

The Hybrid Bootstrap

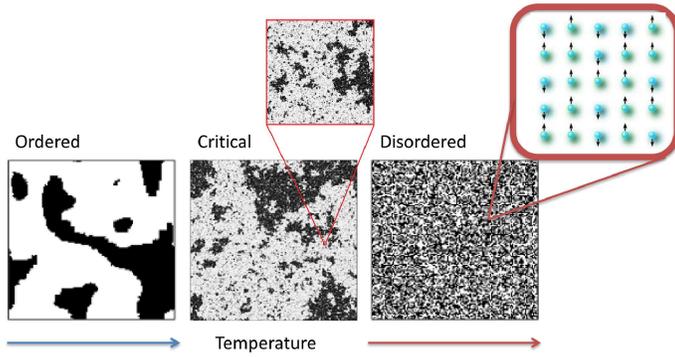
Ning SU
(University of Pisa)

24/02/2022 @ Institut Denis Poisson, Tours

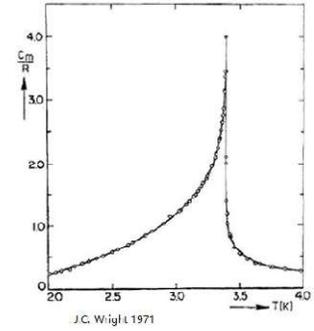
(Based on Ning Su, arXiv:2202.07607)

Critical phenomena : Ising model

$$\mathcal{H}_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$



Conformal invariance



$$C_p \approx (T - T_c)^{-\alpha}$$

critical exponent

$$\alpha = 2 - 3 / (3 - \Delta_\epsilon)$$

$$\eta = 2 \Delta_\sigma - 1$$

Ising model : studied for 100 years!

1d



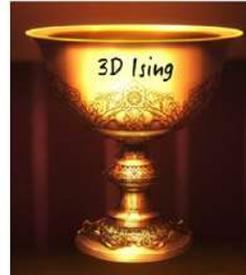
Ernst Ising
1925

2d



Lars Onsager
1944

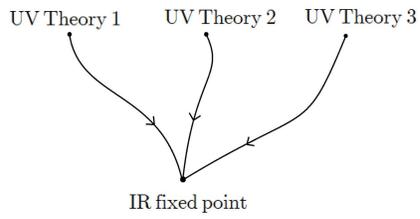
3d



Holy Grail !

Traditional methods

Simulate *a* UV theory



4- ϵ perturbation theory

4D Free Theory

$$\Delta_\sigma = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108} + \frac{109\epsilon^3}{11\,664} + \epsilon^4 \left(\frac{7217}{1\,259\,712} - \frac{2\text{Zeta}[3]}{243} \right) + \dots$$

diverge

3D Ising CFT

Traditional methods don't work universally for every problem

Better starting point for 3D universality class ?

A paradigm shift:

Study the theory **exactly** at the IR fixed point and **non-perturbatively**
(WITHOUT any flow)

Conformal Bootstrap

Conformal Bootstrap :
use **consistency conditions** to constrain the macroscopic theory **without resorting to the UV details**



Credit: Dennis Boos, Leonard Stefanski

"If the 1971 renormalization group ideas had not been developed, the Migdal-Polyakov bootstrap would have been the most promising framework of its time for trying to further understand critical phenomena. However, the renormalization group methods have proved both easier to use and more versatile, and the bootstrap receives very little attention today." --- K.G. Wilson 1982 Nobel Prize Lecture

Revived in 2008. Funded by Simons Foundation in 2017.

An organic mix of High Energy theory, Condensed matter Physics, Computer Science

Constraints for conformal field theory

1, Spectrum: infinite set of operators:

$$\mathcal{O}_{\Delta, \ell}$$

↑ scaling dimension ↙ spin

$$\alpha = 2 - 3/(3 - \Delta_\sigma)$$

$$\eta = 2\Delta_\sigma - 1$$

2, Interaction among operators :

$$\mathcal{O}_i \times \mathcal{O}_j \sim \sum_k \begin{array}{c} \mathcal{O}_i \\ \diagdown \\ \mathcal{O}_j \end{array} \mathcal{O}_k \sim \sum_k f_{ijk} \mathcal{O}_k$$

Operator Product Expansion (OPE) coefficients

3, Crossing symmetry :

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \sim \sum_k \begin{array}{c} 1 \quad 4 \\ \diagdown \quad \diagup \\ \mathcal{O}_k \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} = \sum_{k'} \begin{array}{c} 1 \quad 4 \\ \diagdown \quad \diagup \\ \mathcal{O}_{k'} \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array}$$

All combined :

$$\langle \phi \phi \phi \phi \rangle \Rightarrow \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 F_{\Delta, \ell}(z, \bar{z}) = 0$$

What choices of $\{f_{ijk}, \Delta_i\}$ are consistent ?

Example of the bootstrap equation

Consider four-point function $\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle$ in 2D (Dolan and Osborn 2001, 2004) :

$$\sum_{\mathcal{O} \in \phi \times \phi} f_{\phi\phi\mathcal{O}}^2 F_{\Delta, \ell}(z, \bar{z}) = 0$$

$$F_{\Delta, \ell} = ((1-z)(1-\bar{z}))^{\Delta_\phi} g_{\Delta, \ell}(z, \bar{z}) - (z\bar{z})^{\Delta_\phi} g_{\Delta, \ell}(1-z, 1-\bar{z})$$

$$g_{\Delta, \ell}(z, \bar{z}) = k_{\Delta+l}(z) k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})$$

$$k_\beta(z) = x^{\beta/2} {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta; z\right)$$

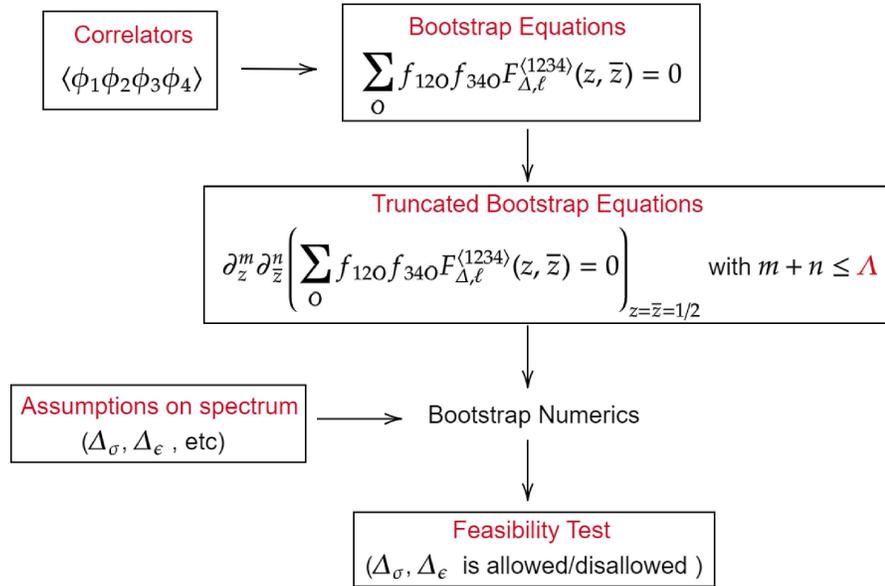
$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z \bar{z}, \quad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

$$0 < z < 1, \quad 0 < \bar{z} < 1$$

Overview

- 50 Introduction
- 50 *Numerical bootstrap*
- 50 Analytic bootstrap
- 50 Hybrid bootstrap
- 50 Outlook

Bootstrap workflow

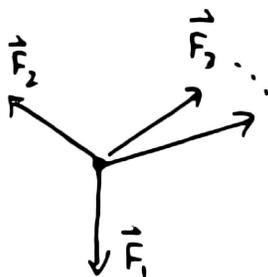


Numerics

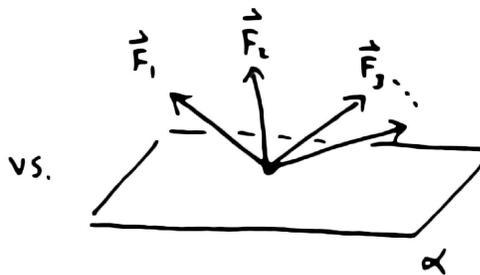
How to solve $\sum_{O \in \phi \times \phi} \int \phi \phi O^2 F_{\Delta, \ell}(z, \bar{z}) = 0$? (bootstrap Equ. from $\langle \phi \phi \phi \phi \rangle$)

Step 1 : $F_{\Delta, \ell}(z, \bar{z}) \approx \sum_{m+n \leq \Lambda} c_{mn} \partial_z^m \partial_{\bar{z}}^n F_{\Delta, \ell}(z, \bar{z})$ $F_{\Delta, \ell}(z, \bar{z}) \rightarrow$ vector $(c_{01}, c_{12}, c_{23} \dots)$

Step 2 :



Can't found α : feasible



Found $\alpha \cdot F_{\Delta, \ell} \geq 0$: infeasible (**rigorous**)

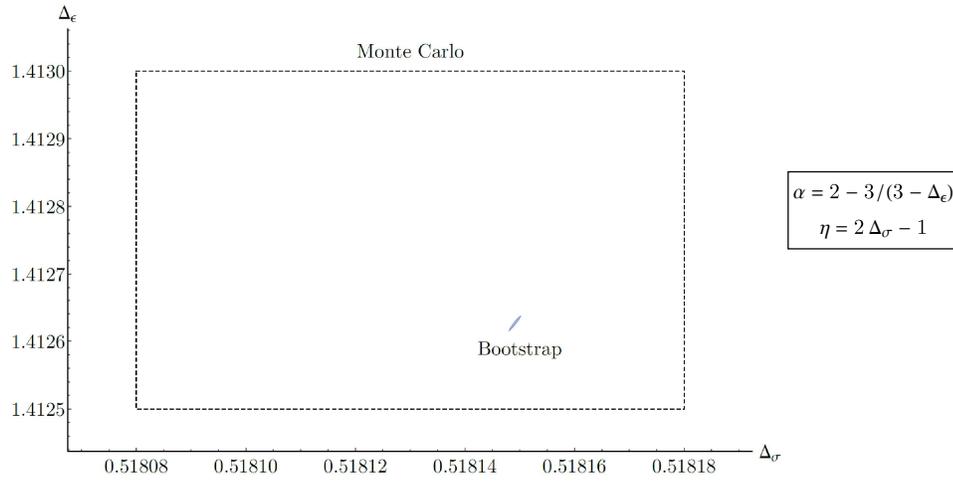
α : a linear functional taking $f(z, \bar{z}) \rightarrow \mathbb{R}$
Numerically it's a Semi-Definite Program(SDP).

Bootstrap 3D Ising CFT

Correlator : $\langle \sigma\sigma\sigma\sigma \rangle$, $\langle \epsilon\epsilon\epsilon\epsilon \rangle$, $\langle \epsilon\epsilon\sigma\sigma \rangle$

Assumptions : σ , ϵ are the only two relevant scalars.

Ising: Scaling Dimensions



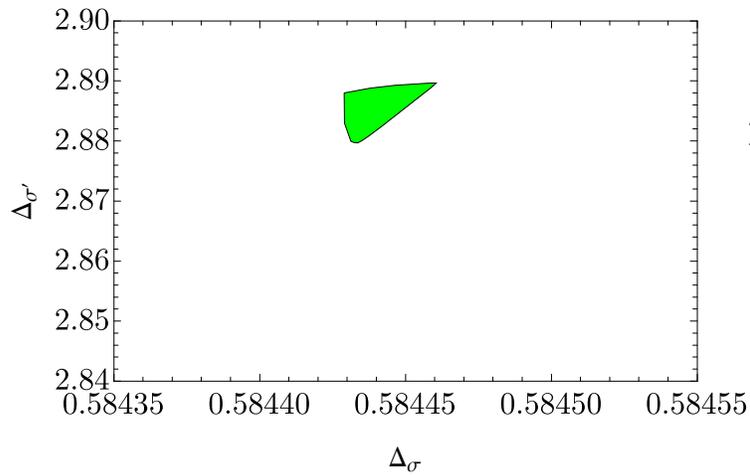
$\eta = 0.036298$ (2), $\alpha = 0.11008$ (1)
(Kos, Poland, Simmons-Duffin, Vichi 2016)

bootstrap 3D super-Ising

$$\mathcal{L}_{\text{SuperIsing}} = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} \bar{\psi} \not{\partial} \psi + \frac{\lambda}{2} \sigma \bar{\psi} \not{\partial} \psi + \frac{\lambda^2}{8} \sigma^4$$

Bootstrapping $\langle \sigma \sigma \sigma \sigma \rangle$ (Rong, NS 2018)

$\Lambda = 35$



$$\Delta_\sigma = 0.584444 \text{ (30)}$$

$$\eta = 0.168888 \text{ (60)}$$

Emergent supersymmetry (SUSY)
(Grover, et al 2014)

Before our work :

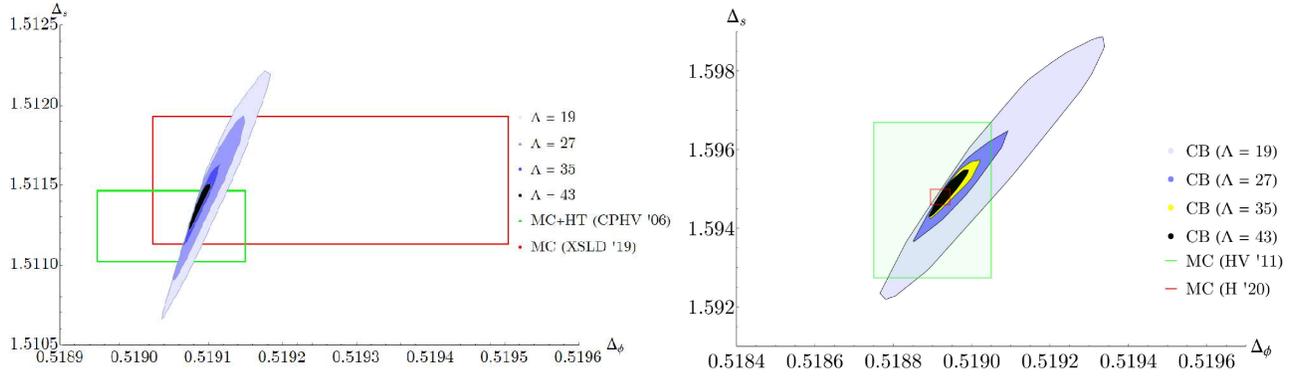
$$\Delta_\phi = 0.585 \text{ (four loop)}$$

no Monte Carlo

Bootstrap 3D O(N) CFTs

Lagrangian : $\mathcal{L} = \partial\phi_i \partial\phi_i + m^2 (\phi_i \phi_i) + \lambda (\phi_i \phi_i)^2$

Bootstrapping all 4pt involves $\{v = \phi_i, s = \phi^2, t = \phi_i \phi_j\}$:



(Chester, Landry, Liu, Poland, Simmons-Duffin, [SN](#), Vichi 2019, 2020)

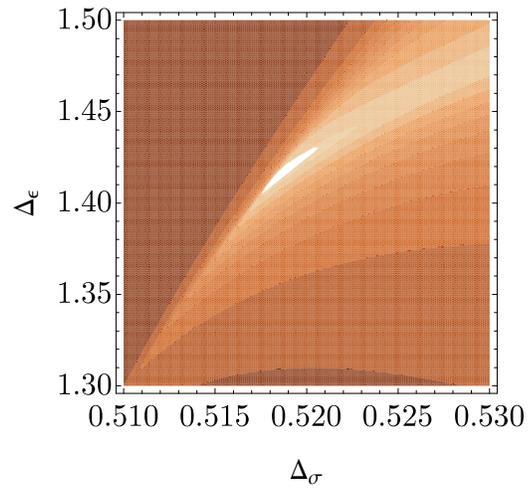
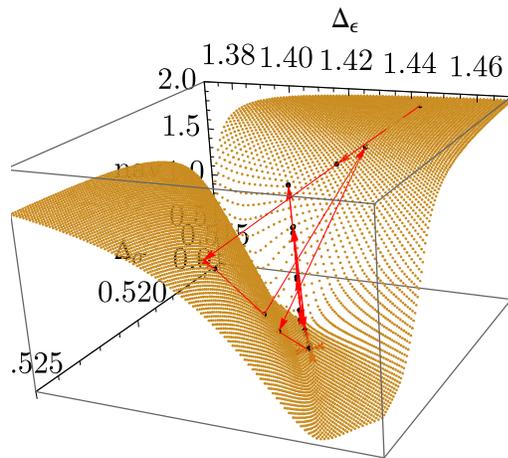
Difficulties in numerical bootstrap

- *Need to scan a parameter space with a large dimensional*
- *Slow convergence at large Λ*

The navigator function

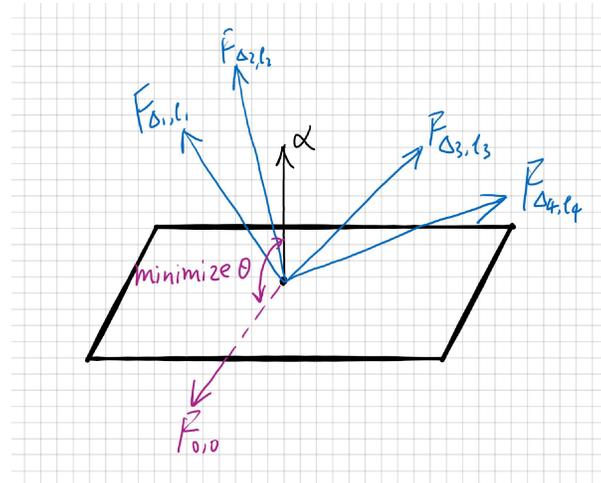
“feasible/infeasible” → a continuous measure of success (navigator function)

Unspecified $\theta_{\sigma\epsilon}$, $\Lambda=11$



(Reehorst, Rychkov, Simmons-Duffin, Sirois, [SN](#), van Reese 2021)

The navigator function : details



minimize θ , s.t. $\alpha(F_{\Delta, l}(z, \bar{z})) \geq 0$ for other operators in the spectrum

If $\min(\theta) \geq 90^\circ$: Infeasible; If $\min(\theta) < 90^\circ$: Feasible

Overview

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Analytic lightcone bootstrap

Bootstrap equation for $\langle \sigma\sigma\sigma\sigma \rangle$:

$$\left(\frac{(1-z)(1-\bar{z})}{z\bar{z}}\right)^{2h_\sigma} \sum_{\mathcal{O} \in \sigma \times \sigma} f_{\sigma\sigma\mathcal{O}}^2 g_{h_\sigma, \bar{h}_\sigma}(z, \bar{z}) = \sum_{\mathcal{O} \in \sigma \times \sigma} f_{\sigma\sigma\mathcal{O}}^2 g_{h_\sigma, \bar{h}_\sigma}(1-z, 1-\bar{z})$$

$$h = \frac{\Delta - \ell}{2}, \quad \bar{h} = \frac{\Delta + \ell}{2}$$

In $z \ll 1 - \bar{z} \ll 1$ limit, $g_{h, \bar{h}}(z, \bar{z}) \approx z^h \bar{z}^{\bar{h}} {}_2F_1(\bar{h}, \bar{h}, 2\bar{h}, \bar{z})$

$$\left(\frac{1-\bar{z}}{z}\right)^{2h_\sigma} \underset{z \ll 1 - \bar{z} \ll 1}{\approx} \sum_{\mathcal{O}} f_{\sigma\sigma\mathcal{O}}^2 (1-\bar{z})^{h_\sigma} {}_2F_1(\bar{h}_\sigma, \bar{h}_\sigma, 2\bar{h}_\sigma, 1-z) \rightarrow \sum_{\mathcal{O}} f_{\sigma\sigma\mathcal{O}}^2 \frac{\Gamma(2\bar{h}_\sigma)}{\Gamma(\bar{h}_\sigma)^2} \log(z)$$

How could it possible that $z^{-2h_\sigma} = x_1 \log(z) + x_2 \log(z) + \dots$?

There exist a family of operators $[\sigma\sigma]_{0, \ell}$ such that at $\ell \rightarrow \infty$

$$\Delta_{[\sigma\sigma]_{0, \ell}} \rightarrow 2\Delta_\sigma + \ell, \quad f_{\sigma\sigma}^2_{[\sigma\sigma]_{0, \ell}} \rightarrow \frac{\sqrt{\pi}}{\Gamma(2h_\sigma)^2} 2^{3-2\ell} \ell^{-\frac{3}{2}+4h_\sigma}$$

Numerics can't capture the $\ell \rightarrow \infty$ behavior!

Solving bootstrap equation at the lightcone limit

Prediction from the Inversion formula (Caron-Huot 2017; Liu, Meltzer, Poland, Simmons-Duffin 2020)

Generating function : $C(z, \bar{h}) = f_{\sigma\sigma[\sigma\sigma]_0}(\bar{h}) z^{2 h_{[\sigma\sigma]_0}(\bar{h})} + \dots$

$$C_{\mathcal{O}}(z, \bar{h}) \approx f_{\sigma\sigma\mathcal{O}}^2 2 \sin(\pi (h_{\mathcal{O}} - 2 h_{\sigma}))^2 \sum_{p=0}^{\infty} \sum_{q=-p}^p \mathcal{A}_{p,q}(h_{\mathcal{O}}, \bar{h}_{\mathcal{O}}) \frac{z^{2 h_{\sigma}} k_{\bar{h}_{\mathcal{O}}+q}^{1-z}}{(1-z)^{2 h_{\sigma}}} \kappa_2 \bar{h} \Omega_{\bar{h}, h_{\mathcal{O}}+p, 2 h_{\sigma}}^{h_{\sigma}}$$

$$C(z, \bar{h}) = \sum_{\mathcal{O} \in \sigma \times \sigma} C_{\mathcal{O}}(z, \bar{h})$$

At large \bar{h} , the main contribution to $C(z, \bar{h})$ come from $C_{\mathcal{O}}(z, \bar{h})$ with small $h_{\mathcal{O}}$.

Information of small twist operators \rightarrow Information about $[\sigma\sigma]_{0,\ell}$ at large ℓ .

Analytic lightcone bootstrap : Example

Using $\{\epsilon, T\}$ to approximate the generating function

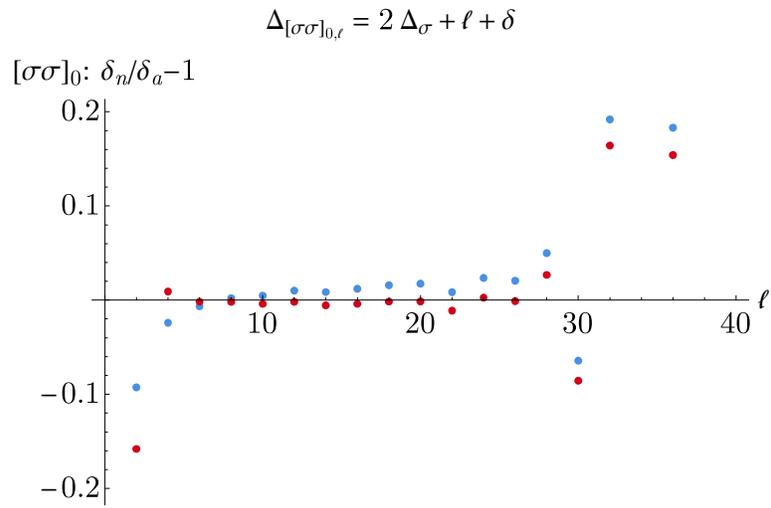
$$\Delta_{[\sigma\sigma]_{0,\ell}} \approx 2\Delta_\sigma + \ell +$$

$$\begin{aligned} & \left(2 \left(- \left((128 f_{\sigma\sigma T}^2 \Gamma(\bar{h})^2 \Gamma(\bar{h} + \Delta_\sigma - \frac{3}{2})) / \right. \right. \right. \\ & \quad \left. \left. \left(3 \pi \Gamma(2\bar{h} - 1) \Gamma(\Delta_\sigma - \frac{1}{2})^2 \Gamma(\bar{h} - \Delta_\sigma + \frac{3}{2}) \right) \right) - \right. \\ & \quad \left. \left(f_{\sigma\sigma\epsilon}^2 \Gamma(\bar{h})^2 \Gamma(\Delta_\epsilon) \Gamma(\bar{h} - \frac{\Delta_\epsilon}{2} + \Delta_\sigma - 1) \right) / \right. \\ & \quad \left. \left. \left. \left(\Gamma(2\bar{h} - 1) \Gamma(\frac{\Delta_\epsilon}{2})^2 \Gamma(\Delta_\sigma - \frac{\Delta_\epsilon}{2})^2 \Gamma(\bar{h} + \frac{\Delta_\epsilon}{2} - \Delta_\sigma + 1) \right) \right) \right) \right) / \\ & \left(- \left((128 (2\gamma + 2\psi^{(0)}(\frac{5}{2})) f_{\sigma\sigma T}^2 \Gamma(\bar{h})^2 \Gamma(\bar{h} + \Delta_\sigma - \frac{3}{2})) / \right. \right. \\ & \quad \left. \left. \left(3 \pi \Gamma(2\bar{h} - 1) \Gamma(\Delta_\sigma - \frac{1}{2})^2 \Gamma(\bar{h} - \Delta_\sigma + \frac{3}{2}) \right) \right) - \right. \\ & \quad \left. \left(f_{\sigma\sigma\epsilon}^2 \Gamma(\bar{h})^2 \Gamma(\Delta_\epsilon) (2\psi^{(0)}(\frac{\Delta_\epsilon}{2}) + 2\gamma) \Gamma(\bar{h} - \frac{\Delta_\epsilon}{2} + \Delta_\sigma - 1) \right) / \right. \\ & \quad \left. \left. \left. \left(\Gamma(2\bar{h} - 1) \Gamma(\frac{\Delta_\epsilon}{2})^2 \Gamma(\Delta_\sigma - \frac{\Delta_\epsilon}{2})^2 \Gamma(\bar{h} + \frac{\Delta_\epsilon}{2} - \Delta_\sigma + 1) \right) \right) + \right. \\ & \quad \left. \left. \left. \left(\Gamma(\bar{h})^2 \Gamma(\bar{h} + \Delta_\sigma - 1) \right) / \left(\Gamma(2\bar{h} - 1) \Gamma(\Delta_\sigma)^2 \Gamma(\bar{h} - \Delta_\sigma + 1) \right) \right) \right) \right) \end{aligned}$$

(depends on $\Delta_\sigma, \Delta_\epsilon, f_{\sigma\sigma\epsilon}, f_{\sigma\sigma T}$)

$$\bar{h} = (\Delta_{[\sigma\sigma]_{0,\ell}} + \ell) / 2$$

Analytic lightcone bootstrap : Example



Overview

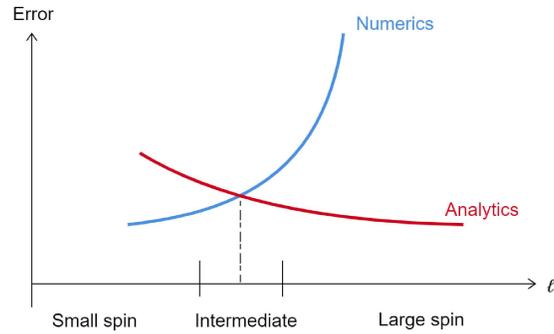
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The hybrid bootstrap

(Based on Ning Su, arXiv:2202.07607)

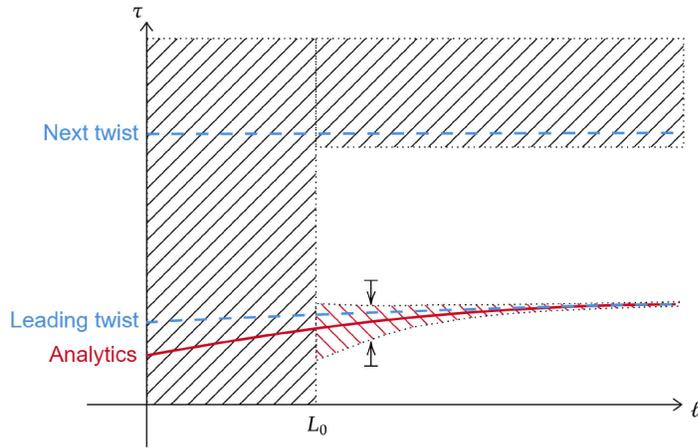
Numerics : around $z = \bar{z} = 1/2$: sensitive to small Δ , **small** l .

Analytics : around $z \ll 1 - \bar{z} \ll 1$: $\Delta_{[\sigma\sigma]_{0,\ell}} \approx f(\Delta_\sigma, \Delta_\epsilon, f_{\sigma\sigma\epsilon}, f_{\sigma\sigma T})$ accurate for **large** l .



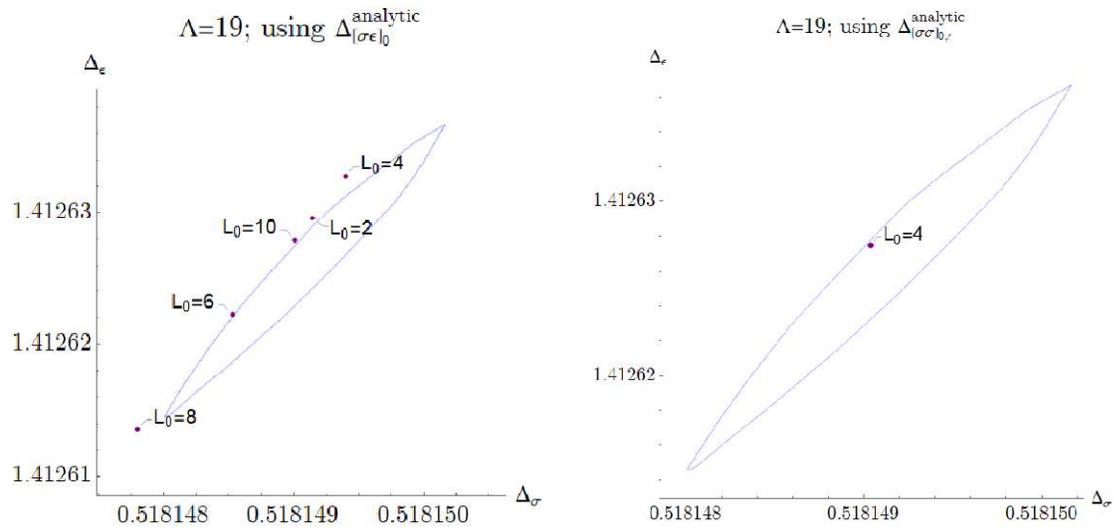
We should combine the numerics & analytics :
two difficulties in the past (before the navigator was invented)...

The hybrid bootstrap



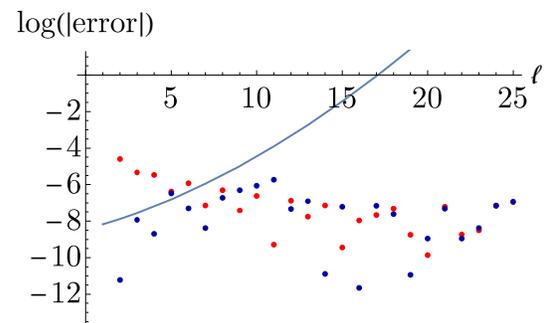
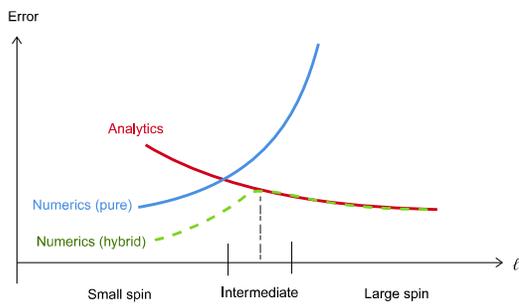
- 1, Choose $(\Delta_\sigma, \Delta_\epsilon, f_{\sigma\sigma\epsilon}, f_{\sigma\sigma T}, \text{Area of red})$
- 2, Compute analytics
- 3, Run the numerics (navigator),
demanding operators exist in the red region
- 4, Repeat and minimize the *Area*
while navigator function ≤ 0

The hybrid bootstrap



hybrid bootstrap at $\Lambda = 19$ predicts very precise values that are within the previous $\Lambda = 43$ rigorous error bars!

The hybrid bootstrap



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Outlook

Huge potential ! and it's really just the beginning!

☞ *Towards the full-fledged hybrid bootstrap*

- ☞ Systematically improve the analytics
- ☞ Find a reliable way to estimate error
- ☞ Find a way to using multiple analytics information at the same time
- ☞ Run the hybrid bootstrap at larger Λ

☞ *Applications*

- ☞ Push the limit of bootstrapping 3d Ising, Super-Ising, O(N) ...
- ☞ Apply hybrid bootstrap to new models:
cubic model, scalar QED, $\mathcal{N}_f = 4$ QED (Dirac spin liquid)