mathrm{ext}}(x_i) + \frac{1}{N} \sum_{1 \le i < j \le N} V(x_i - x_j) \quad \text{on} \quad \bigotimes_{s}^{N} L^2(\mathbb{R}^3)$$

- $\Psi(x_1, \dots, x_N) = u(x_1) \dots u(x_N)$ captures the leading order

$$\lambda_1(H_N^{\mathrm{MF}}) = N \inf_{\|\varphi\|_2 = 1} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 + V_{\mathrm{ext}} |u|^2 + \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |u(x)|^2 V(x-y) |u(y)|^2 \right\} + \mathcal{O}(1)$$

- Excitation spectrum described by the **Bogoliubov's approximation**

$$\left\{ \lambda_i(H_N^{\mathrm{MF}}) - \lambda_1(H_N^{\mathrm{MF}}) \right\}_{i \ge 2} = \sum_{i \ge 1} n_i e_i,$$

for $\{n_i\}_{i \ge 1} \subset \{1, 2, \dots\}$ and $\{e_i\}_{i \ge 1}$ the eigenvalues of

$$E = (D^{1/2}(D + 2K)D^{1/2})^{1/2}$$

where $D = -\Delta + V_{\mathrm{ext}} + V * \varphi^2 - \mu$ $(D\varphi = 0)$ and $K(x, y) = \varphi(x) V(x-y) \varphi(y)$.

$\longrightarrow$ in the translational invariant case $E(p) = \sqrt{p^4 + 2p^2 \widehat{V}(p)}$.

$\longrightarrow$ [Seiringer '11], [Grech-Seiringer '13], [Lewin-Nam-Serfaty-Solovej '13], [Lewin-Nam-Rougerie '14]

Interpolating MF and Gross–Pitaevskii

For $0 < \beta \leq 1$

$$H_N^\beta = \sum_{i=1}^N -\Delta_{x_i} + V_{\text{ext}}(x_i) + \frac{1}{N} \sum_{1 \leq i < j \leq N} N^{3\beta} V(N^\beta (x_i - x_j)) \quad \text{on} \quad \bigotimes_s^N L^2(\mathbb{R}^3)$$

- $\Psi(x_1, \dots, x_N) = u(x_1) \dots u(x_N)$ captures the leading order

$$\lambda_1(H_N^\beta) = N \inf_{\|\varphi\|_2 = 1} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 + V_{\text{ext}} |u|^2 + \frac{1}{2} \widehat{V}(0) \int_{\mathbb{R}^3} |u|^4 \right\} + \mathcal{O}(N^\beta)$$

- Excitation spectrum described by the **Bogoliubov's approximation**

$$\left\{ \lambda_i(H_N^\beta) - \lambda_1(H_N^\beta) \right\}_{i \geq 2} = \sum_{i \geq 1} n_i e_i,$$

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where $D = -\Delta + V_{\text{ext}} + \widehat{V}(0)\varphi^2 - \mu$ $(D\varphi = 0)$ and $K(x, y) = \widehat{V}(0)\delta_{x,y}\varphi(x)^2$.

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The Gross–Pitaevskii regime

For $0 < \beta \leq 1$

$$H_N^{GP} = \sum_{i=1}^{N} -\Delta_{x_i} + V_{\text{ext}}(x_i) + \sum_{1 \leq i < j \leq N} N^2 V(N(x_i - x_j)) \quad \text{on} \quad \bigotimes_s^N L^2(\mathbb{R}^3)$$

- $\Psi(x_1, \dots, x_N) = u(x_1) \dots u(x_N)$ captures the leading order

$$\lambda_1(H_N^{GP}) = N \inf_{\|\varphi\|_2 = 1} \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 + V_{\text{ext}} |u|^2 + 4\pi \mathfrak{a}_0 \int_{\mathbb{R}^3} |u|^4 \right\} + \mathcal{O}(1)$$

- Excitation spectrum described by the **Bogoliubov's approximation**

Theorem (Nam-A.T. '21)

$$\left\{ \lambda_i(H_N^{GP}) - \lambda_1(H_N^{GP}) \right\}_{i \geq 2} = \sum_{i \geq 1} n_i e_i + \mathcal{O}(N^{-1/12}),$$

for $\{n_i\}_{i \geq 1} \subset \{1, 2, \dots\}$ *and* $\{e_i\}_{i \geq 1}$ *the eigenvalues of* $E = (D^{1/2}(D + 2K)D^{1/2})^{1/2}$, *where* $D = -\Delta + V_{\text{ext}} + 8\pi \mathfrak{a}_0 \varphi^2 - \mu$ $(D\varphi = 0)$ *and* $K(x, y) = 8\pi \mathfrak{a}_0 \delta_{x,y} \varphi(x)^2$.

$\longrightarrow$ in the translational invariant case $E(p) = \sqrt{p^4 + p^2 16\pi \mathfrak{a}_0}$.

$\longrightarrow$ [Lieb-Yngvason '99], [L-Y-Seiringer '00], [Boccato-Brennecke-Cenatiempo-Schlein '18,'19',20'], [Nam-Napiórkowski-Ricaud-A.T. '22], [Schraven-Brennecke-Schlein '22], [Nam-A.T. '21], [Schraven-Brennecke-Schlein '22], [Hainzl-Schlein-A.T. '22+]

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- Excitation spectrum described by the **Bogoliubov's approximation**

Theorem (Nam-A.T. '21)

$$\text{“}\mathcal{U}^\dagger H_N^{GP} \mathcal{U} = \lambda_1(H_N) + \sum_{i=1}^{N} E_{x_i} + o(1).\text{”}$$

with $E = (D^{1/2}(D + 2K)D^{1/2})^{1/2}$, *where* $D = -\Delta + V_{\text{ext}} + 8\pi \mathfrak{a}_0 \varphi^2 - \mu$ $(D\varphi = 0)$ *and* $K(x, y) = 8\pi \mathfrak{a}_0 \delta_{x,y} \varphi(x)^2$.

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Bogoliubov's approximation

Let φ the GP minimizer, and $P = |\varphi\rangle\langle\varphi|$, $Q = 1 - P$

▷ Bogoliubov's approximation: $\langle Q_1 + \cdots + Q_N \rangle_{\Psi_N} = \mathcal{O}(1)$, $\|Q_i \Psi\| \simeq N^{-1/2}$

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + V_{\mathrm{ext}}(x_i) + \sum_{1 \leq i < j \leq N} N^2 V(N(x_i - x_j))$$

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$$H_N = \sum_{i=1}^{N} (P_i + Q_i)\left(-\Delta_{x_i} + V_{\mathrm{ext}}(x_i)\right)(P_i + Q_i)$$
$$+ \sum_{1 \le i < j \le N} N^2 (P_i + Q_i)(P_j + Q_j) V(N(x_i - x_j))(P_i + Q_i)(P_j + Q_j)$$

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$$
\begin{aligned}
H_N \simeq & \left(\sum_{i=1}^{N} P_i\right)\left(\int |\nabla\varphi|^2 + V_{\mathrm{ext}}|\varphi|^2 + \frac{1}{2}\int N^3 V(N\cdot) * |\varphi|^2|\varphi|^2\right) \\
& + \sum_{i=1}^{N} P_i(-\Delta + V_{\mathrm{ext}} + 2\|V\|_{L^1}\varphi^2)Q_i + h.c. \\
& + \sum_{i=1}^{N} Q_i(-\Delta + V_{\mathrm{ext}} + 2\|V\|_{L^1}\varphi^2)Q_i + \sum_{1\le i<j\le N} N^2 P_i P_j V(N(x_i - x_j))Q_i Q_j + h.c. \\
& + \sum_{1\le i<j\le N} N^2 P_i Q_j V(N(x_i - x_j))Q_i Q_j + h.c. + sym. \\
& + \sum_{1\le i<j\le N} N^2 Q_i Q_j V(N(x_i - x_j))Q_i Q_j.
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$$+ \sum_{i=1}^{N} P_i(-\Delta + V_{\mathrm{ext}} + 2\|V\|_{L^1}\varphi^2)Q_i + h.c.$$

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▷ Bogoliubov's approximation: $\langle Q_1 + \cdots + Q_N \rangle_{\Psi_N} = \mathcal{O}(1)$, $\|Q_i \Psi\| \simeq N^{-1/2}$

$$H_N \simeq N\mathcal{E}_{MF}(\varphi) + \sum_{i=1}^{N} Q_i(-\Delta + V_{\mathrm{ext}} + 2\|V\|_{L^1}\varphi^2 - \mathcal{E}_{MF}(\varphi))Q_j$$
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$$(P+Q)^{\otimes N} = \sum_{k=0}^{N} \mathfrak{P}_k \qquad \Longleftrightarrow \qquad L^2(\mathbb{R}^3)^{\otimes_s N} \simeq \bigoplus_{k=0}^{N} \mathcal{H}_+^k := \mathcal{F}^{\le N}(\mathcal{H}_+).$$

On the Fock space, physical quantities are expressed in terms of creation and annihilation operators

$$\boxed{a^\dagger(f)\Psi = f \otimes_s \Psi} \quad \in \mathcal{H}_+^{k+1} \quad \text{for all } k \ge 0,\ \Psi \in \mathcal{H}_+^k,\ f \in \mathcal{H}_+$$

and $a(f)$ is the adjoint of $a^\dagger(f)$. They satisfy the Canonical Commutation Relations (CCR)

$$a^\dagger(f) = \int f(x) a_x^\dagger \mathrm{d}x, \qquad a(f) = \int \overline{f(x)} a_x \mathrm{d}x,$$

$$[a_x, a_y^\dagger] = \delta_{x,y} \qquad \Longleftrightarrow \qquad [a(f), a^\dagger(g)] = \langle f, g\rangle_{L^2}$$

$$[a_x, a_y] = [a_x^\dagger, a_y^\dagger] = 0 \qquad \Longleftrightarrow \qquad [a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0$$

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$$[a_x, a_y] = [a_x^\dagger, a_y^\dagger] = 0 \qquad \Longleftrightarrow \qquad [a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0$$

Landau's correction

Symplectic diagonalization: Quadratic Hamiltonians can be diagonalized by a unitary operator $\mathcal{U}^*_{\mathrm{Bog}}$ on $\mathcal{F}(\mathcal{H}_+)$ preserving the CCR

$$H_N \simeq N\mathcal{E}_{MF}(u) + \int a_x^\dagger(-\Delta_x + V_{\mathrm{ext}}(x) + 2\|V\|_{L^1}\varphi^2(x) - \mathcal{E}_{MF}(\varphi))a_x + \frac{1}{2}\int N^3 V(N(x-y))\varphi(x)\varphi(y)a_x a_y + h.c.$$

$$\simeq N\mathcal{E}_{MF}(u) + \int E(x,y)(a^\dagger(\sqrt{1+s^2}_x) + a(s_x))(a(\sqrt{1+s^2}_y) + a^\dagger(s_y)) - \mathrm{tr}(sEs)$$

$$\simeq N\mathcal{E}_{\mathrm{GP}}(u) + \int E(x,y)\mathcal{U}^*_{\mathrm{Bog}} a_x^* a_y \mathcal{U}_{\mathrm{Bog}}$$

- ▷ $\mathcal{U}^*_{\mathrm{Bog}} a_x^\dagger \mathcal{U}_{\mathrm{Bog}} = a^\dagger(\sqrt{1+s^2}_x) + a(s_x)$, $\quad s(x,y) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \simeq \mathrm{HS}(\mathcal{H}_+)$
- ▷ $\int E(x,y)a_x^\dagger a_y = \bigoplus_{n\geq 0}\left(\sum_{k=1}^n E_{x_k}\right)$

$$\boxed{E = \left(D_{\mathrm{MF}}^{1/2}(D_{\mathrm{MF}} + 2\|V\|_{L^1}\varphi^2)D_{\mathrm{MF}}^{1/2}\right)^{1/2}}$$

- ▷ Landau: $\int V$ is unphysical and should be replaced by $8\pi\mathfrak{a}(V)$.
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Mathematical proof

▷ extract the **cubic** contribution in order to implement Landau's correction

$$\sum_{1\leq i<j\leq N} N^2 P_i Q_j V(N(x_i - x_j)) Q_i Q_j + h.c. + sym.$$

▷ in second quantized form, this becomes

$$\begin{aligned} H_N \simeq N\mathcal{E}_{MF}(u) &+ \int D_{MF}(x,y) a_x^\dagger a_y + N^{1/2} a^\dagger (D_{MF}\varphi) + h.c. \\ &+ \frac{1}{2}\int N^3 V(N(x-y))\varphi(x)\varphi(y) a_x a_y + h.c. \\ &+ \frac{1}{2}\int N^2 V(N(x-y)) a_x^* a_x^* a_y a_y \end{aligned}$$

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\begin{aligned}
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&+ \int N^{3/2} V(N(x-y))\varphi(y) a_x^* a_x a_y + h.c. \\
&+ \frac{1}{2}\int N^2 V(N(x-y)) a_x^* a_x^* a_y a_y
\end{aligned}
$$

Mathematical proof

▷ in second quantized form, this becomes

$$\begin{aligned}\mathcal{U}_{\mathrm{Bog},1}^{\dagger} H_N \mathcal{U}_{\mathrm{Bog},1} \simeq N\mathcal{E}_{\mathrm{GP}}(u) &+ \int D_{MF}(x,y) a_x^{\dagger} a_y + \mathcal{O}(N^{\beta}) \\ &+ \frac{1}{2}\int N^{3\beta} U(N^{\beta}(x-y))\varphi(x)\varphi(y) a_x a_y + h.c. \\ &+ \int N^{3/2} V(N(x-y))\varphi(y) a_x^* a_x a_y + h.c. \\ &+ \frac{1}{2}\int N^2 V(N(x-y)) a_x^* a_x^* a_y a_y\end{aligned}$$

▷ we implement a first Bogoliubov transformation $\mathcal{U}_{\mathrm{Bog},1}$ to replace $N^3 V(N\cdot)$ by a softer potential $N^{3\beta} U(N^{\beta}\cdot)$ for some small $\beta > 0$ and $\int U = 8\pi\mathfrak{a}(V)$

Mathematical proof

▷ in second quantized form, this becomes

$$\mathcal{U}_{\mathrm{cub}}^{\dagger}\mathcal{U}_{\mathrm{Bog},1}^{\dagger}H_N\mathcal{U}_{\mathrm{Bog},1}\mathcal{U}_{\mathrm{cub}} \simeq N\mathcal{E}_{\mathrm{GP}}(u) + \int D_{\mathrm{GP}}(x,y)a_x^{\dagger}a_y + \mathcal{O}(N^{\beta})$$
$$+ \frac{1}{2}\int N^{3\beta}U(N^{\beta}(x-y))\varphi(x)\varphi(y)a_x a_y + h.c.$$
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▷ we implement a cubic transformation $\mathcal{U}_{\mathrm{cub}} = \exp(S - S^{\dagger})$

$$[\int D_{\mathrm{HF}}(x,y)a_x^{*}a_y + \frac{1}{2}\int N^{2}V(N(x-y))a_x^{*}a_x^{*}a_y a_y, S] = \int N^{3/2}V(N(x-y))\varphi(y)a_x^{*}a_x a_y$$

Mathematical proof

▷ in second quantized form, this becomes

$$\mathcal{U}_{\mathrm{Bog},2}^{\dagger}\mathcal{U}_{\mathrm{cub}}^{\dagger}\mathcal{U}_{\mathrm{Bog},1}^{\dagger}H_N\mathcal{U}_{\mathrm{Bog},1}\mathcal{U}_{\mathrm{cub}}\mathcal{U}_{\mathrm{Bog},1}^{\dagger}$$
$$\simeq N\mathcal{E}_{\mathrm{GP}}(u)+\int E(x,y)a_x^{\dagger}a_y+\mathcal{O}(N^{\beta})-\mathcal{O}(N^{\beta})$$
$$+\frac{1}{2}\int N^2V(N(x-y))a_x^*a_x^*a_ya_y$$

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▷ finally we implement a last Bogoliubov transform $\mathcal{U}_{\mathrm{Bog},2}$ to obtain the excitation spectrum and kill the error $\mathcal{O}(N^{\beta})$

Main result

Theorem (Nam-T' 2021, arXiv:2106.11949)

For every $k \in \mathbb{N}$, when $N \to \infty$,

$$\lambda_k(H_N) - \lambda_1(H_N) = \sum_{i \geq 1} n_i e_i + \mathcal{O}(N^{-1/12}), \quad n_i \in \{0, 1, 2, \ldots\}$$

where the **elementary excitations** $e_1 \leq e_2 \leq \ldots$ are the positive eigenvalues of

$$E = \left(D^{1/2}(D + 16\pi\mathfrak{a}(V)\varphi^2)D^{1/2}\right)^{1/2}$$

with $$D = -\Delta + V_{\mathrm{ext}} + 8\pi\mathfrak{a}(V)\varphi^2 - \mu, \quad D \geq 0, \quad D\varphi = 0$$

Similar results obtained by Brennecke–Schlein–Schraven ['21]

▷ Heuristically, H_N is unitarily equivalent to a non-interacting operator

$$\mathcal{U}^* H_N \mathcal{U} - \lambda_1(H_N) \approx \mathrm{d}\Gamma(E) = \bigoplus_{n=0}^{\infty} \left(\sum_{i=1}^{n} E_{x_i} \right)$$

where the right side operator acts on Fock space of excited particles $\mathcal{F}(\{\varphi\}^\perp)$

Earlier results

- Seiringer ['11] and Grech–Seiringer ['13]: in the **mean-field regime** $N^2 V(Nx) \mapsto N^{-1} V(x)$, the elementary excitations are described by

$$E_{MF} = \left(D_{MF}^{1/2} (D_{MF} + K_{MF}) D_{MF}^{1/2} \right)^{1/2}$$

with $D_{MF} = -\Delta + V_{\mathrm{ext}} + V * \varphi^2 - \mu_{MF}, \quad K_{MF}(x,y) = \varphi(x) V(x-y) \varphi(y)$

Further extensions by Lewin–Nam–Serfaty–Solovej, Derezinśki-Napiórkowski, Nam–Seiringer, Rougerie–Spehner, Pizzo, Bossmann–Petrat–Seiringer

- Boccato–Brennecke–Cenatiempo–Schlein ['19]: the **homogeneous gas** on $\mathbb{T}^3$ has $\varphi = 1$ and

$$e_p = \sqrt{|p|^4 + 16\pi \mathfrak{a}(V)|p|^2}, p \in 2\pi\mathbb{Z}^3$$

This implies **Landau's criterion for superfluidity**

$$\mathrm{d}\Gamma(E) \geq c_0 |\text{total momentum}|, \quad c_0 := \inf_{p \neq 0} \frac{e_p}{|p|} > 0$$

Outlooks

- Good understanding and simple techniques $\Longrightarrow$ application to other settings, eg. Neumann Laplacian
- Open Problem: Proof of BEC in the **thermodynamic limit** $\Omega = [0, L]^3$, $\varphi = L^{-3/2}\mathbb{1}_\Omega$

$$H_N^{\mathrm{TL}} = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} V(x_i - x_j)$$

Conjecture (BEC in the thermodynamic limit)

$$\frac{\langle \Psi_N a^\dagger(\varphi) a(\varphi) \Psi_N \rangle}{N} \geq c_0 > 0 \text{ as } N \underset{NL^{-3}=\rho}{\longrightarrow} \infty$$

- **Intermediate length scales** between TL and GP

$$H_N^\kappa = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^{2-2\kappa} V(N^\kappa (x_i - x_j))$$

for $0 \leq \kappa \leq 2/3$. $\longrightarrow$ [Adhikari - Brennecke - Schlein '20], [Brennecke - Caporaletti- Schlein '21], [Fournais '20]

- study the **positive temperature** case

$$f(T, \rho) = 4\pi\rho^2 a \left(1 + \sqrt{\rho a^3}\, C_{\mathrm{LHY}}\left(\frac{T}{a\rho}\right) + o(\sqrt{\rho a^3})\right), \quad C_{\mathrm{LHY}}(\alpha) \to_{\alpha\to 0} \frac{128}{15\sqrt{\pi}},$$

Thank you for your attention.

Comparison with previous works

- ▷ Both Grech-Seiringer ['13], Boccato–Brennecke–Cenatiempo–Schlein ['19], Brennecke-Schlein-Schraven ['21] ×2 use approximate CCR:

$$b^\dagger(g) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (|Qg\rangle \langle \varphi|)_i \quad \Longrightarrow \quad [b_x^\dagger, b_y] = (1 - \frac{\mathcal{N}}{N})\delta_{x,y} - \frac{1}{N} a_x^* a_y$$

$$\Longrightarrow \widetilde{\mathcal{U}}_{Bog}^\dagger b^\dagger(g) \widetilde{\mathcal{U}}_{Bog} = b^\dagger(\sqrt{1+s^2}g) + b(s^2 g) + \text{error}$$

They work in fact in $\mathcal{H}^N$ (Canonical picture).

▷ We use the embedding $\mathcal{H}^N \simeq \mathcal{F}^{\leq N}(\mathcal{H}_+) \hookrightarrow \mathcal{F}(\mathcal{H})$ (full Fock space: Grand-Canonical picture), where

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- We renormalize the interaction potential as an intermediate step $N^3 V(N(x-y)) \longrightarrow N^{3\beta} U(N^\beta(x-y))$ for $0 < \beta < 1$

$$2N^3(-\Delta f_U^\beta)(N(x-y)) + N^3 V(N(x-y)) = N^{3\beta} U(N^\beta(x-y))$$

the idea goes back to Dyson. Recall

$$\Psi \simeq Z_{f_U^\beta}^{-1} \prod_{1 \leq i < j \leq N} f_U^\beta(N(x_i - x_j)) \prod_{i=1}^{N} u(x_i).$$

▷ $\beta \simeq 1$ very good approximation but the new potential is still very singular
▷ $\beta \simeq 0$ big error terms but we can apply mean-field techniques

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Comparison with previous works

- Working on the full Fock space $\mathcal{F}(\mathcal{H})$, we can do very precise computations

Theorem (Nam, Napiórkowski, Ricaud & T' ['21]): refined lower bound on quadratic Hamiltonians

Assuming $H \geq (1+\varepsilon)\|K\|_{\mathrm{op}}$, we have

$$\int H(x,y)a_x^\dagger a_y + \frac{1}{2}\iint K(x,y)(a_x^* a_y^* + a_x a_y)dxdy$$
$$\geq -\frac{1}{4}\,\mathrm{tr}\,(H^{-1}K^2) - C_\varepsilon\|K\|_{\mathrm{op}}\,\mathrm{Tr}(H^{-2}K^2)$$

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Bogoliubov's excitations

▷ We want to describe

$$\lambda_k(H_N) - \lambda_1(H_N) = \mathcal{O}(1) \qquad \text{for every } k \in \mathbb{N}$$

The GP theory captures also the 2nd order ! Taylor expansion of the GP functional with $v \perp \varphi$

$$\mathcal{E}_{\mathrm{GP}}\left(\frac{\varphi + v}{\sqrt{1 + \|v\|_{L^2}^2}}\right) = \mathcal{E}_{\mathrm{GP}}(\varphi) + \frac{1}{2}\left\langle \begin{pmatrix} v \\ \overline{v} \end{pmatrix}, \mathcal{E}''_{\mathrm{GP}}(\varphi) \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \right\rangle_{\mathcal{H}_+ \oplus \overline{\mathcal{H}_+}} + o(\|v\|_{H^1}^2)$$

The Hessian matrix

$$\mathcal{E}''_{\mathrm{GP}}(\varphi) = \begin{pmatrix} D + 8\pi\mathfrak{a}(V)\varphi^2 & 8\pi\mathfrak{a}(V)\varphi^2 \\ 8\pi\mathfrak{a}(V)\varphi^2 & D + 8\pi\mathfrak{a}(V)\varphi^2 \end{pmatrix}$$

with $D = -\Delta + V_{\mathrm{ext}} + 8\pi\mathfrak{a}(V)\varphi^2 - \mu$, $D\varphi = 0$, can be diagonalized by a real, symplectic matrix

$$\mathcal{V}^* \mathcal{E}''_{\mathrm{GP}}(\varphi)\mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \sqrt{1+s^2} & s \\ s & \sqrt{1+s^2} \end{pmatrix}$$

with the one-body **excitation operator** on $\mathcal{H}_+ = \{\varphi\}^\perp \subset L^2(\mathbb{R}^3)$

$$E = \left(D^{1/2}(D + 16\pi\mathfrak{a}(V)\varphi^2)D^{1/2}\right)^{1/2}$$