Elementary geometry and gravitational energy

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Outline

Introduction

Gravity is curvature.

How does curvature affect area/volume?

What does matter do to geometry?

Pauli: Theory of Relativity

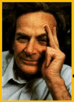
In an arbitrary [3-dimensional] Riemannian manifold, [the volume of a sphere of radius ℓ] becomes a complicated function of ℓ. We can imagine it to be expanded in a power series in ℓ and retain only the [first non-trivial] term. This gives

$$V = \frac{4\pi}{3}\ell^3\left(1 + \frac{\mathcal{R}}{30}\ell^2 + \dots\right)$$

[...] Differentiating, one obtains [...] the formula for the surface of the sphere

$$A = 4\pi\ell^2\left(1 + \frac{\mathcal{R}}{18}\ell^2 + \dots\right)$$

Here, V is the volume of the ball, A is the "area" of its boundary, ℓ its radius, and $\mathcal{R}$ the scalar curvature at the ball's center.



What matter does to geometry

The rule that Einstein gave for the curvature is the following: If there is a region of space with matter in it and we take a sphere small enough that the density ϱ of matter inside it is effectively constant, then the radius excess for the sphere is proportional to the mass inside the sphere. Using the definition of excess radius, we have

$$\delta\ell|_A = \ell - \sqrt{\frac{A}{4\pi}} = \frac{G}{3c^2}M\left(= \frac{G}{3c^2}\frac{4\pi}{3}\varrho\ell^3\right)$$

Here M is the mass inside the sphere, and $\delta\ell|_A$ is the "excess" radius to keep the area fixed at its flat spacetime value.

Exponential map: Riemann normal coordinates

- In any semi-Riemannian manifold (M, g) geodesic curves are uniquely defined by having vanishing acceleration:

$$\nabla_{\vec{v}}\vec{v} = 0 \qquad (\Longleftrightarrow v^\mu \nabla_\mu v^\alpha = 0).$$

- In local coordinates the curve is given by $x^\mu = X^\mu(\tau)$ and this reads

$$\frac{d^2 X^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dX^\rho}{d\tau} \frac{dX^\sigma}{d\tau} = 0$$

  where $\Gamma^\mu_{\rho\sigma}$ are the Christoffel symbols of the connection ∇.
- From the standard theory of ODEs, given an initial point $p := X^\mu(0)$ and initial 'velocity' $v^\mu := dX^\mu/d\tau(0)$ there is a unique solution, that is, a unique geodesic.
- This allows to define a system of local coordinates in a neighbourhood of p by using the *exponential map* from the tangent space at p into the manifold: for every tangent vector v^μ at p take the point in the manifold at $\tau = 1$ along the unique geodesic defined by v^μ.

Riemann normal coordinates

The deviations away from flatness near p can be characterized by the curvature at p via the standard *Riemann Normal Coordinate* (RNC) expansion

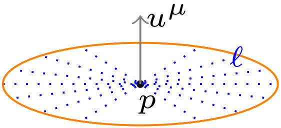
$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} - \frac{1}{3}x^\mu x^\nu R_{\alpha\mu\beta\nu} - \frac{1}{6}x^\mu x^\nu x^\rho \nabla_\mu R_{\alpha\nu\beta\rho}$$
$$+ x^\mu x^\nu x^\rho x^\sigma \left(\frac{2}{45} R^\gamma{}_{\mu\alpha\nu} R_{\gamma\rho\beta\sigma} - \frac{1}{20} \nabla_\mu \nabla_\nu R_{\alpha\rho\beta\sigma} \right) + O(x^5).$$

Here, $\eta_{\alpha\beta}$ is the flat (Minkowski) metric and $R_{\alpha\mu\beta\nu}$, $\nabla_\mu R_{\alpha\nu\beta\rho}$, and $\nabla_\mu \nabla_\nu R_{\alpha\rho\beta\sigma}$ represent the components of the curvature and its covariant derivatives at p.

$$\mu, \nu, \cdots = 0, 1, 2, 3, \quad i, j, \cdots = 1, 2, 3, \quad A, B, \cdots = 2, 3$$

Spatial geodesic balls

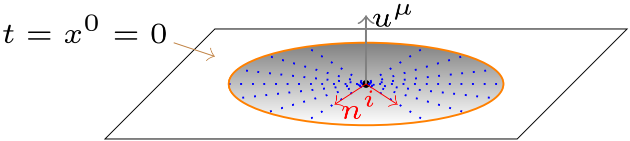
Choose $p \in \mathcal{M}$ and then choose $u^\mu \in T_p\mathcal{M}$, $u^\mu u_\mu = -1$.



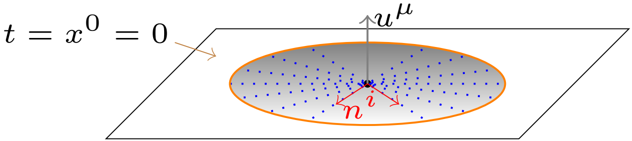
- Adapt the RNC $\{x^\mu\}$ based at $p := \{x^\mu = 0\}$ so that $u^\mu = \delta^\mu_0$
- The spatial geodesic ball lies on the hypersurface $t \equiv x^0 = 0$ and $\{x^i\}$ can be used as coordinates on the ball.
- The spacelike geodesics generating it have

$$x^\mu = r n^\mu, \quad u_\mu n^\mu = 0, \quad \Longrightarrow n^\mu = n^i \delta^\mu_i$$

where r is the affine parameter and we set $\delta_{ij} n^i n^j = 1$



Spatial geodesic balls' metric



The metric on the ball reads (careful with the notation!)

$$h_{ij}(x) = \delta_{ij} - \frac{1}{3}x^k x^l R_{ikjl} - \frac{1}{6}x^k x^l x^m \nabla_k R_{iljm}$$
$$+x^k x^l x^m x^n \left(-\frac{2}{45}R_{0kil}R_{0mjn} + \frac{2}{45}R^p_{kil}R_{pmjn} - \frac{1}{20}\nabla_k\nabla_l R_{imjn}\right) + \ldots$$

Spatial geodesic ball's boundary

- Denote by $\{\theta^A\} := \{\vartheta, \varphi\}$ the local $\mathbb{S}^2$ coordinates on the ball's boundary,
- Such boundary is described by giving its "radius" r as a function of the initial direction n^i, or equivalently of $\{\vartheta, \varphi\}$.
- Observe that $n^i(\vartheta, \varphi)$ are given by

$$n^1 = \sin\vartheta\cos\varphi, \qquad n^2 = \sin\vartheta\sin\varphi, \qquad n^3 = \cos\vartheta$$

(the standard embedding of the unit 2-sphere in Euclidean space).

- The induced metric on the boundary of the ball reads

$$q_{AB} = h_{ij}(rn^k)\frac{\partial x^i}{\partial\theta^A}\frac{\partial x^j}{\partial\theta^B}.$$

Area and volume at first order

- Define the ball's boundary at first order by $r = \ell + \delta\ell^{(1)}$.
- $\delta\ell^{(1)}$ is a function defined on the 2-sphere so that it can be expanded in spherical harmonics. Letting s denote the "spin":

$$\delta\ell^{(1)} = \sum_{s=0}^{\infty} Y_{i_1\ldots i_s} n^{i_1} \ldots n^{i_s} := \delta\ell_1 + \sum_{s=1}^{\infty} Y_{i_1\ldots i_s} n^{i_1} \ldots n^{i_s}$$

  where $Y_{i_1\ldots i_s}$ are totally symmetric and traceless for $s > 1$.
- A calculation at linear order in the curvature gives, for the volume of the geodesic ball and the area of its boundary

$$V - V^\flat = 4\pi\ell^2 \left(\delta\ell_1 - \frac{\mathcal{R}}{90}\ell^3\right) := \delta^{(1)} V$$

$$A - A^\flat = 4\pi\ell \left(2\delta\ell_1 - \frac{\mathcal{R}}{18}\ell^3\right) := \delta^{(1)} A$$

  where $V^\flat = 4\pi\ell^3/3$ is the volume of a radius ℓ round ball in Euclidean space, $A^\flat = 4\pi\ell^2$ has a similar meaning and $\mathcal{R}$ is the intrinsic scalar curvature of the $t = 0$ hypersurface at p.

Using the Einstein field equations

- Note: at first order, the volume and area depend only on the spherically symmetric "excess" $\delta\ell_1$.
- Observe: we recover Pauli's remark by just setting $\delta\ell_1 = 0$ (keep the radius of the ball fixed!).
- We also recover Feynman's interesting remark by keeping the area fixed ($A = A^\flat$), recalling the constraint (or Gauss) relation ($G_{\mu\nu}$ is the Einstein tensor)

$$\mathcal{R} = 2G_{\mu\nu}u^\mu u^\nu + K_{\mu\nu}K^{\mu\nu} - K^2 \quad \Longrightarrow \quad \mathcal{R} = 2G_{00}$$

and using Einstein's field equations !

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

- Then

$$< \delta r > |_A = \delta\ell_1|_A = \frac{\ell^3}{18} G_{00} = \frac{8\pi G}{c^4}\frac{\ell^3}{18} T_{00} = \frac{G}{3c^2}\frac{4\pi}{3}\underbrace{\frac{T_{00}}{c^2}}_{\varrho}\ell^3$$

Variations of volume and the energy density

- Similarly,

$$< \delta r > |_V = \delta \ell_1|_V = \frac{\ell^3}{45} G_{00} = \frac{8\pi G}{c^4} \frac{\ell^3}{45} T_{00} = \frac{2G}{15c^2} \frac{4\pi}{3} \underbrace{\frac{T_{00}}{c^2}}_{\varrho} \ell^3.$$

- What is to be compared? One can keep one quantity (radius, area and volume) fixed and see the effect on the other two.
- At this order, these variations are

$$\delta^{(1)} V|_r = \delta^{(1)} V|_{<r>} = -\frac{8\pi G}{c^4} 4\pi \frac{\ell^5}{45} T_{00}, \qquad \delta^{(1)} V|_A = -\frac{8\pi G}{c^4} 4\pi \frac{\ell^5}{30} T_{00}$$

$$\delta^{(1)} A|_r = \delta^{(1)} A|_{<r>} = -\frac{8\pi G}{c^4} 4\pi \frac{\ell^4}{9} T_{00}, \qquad \delta^{(1)} A|_V = -\frac{8\pi G}{c^4} 4\pi \frac{\ell^4}{15} T_{00}$$

- The variation is, in all cases, proportional to the energy density at the center of the ball, with a different proportionality factor.
- Is there a correct, or physically preferred, factor? Why Feynman chose the radius excess?

Vacuum!

What does pure gravity do to geometry?

Area deficit in vacuum

- If $G_{\mu\nu} = 0$ the radius excess and the area and volume deficits all vanish at first order in the curvature.
- However, the gravitational field is itself a source of curvature, and this "purely gravitational" curvature affects geometric quantities too.
- If we trust any of the relations between radius/area/volume changes and energy density at first order, the corresponding radius/area/volume changes in vacuum — which arise at higher orders— should be related, in one way or another, to the pure gravitational energy density
- Alternatively, such variations could help provide a notion of quasilocal energy for the gravitational field.
- At second order in the RNC expansion, the volume of the ball and the area of its boundary receive corrections depending quadratically on the curvature.

Interlude: the electromagnetic field

- The electromagnetic field is given by a 2-form $\boldsymbol{F}$ that satisfies Maxwell equations

$$d\boldsymbol{F} = 0, \qquad \delta\boldsymbol{F} = \boldsymbol{j}$$

(Here $\delta := \star d \star$ is the co-differential, and $\star$ the Hodge dual operation for the canonical volume-element 4-form: $(\star F)_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu\rho\sigma}F^{\rho\sigma}$).

- The electric and magnetic fields relative to an observer u^μ are defined by

$$\boldsymbol{\mathcal{E}} := i_u \boldsymbol{F}, \qquad \mathcal{E}_\mu := -F_{\mu\nu}u^\nu$$

$$\boldsymbol{\mathcal{B}} := i_u \star \boldsymbol{F}, \qquad \mathcal{B}_\mu := -(\star F)_{\mu\nu}u^\nu$$

- These are spacelike vectors and spatial (relative to u^μ): $u^\mu \mathcal{E}_\mu = u^\mu \mathcal{B}_\mu = 0$.
- At the point p in RNC these read $\mathcal{E}_i = F_{0i}$ and $\mathcal{B}_i = \epsilon_{ijk}F^{jk}$

The Weyl tensor and its decomposition

- The Weyl tensor is the part of the curvature tensor not in the Ricci tensor:

$$R_{\alpha\beta\lambda\mu} = C_{\alpha\beta\lambda\mu} + R_{\alpha[\lambda}g_{\mu]\beta} - R_{\beta[\lambda}g_{\mu]\alpha} - \frac{R}{6}\left(g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda}\right)$$

- $C_{\alpha\beta\lambda\mu} = -C_{\alpha\beta\mu\lambda} = -C_{\beta\alpha\lambda\mu}, \quad C_{\alpha[\beta\lambda\mu]} = 0, \quad C^{\rho}{}_{\beta\rho\mu} = 0$
- Given that $R_{\mu\nu}|_p = 0$, $R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}$ at p.
- $C_{\alpha\beta\mu\nu}$ may be decomposed into their **electric and magnetic** parts with respect to u^μ (we only need them at p)

$$E_{\beta\mu} := C_{\alpha\beta\lambda\mu}u^\alpha u^\lambda, \quad E_{\beta\mu} = E_{\mu\beta}, \quad E_{\beta\mu}u^\mu = 0, \quad E^{\mu}{}_{\mu} = 0$$
$$B_{\beta\mu} := \star C_{\alpha\beta\lambda\mu}u^\alpha u^\lambda, \quad B_{\beta\mu} = B_{\mu\beta}, \quad B_{\beta\mu}u^\mu = 0, \quad B^{\mu}{}_{\mu} = 0.$$

where

$$\star C_{\alpha\beta\lambda\mu} = \frac{1}{2}\eta_{\alpha\beta\rho\sigma}C^{\rho\sigma}{}_{\lambda\mu}$$

is the Hodge dual of the Weyl tensor, $\eta_{\alpha\beta\rho\sigma}$ being the canonical volume element 4-form.

The Electric-magnetic decomposition of $C_{\alpha\beta\mu\nu}$

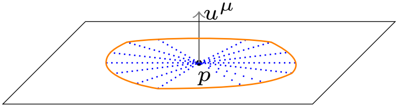
- $E_{\beta\mu}$ and $B_{\beta\mu}$ are purely spatial tensors (relative to u^μ)
- At the point p in the chosen RNC one has

$$\begin{aligned} E_{ij} &= C_{0i0j} && \text{“electric”} \\ B_{ij} &= \frac{1}{2}\epsilon_{jkl}C_{0i}{}^{kl} && \text{“magnetic”} \\ C_{ijkl} &= E_{ik}h_{jl} - E_{jk}h_{il} - E_{il}h_{jk} + E_{jl}h_{ik} && h^{ij}C_{ikjl} = E_{kl}. \end{aligned}$$

(h_{ij} is the metric on the hypersurface $t = 0$)

- Remember, E_{ij} and B_{ij} are traceless and symmetric.

The ball at second order



- Define the ball at second order by

$$r = \underbrace{\ell + \delta\ell_1 + \sum_{s=1}^{\infty} Y_{i_1\dots i_s} n^{i_1}\dots n^{i_s}}_{O(1)} + \underbrace{\delta\ell_2 + \tilde{\delta}\ell_2(\vartheta,\varphi)}_{O(2)}.$$

- $\delta\ell_2$ is the spherically symmetric piece $(s=0)$ of the 2nd-order perturbation to r
- We are going to prove that that $\tilde{\delta}\ell_2$ is irrelevant for the volume and the area at quadratic order in curvature.

Volume of geodesic balls at quadratic order

The volume of the ball at this order (and with $R_{\mu\nu} = 0$) is

$$\begin{aligned} V - V^\flat \quad = \quad & 4\pi \left\{ \underbrace{\ell^2 \delta\ell_1}_{O(1)} + \frac{\ell^7}{1575}\left(-B^2 - \frac{E^2}{6}\right) \right. \\ & \left. + \ell^2\delta\ell_2 + \ell\delta\ell_1^2 + 2\ell\sum_{s=1}^{\infty} c_s Y_{[s]}^2 - \frac{\ell^4}{45} Y^{ij} E_{ij} \right\} \end{aligned}$$

where c_s are known constant factors depending on s. In particular

$$c_2 = \frac{1}{15}.$$

($Y_{[s]}^2 \equiv Y_{i_1 \ldots i_s} Y^{i_1 \ldots i_s}$, $E^2 \equiv E_{ij}E^{ij}$, and $B^2 \equiv B_{ij}B^{ij}$.)

Area of geodesic balls at quadratic order

Analogously, the area of the ball's boundary at this order (and with $R_{\mu\nu} = 0$) is

$$
\begin{aligned}
A - A^{\flat} \quad &= \quad 4\pi \left\{ \underbrace{2\ell\delta\ell_1}_{O(1)} + \frac{\ell^6}{225}\left(-B^2 - \frac{E^2}{6}\right) \right. \\
&\qquad \left. + 2\ell\delta\ell_2 + \delta\ell_1^2 + \sum_{s=1}^{\infty} b_s Y_{[s]}^2 - \frac{4\ell^3}{45} Y^{ij} E_{ij} \right\}
\end{aligned}
$$

where b_s are known constant factors depending on s. In particular

$$b_2 = \frac{8}{15}.$$

($Y_{[s]}^2 \equiv Y_{i_1 \dots i_s} Y^{i_1 \dots i_s}$, $E^2 \equiv E_{ij} E^{ij}$, and $B^2 \equiv B_{ij} B^{ij}$.)

Only spin $s = 2$ is relevant!

- Only the spin-2 deformation gives a different contribution to the area and volume in curved spacetime than in flat spacetime: the term $Y^{ij}E_{ij}$
- Given E_{ij} as data, Y_{ij} can always be split

$$Y_{ij} = \ell^3 \left(\gamma E_{ij} + Z_{ij}\right), \qquad E_{ij}Z^{ij} = 0.$$

- Hence, it is only the component of Y_{ij} aligned with E_{ij} that contributes differently than in flat space.
- Thus, $Y_{[s]}$ for all $s \neq 2$ and Z_{ij} cannot be fixed in terms of the local gravitational field at this order in perturbations.
- Put another way, the only relevant non-spherical deformation at this order is given by Y_{ij} <u>aligned</u> with E_{ij} —so that the gravitational field itself determines the ball's shape at this order.

Variations with Y_{ij} aligned with E_{ij}

With this in mind, setting $\delta\ell_1 = 0$ and $Y_{i_1 \dots i_s} = 0$ for all $s \neq 2$ and

$$Y_{ij} = \gamma \ell^3 E_{ij}$$

and using the explicit values of $b_2 = 8/15$ and $c_2 = 1/15$ one gets

$$V - V^\flat = 4\pi \left\{ \frac{\ell^7}{1575} \left(-B^2 - \frac{E^2}{6} \right) + \ell^2 \delta\ell_2 + 2\gamma^2 \frac{\ell^7}{15} E^2 - \gamma \frac{\ell^7}{45} E^2 \right\}$$

$$A - A^\flat = 4\pi \left\{ \frac{\ell^6}{225} \left(-B^2 - \frac{E^2}{6} \right) + 2\ell \delta\ell_2 + 8\gamma^2 \frac{\ell^6}{15} E^2 - 4\gamma \frac{\ell^6}{45} E^2 \right\}$$

The magenta terms give $\delta^{(2)} A|_r$ and $\delta^{(2)} V|_r$, while the red terms are due to the spin-2 deformation aligned with E_{ij}.
Observe that the magenta terms alone give an expression which is negative definite..., but is it a correct "energy density"?

Variations keeping A or $< r >$ fixed

- Keeping $A = A^\flat$ (so that $\delta\ell_1 = 0$) we derive

$$< \delta r > |_A = \delta\ell_2|_A = \frac{\ell^5}{450}\left[B^2 + E^2 - \frac{5}{6}E^2\,(12\gamma - 1)^2\right]$$

- One can also keep the averaged radius fixed. Recall that, at first order, it was enough to set $\delta\ell_1 = 0$ (the spin-0 part), as the rest did not contribute. Analogously, now only the spin-0 part of the second-order perturbation to the radius contributes at this order, and it is enough to fix $\delta\ell_2 = 0$. Then

$$\delta^{(2)}A|_{<r>} = -4\pi\frac{\ell^6}{225}\left[B^2 + E^2 - \frac{5}{6}E^2\,(12\gamma - 1)^2\right]$$

A pure gravitational energy formula?

Does any of these formulae contain a quasi-local gravitational energy?

What should we expect as the correct answer at this quadratic order, in vacuum?

Required properties

There are several desirable and expected properties for the sought expression if it is to describe gravitational strength:

1. It should be positive definite, zero if and only if $C_{\alpha\beta\mu\nu} = 0$
2. quadratic in the curvature (that is, in $C_{\alpha\beta\mu\nu}$)
3. the timelike component (with respect to u^μ) of a tensor field
4. This tensor field should have the properties ensuring that the putative energy —its totally timelike component— propagates causally, in the sense that it vanishes in the entire domain of dependence of any spacelike region in which it initially vanishes. This is known to require two important ingredients
   - some control of the tensor-field divergence (usually ensured by the underlying field equations if they are hyperbolic)
   - the dominant property, which states that the tensor contracted on any future-pointing vectors is non-negative. This dominant property also guarantees that the 'momentum density' vector (the tensor contracted on u^μ on all indices but one) is future-pointing casual. This momentum density points in the direction of propagation of the putative energy

Interlude: Bel-Robinson super-energy tensor

There is a unique (symmetric) tensor with the above properties
(JMMS , Class. Quantum Grav. **17** (2000) 2799):

the **Bel-Robinson tensor** $\mathcal{T}_{\alpha\beta\mu\nu}$.

Recall: the electromagnetic field

- $T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\left(F_{\mu\rho}F_{\nu}{}^{\rho} + \star F_{\mu\rho}\star F_{\nu}{}^{\rho}\right)$
- $T_{\mu\nu} = T_{\nu\mu}$
- $T^{\rho}{}_{\rho} = 0$
- $$T_{\mu\rho}T_{\nu}{}^{\rho} = \frac{1}{4}g_{\mu\nu}T_{\rho\sigma}T^{\rho\sigma}$$
- $$T_{\mu\nu}u^{\mu}v^{\nu} \geq 0$$

  for arbitrary future-pointing vectors u^{μ} and v^{ν} (inequality is strict if all of them are timelike). This is the Dominant Energy *property*. The energy density of the e.m. field relative to u^{μ} is $T_{\mu\nu}u^{\mu}u^{\nu} = T_{00} = \mathcal{E}^2 + \mathcal{B}^2$.
- $\nabla^{\mu}T_{\mu\nu} = F_{\nu\rho}j^{\rho}$ and therefore $\nabla^{\mu}T_{\mu\nu} = 0$ if there are no charge nor currents ($j^{\mu} = 0$).
- This provides conserved quantities if there are (conformal) Killing vector fields.

Bel-Robinson tensor

- $\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma} C_\beta{}^\rho{}_\mu{}^\sigma + C_{\alpha\rho\mu\sigma} C_\beta{}^\rho{}_\lambda{}^\sigma - \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} C_{\rho\tau\sigma\nu} C^{\rho\tau\sigma\nu}$
- $\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda\sigma} C_\beta{}^\rho{}_\mu{}^\sigma + \star C_{\alpha\rho\lambda\sigma} \star C_\beta{}^\rho{}_\mu{}^\sigma$
- $\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}$
- $\mathcal{T}^\rho{}_{\rho\lambda\mu} = 0$
- $$\mathcal{T}_{\alpha\beta\lambda\mu} \mathcal{T}_\gamma{}^{\beta\lambda\mu} = \frac{1}{4} g_{\alpha\gamma} \mathcal{T}_{\rho\beta\lambda\mu} \mathcal{T}^{\rho\beta\lambda\mu}$$
- $$\mathcal{T}_{\alpha\beta\lambda\mu} u^\alpha v^\beta w^\lambda z^\mu \geq 0$$
  for arbitrary future-pointing vectors u^α, v^β, w^λ, and z^μ (inequality is strict if all of them are timelike). This is called the Dominant property. ($\mathcal{T}_{0000} = 0 \Longrightarrow C_{\alpha\beta\lambda\mu} = 0$).
- $$\nabla^\alpha \mathcal{T}_{\alpha\beta\lambda\mu} = 0$$
  if the vacuum Einstein's field equations $G_{\mu\nu} = 0$ hold.
- This provides conserved quantities if there are (conformal) Killing vector fields.

Equivalence Principle!

Bel-Robinson versus energy

- The Bel-Robinson tensor is reminiscent of energy-momentum tensors, yet it is not such a thing –it cannot be!
- Due to the equivalence principle, the gravitational field does not possess any pointwise-defined energy-momentum tensor.
- Gravitational energy is not localizable!
- Still, $\mathcal{T}_{\alpha\beta\lambda\mu}$ seems related somehow to the energy-momentum properties of the the gravitational field —but its geometric units (L^{-4}) are wrong
- The correct physical units (multiplying by c^4/G) are *energy density per unit area*.
- is there any relation with gravitational energy?
- The only sensible definitions of gravitational energy are either global (ADM, Bondi-Trautman) or quasi-local.

Quasilocal energies

- Quasilocal energies are defined associated to closed surfaces —usually a topological sphere: they are intended to represent the total gravitational energy contained inside that surface.
- Unfortunately, there are (too) many definitions of quasilocal energy (-momentum) in GR with no consensus on any particular choice (see L.B. Szabados, Living Rev. Relativ. 2009).
- As an example, we can consider the Hawking-Hayward-type definition

$$E_H(S) := \frac{c^4}{G}\sqrt{\frac{\mathrm{Area}(S)}{16\pi}}\frac{1}{16\pi}\int_S\left(\mathcal{K} + \frac{2-\epsilon}{2}H_\mu H^\mu + \epsilon K^\mu_{AB}K^{AB}_\mu\right)$$

where $\mathcal{K}$ is the Gaussian curvature, K^μ_{AB} the shape tensor and $H^\mu = q^{AB}K^\mu_{AB}$ the mean curvature vector of S; $\epsilon \in [0,1]$ ($\epsilon = 0, 1$ give the "Hawking" and "Hayward" masses, respectively).

Quasilocal energy in the small sphere limit

- Any of the quasilocal energies for closed surfaces applied to very small spheres $\oplus$ of radius ℓ gives, at first non-trivial order in ℓ:

$$E_{\oplus} = \frac{4\pi}{3}\ell^3 T_{00} + O(\ell^4)$$

(in a basis with $\vec{e}_0$ orthogonal to the sphere).

- But, what happens if we are in vacuum? That is, if $T_{\mu\nu} = 0$.
- Then, as first proven by Horowitz and Schmidt (1982)

$$E_{\oplus} = (\text{constant})\frac{c^4}{G}\ell^5 \mathcal{T}_{0000} + O(\ell^6)$$

The proportionality constant depends on the particular choice of quasilocal energy definition (see Szabados, Living Rev. Relativ. 2009).

- For instance, for $E_H(S)$ one derives (J Wang, CQG 37 (2020) 085004)

$$E_H(\oplus) = \frac{1-4\epsilon}{90}\frac{c^4}{G}\ell^5 \mathcal{T}_{0000} + O(\ell^6)$$

still depending on ϵ.

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The area deficit in terms of $\mathcal{T}_{0000}$

Notice:

$$\mathcal{T}_{0000} = E^2 + B^2$$

The previously computed formulas that were shown in blue can be recast in the following appealing form:

$$< \delta r > |_A = \delta \ell_2 |_A = \frac{\ell^5}{450} \left[\mathcal{T}_{0000} - \frac{5}{6} E^2 \left(12\gamma - 1\right)^2 \right]$$

$$\delta^{(2)} A|_{<r>} = -4\pi \frac{\ell^6}{225} \left[\mathcal{T}_{0000} - \frac{5}{6} E^2 \left(12\gamma - 1\right)^2 \right]$$

Hence, if $\gamma = 1/12$ the result is the expected and natural Bel-Robinson 'super-energy'!

The magic of $\gamma = 1/12$

The value $\gamma = 1/12$, that is to say, the deformation fixed to be

$$\delta\ell = \delta\ell_1 + \frac{\ell^3}{12} E_{ij} n^i n^j + \delta\ell_2 + \tilde{\delta}\ell_2(\vartheta, \varphi)$$

provides at first order

$$< \delta r > |_A = \delta\ell_1|_A = \frac{8\pi G}{c^4} \frac{\ell^3}{18} T_{00}, \qquad \delta^{(1)} A|_{<r>} = -\frac{8\pi G}{c^4} 4\pi \frac{\ell^4}{9} T_{00}$$

and at second order ($T_{00} = 0 = \delta\ell_1 = \delta^{(1)} A$)

$$< \delta r > |_A = \delta\ell_2|_A = \frac{\ell^5}{450} \mathcal{T}_{0000}, \qquad \delta^{(2)} A|_{<r>} = -4\pi \frac{\ell^6}{225} \mathcal{T}_{0000}$$

Observe that the relative factors agree.

The magic of $\gamma = 1/12$ (2)

- Concerning the rest of the variations (volume changes, or variations of area/radius keeping volume fixed) in all of them the value of $\gamma = 1/12$ makes the variation extreme
- It maximizes $\delta\ell_2|_V$ and $\delta^{(2)}V|_A$, while it minimizes $\delta^{(2)}A|_V$ and $\delta^{(2)}V|_{<r>}$.
- Still, is there any independent argument leading to such a value of γ? Which balls are to be compared?
- At quadratic order in the curvature, and given the anisotropy of the gravitational field around generic points, the radius r of the ball can be different for the various directions $n^i(\vartheta, \varphi)$; but, how to give a natural prescription that compares to the round ball in flat space?

An independent argument

- The mean curvature vector of the ball's boundary is at linear order in the curvature (N^μ is the normal to the boundary within the $t=0$ slice)

$$H^\mu = N^\mu \left[\frac{2}{\ell} + \frac{1}{\ell^2} \sum_{s\neq 2} (s+2)(s-1) Y_{i_1\ldots i_s} n^{i_1} \ldots n^{i_s} \right. \\ \left. + n^i n^j \left(\frac{4}{\ell^2} Y_{ij} - \frac{\ell}{3} E_{ij} \right) \right].$$

- Therefore, $\vec{H}$ is constant on the entire boundary (and equal to its flat spacetime value $\vec{H}^\flat = \frac{2}{\ell}\vec{N}$) if and only if $Y_{[s]} = 0$ for all $s \neq 2$ and

$$\boxed{Y_{ij} = \ell^3 E_{ij}/12}$$

- This provides an intrinsic definition, independent of the spacetime, of the boundary of the ball fixing $Y_{[s]} = 0$ for all $s \neq 2$ and $\gamma = 1/12$ at this order.

The quasilocal energy (or mass)

Recall (and notice that Feynman was right with his choice!):

$$\delta\ell_1|_A = \frac{8\pi G}{c^4}\frac{\ell^3}{18}T_{00}, \qquad \delta^{(1)}A|_{<r>} = -\frac{8\pi G}{c^4}4\pi\frac{\ell^4}{9}T_{00}$$

$$\delta\ell_2|_A = \frac{\ell^5}{450}\mathcal{T}_{0000}, \qquad \delta^{(2)}A|_{<r>} = -4\pi\frac{\ell^6}{225}\mathcal{T}_{0000}$$

Thus, one can define the (quasilocal) energy $\mathcal{E}$ or mass $\mathcal{M}$ at first non-trivial order by the averaged excess radius

$$\mathcal{E} = \mathcal{M}c^2 = \frac{3c^4}{G} < \delta r >|_A$$

or, alternatively, by the area deficit

$$\mathcal{E} = \mathcal{M}c^2 = -\frac{3c^4}{8\pi G}\frac{1}{\ell}\delta A|_{<r>}$$

and these formulas are valid <u>with and without</u> matter.

The quasilocal mass of geodesic balls

These definitions lead to (ϱ is the mass density relative to u^μ)

$$\frac{\mathcal{E}}{c^2} = \mathcal{M} = \frac{4\pi}{3}\varrho\ell^3 + O(\ell^5)$$

and, if we are in vacuum with $\varrho = 0$, to

$$\frac{\mathcal{E}}{c^2} = \mathcal{M} = \frac{1}{150}\frac{c^2}{G}\ell^5(E^2 + B^2) + O(\ell^6)$$

Observe that this calculation fixes univocally the constant in blue!

Conclusion

Therefore, a general prescription for the quasilocal energy is:

1. At any point p choose a unit timelike vector u^μ
2. Build a tiny spatial 2-sphere S (that is, locally orthogonal to u^μ) around p with a (angle dependent) radius determined by the constancy of its mean curvature vector
3. Compute the area A of S, set $\ell^2 := \frac{A}{4\pi}$ and compute the averaged excess radius of S with respect to ℓ
4. Multiply by $3c^4/G$ to get the quasilocal energy relative to the "observer" determined by u^μ around p
5. Alternatively, adjust the averaged radius of S to a fixed very small value ℓ and compute the area deficit of S with respect to the Euclidean area $4\pi\ell^2$
6. Multiply by $-3c^4/(8\pi G\ell)$ to get the same quasilocal energy relative to the "observer" u^μ around p

The quasilocal mass/energy in the Schwarzschild metric

For the spherically symmetric vacuum Schwarzschild metric, the energy $\mathcal{E}$ relative to *any* observer u^μ orthogonal to the $\mathbb{S}^2$ orbits of the symmetry group (and at any given point with area coordinate $\bar{r}$) is

$$\mathcal{E} = \mathcal{M}c^2 = \frac{1}{25}\frac{m\ell^5}{\bar{r}^6}Mc^2 = \frac{GM(M\frac{25\ell^5}{\bar{r}^5})}{\bar{r}}$$

where $m := GM/c^2$ and M is the (ADM) mass of the spacetime.

This formula should only be considered valid for $\ell \ll \bar{r}$ ($\ell \to 0$).

Inside a ball with the Earth's radius, at 1 AU from the Sun, the total pure gravitational energy due to the Sun's gravitational field is about

$$120g \sim 1\,\text{black truffle de Bourgogne} \quad (\times c^2)$$

(Compare to Earth's mass: $6 \times 10^{27}g$).

Thanks!

Merci beaucoup pour votre attention

Thank you very much for your attention