Gravity Vacua, Gravitational Radiations and their Cartan Geometry

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Introduction and motivations

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- gravitational memory effect is tightly related to the transition between gravity vacua (Strominger et al 2016, ...)
- Hawking’s initial derivation of the “information paradox” implicitly relied on the uniqueness of the gravity vacua (Hawking, Perry, Strominger 2016 and 2017)
- There has also been a surge of interest for the possibility of realising “flat holography” on the celestial sphere (Strominger et al 2019, ...)

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Many of these results rely on subtle geometrical features of asymptotically flat-space-times, such as “degeneracy of gravity vacua”, which are already present at classical level.

These deserve a deep investigation and full conceptual clarity.

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Asymptotically flat space-times are essential tools in our physical understanding of General Relativity (e.g model gravitational radiations as seen in LIGO), as such they have a venerable history...

- Bondi–Van-der-Burg–Metzner–Sachs (1962)
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At technical level everything is of course understood (see however surprising new results from previous slide), conceptually however the subject typically appears intricate which – one can argue – should not be the case for one which is 60 year old.

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Once again there are classical (technical) answers to these questions, they are however not particularly illuminating.

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I want to highlight that, building up (with necessary adaptations) on relatively recent results from Gover et al (2010–2018) on tractor calculus (see also Penrose–MacCallum 1973 “asymptotic local twistors”), one can shed a new light on the subject (YH 2010, YH 2021):

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What is more...

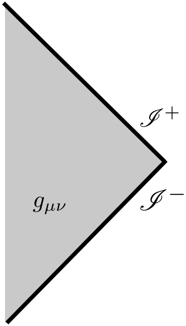
This perspective on the subject suggests generalisations to super- and higher- geometry that would otherwise be very difficult to imagine.

Gravity Vacua :
homogeneous space perspective

How unique is Minkowski space?

Let this be “a” (conformally compactified) Minkowski space-time :

Is this unique?



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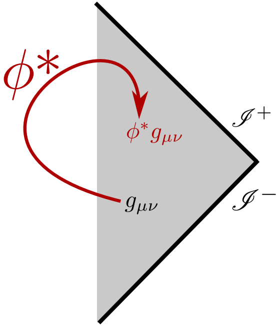

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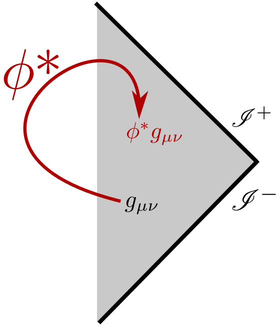

What if we quotient by diffeomorphisms?

- quotienting by all diffeomorphisms will give you a unique remaining Minkowski space,
- quotienting only by diffeomorphisms fixing the conformal boundary $\phi\big|_{\mathscr{I}} = Id$ results in a moduli of “gravity vacua”!

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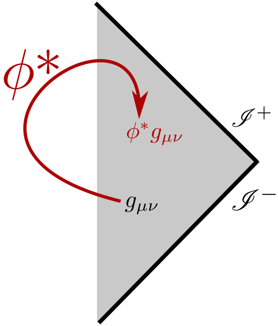

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This is rather sketchy, can we be more precise?

Homogenous space perspective

Let us define Minkowski space $M^{3,1}$ as the homogeneous space

$$M^{3,1} := {}^{ISO(3,1)}\!\diagup_{SO(3,1)}$$

then its conformal boundary $\mathscr{I}^3$ is also an homogeneous space:

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Compare with

$$AdS_4 := {SO(3,2)}\big/{SO(3,1)}$$

with conformal boundary

$$S^1 \times S^2 := {SO(3,2)}\big/{(\mathbb{R} \times SO(2,1)) \ltimes \mathbb{R}^3}$$

Homogenous space perspective

To see the conformal compactification explicitly, introduce

- the conformal group $SO(4,2)$

then

$SO(4,2)$ acts transitively on "conformally compactified Minkowski space",

$$\overline{M}^{3,1} = S^1 \times S^3\big|_{\mathbb{Z}^2} := {SO(4,2)}\big/{(\mathbb{R} \times SO(3,1)) \ltimes \mathbb{R}^3}$$

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To see the conformal compactification explicitly, introduce

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- and a fixed <u>null</u> vector I^I in $\mathbb{R}^{4,2}$ ($I^2 = 0$)

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$$\boxed{ISO(3,1) = Stab(I)} := \left\{ g^I{}_J \in SO(4,2) \,|\, g^I{}_J I^J = I^I \right\}.$$

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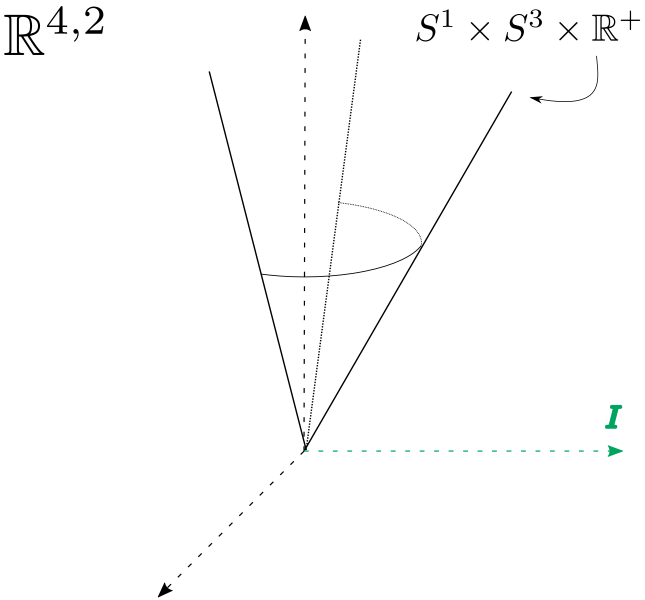
The action of $ISO(3,1)$ decomposes $\overline{M}^{3,1}$ into three orbits

$$\boxed{S^1 \times S^3\big|_{\mathbb{Z}^2} = M^{3,1} \sqcup \mathscr{I}^3 \sqcup \iota}$$

- Minkowski space : $M^{3,1} \simeq \mathbb{R}^4$
- Null-infinity : $\mathscr{I}^3 \simeq S^2 \times \mathbb{R}$
- time/spatial infinity (here identified) : $\iota \simeq \{pt\}$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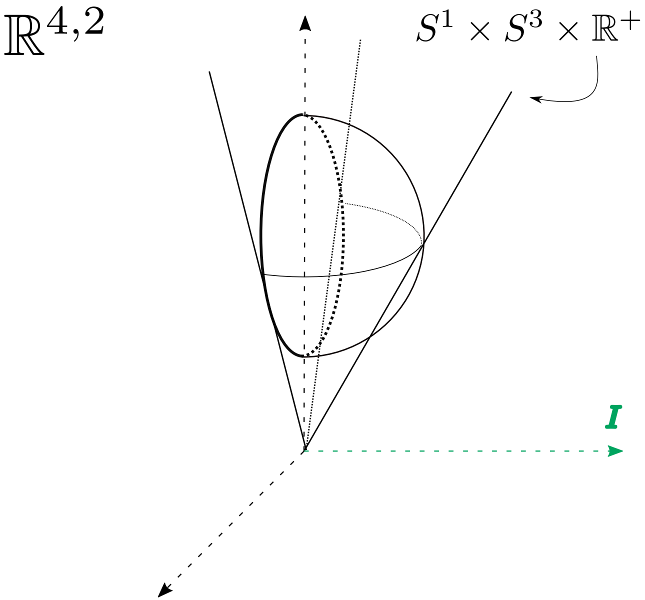

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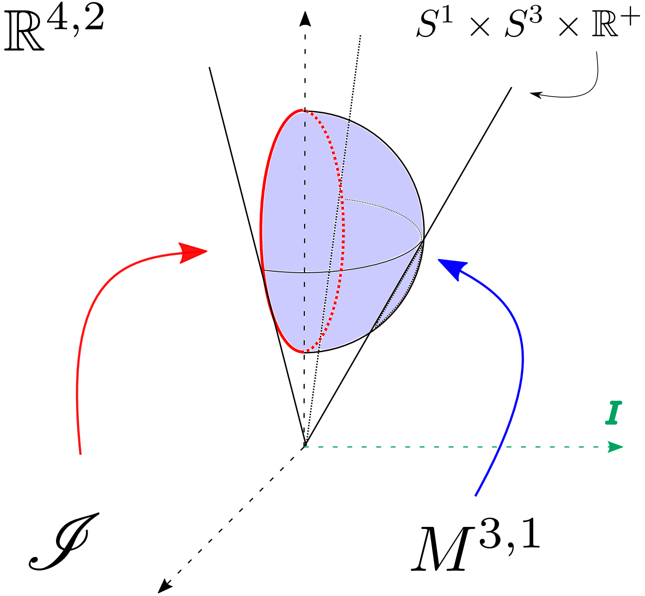

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Gravity Vacua (1)

Let $\mathscr{I} \simeq S^2 \times \mathbb{R}$ be a three-dimensional manifold, and let us consider the space Γ of maps to the homogeneous model

$$\Gamma := \left\{ \phi \colon \mathscr{I} \to {}^{ISO\,(3,1)} \big/ {}_{(\mathbb{R} \times \mathrm{ISO}(2)) \ltimes \mathbb{R}^3} \,\middle|\, \phi \text{ diffeomorphism} \right\}$$

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⚠ With this definition the symmetry group for gravity vacua is huge (all diffeomorphisms of $\mathscr{I}$!):
Diffeomorphisms of $\mathscr{I}$, $f \colon \mathscr{I} \to \mathscr{I}$, act on Γ via pull-back : $f.\phi := \phi \circ f$.

One obtains BMS_4 by requiring that ϕ are isomorphisms of conformal Carrollian geometry. There are all sort of possible definitions in between (Barnich–Troessart 2010, Campiglia–Laddha 2015 , Freidel et al 2020 ...)

Asymptotically flat space-times and BMS

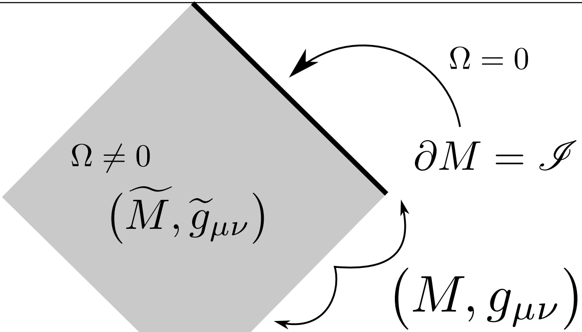
Asymptotically simple space-times (Penrose 1962)

A space-time $\left(\widetilde{M}, \tilde{g}_{\mu\nu}\right)$ is **asymptotically simple** if there exists a space-time $(M, g_{\mu\nu}, \Omega)$ with boundary $\partial M = \mathscr{I}$ such that

- $\widetilde{M}$ is diffeomorphic to the interior $M \backslash \mathscr{I}$ of M
- $\Omega \in C^{\infty}(M)$ is a boundary defining function for $\mathscr{I}$ i.e

$$\Omega > 0 \text{ on } M, \qquad \Omega = 0,\ d\Omega \neq 0 \text{ on } \mathscr{I}$$

- $\tilde{g}_{\mu\nu} = \dfrac{1}{\Omega^2} g_{\mu\nu}$ on $\widetilde{M}$



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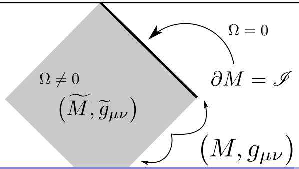
It is **asymptotically dS/flat/AdS** if on top of this

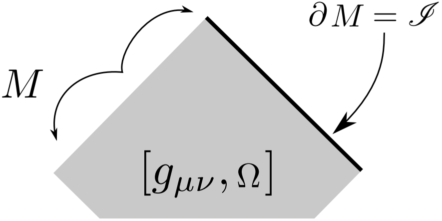
- $\tilde{g}_{\mu\nu}$ is Einstein
- $g_{\mu\nu} n^\mu n^\nu = g^{\mu\nu}\left(d\Omega_\mu, d\Omega_\nu\right) = -\Lambda$ is a constant on $\mathscr{I}$



⚠ ⚠ There is nothing unique about Ω nor $g_{\mu\nu}$ (only $\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$ is) !

Rather one is working with an equivalence class $[g_{\mu\nu}, \Omega]$ for the equivalence relation:

$$(g_{\mu\nu}, \Omega) \sim \left(\lambda^2 g_{\mu\nu}, \lambda\Omega\right) \qquad \lambda \in C^{\infty}(M)$$



$$= [\lambda^2 g_{\mu\nu}, \lambda\Omega] = [\tilde{g}_{\mu\nu}, 1]$$

$$\iota^*_{\mathscr{I}}[g_{\mu\nu}, \Omega] = [h_{ab}, 0] = [\lambda_0^2 h_{ab}, 0]$$

$$\lambda_0 := \iota^*_{\mathscr{I}} \lambda$$

Conformal Boundaries

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The boundary $\mathscr{I}$ of an asymptotically simple space-time $(M, [g_{\mu\nu}, \Omega])$

$(g_{\mu\nu}, \Omega) \sim \left(\lambda^2 g_{\mu\nu}, \lambda\Omega\right)$ is equipped with :

- a conformal metric $[h_{ab}] := \iota^*_{\mathscr{I}}[g_{\mu\nu}]$,

  with $h_{ab} \sim (\lambda_0)^2 h_{ab} \quad \lambda_0 := \iota^*_{\mathscr{I}}\lambda \in \mathcal{C}^\infty(\mathscr{I})$

- a weighted normal $[n^\mu] := [g^{\mu\nu}(d\Omega)_\nu]\big|_{\mathscr{I}}$,

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Recall that, for $(M, [g_{\mu\nu}, \Omega])$ to be *asymptotically dS/flat/AdS*, the weighted normal must have constant norm:

$$g_{\mu\nu} n^\mu n^\nu \big|_{\mathscr{I}} = -\Lambda$$

(NB: this is invariant under $g_{\mu\nu} \mapsto \lambda^2 g_{\mu\nu}$, $n^\mu \mapsto \lambda^{-1} n^\mu$.)

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If $g_{\mu\nu}$ has Lorentzian signature $(-, +, ..., +)$

- if $(M, [g_{\mu\nu}, \Omega])$ is asymptotically dS ($\Lambda = 1$)
  then $g_{\mu\nu} n^\mu n^\nu = -1 \Rightarrow h_{ab}$ is Riemannian
- if $(M, [g_{\mu\nu}, \Omega])$ is asymptotically AdS ($\Lambda = -1$)
  then $g_{\mu\nu} n^\mu n^\nu = 1 \Rightarrow h_{ab}$ is Lorentzian
- if $(M, [g_{\mu\nu}, \Omega])$ is asymptotically flat ($\Lambda = 0$)
  then $g_{\mu\nu} n^\mu n^\nu = 0 \Rightarrow h_{ab}$ is “Carrollian” (null):

$$det(h) = 0, \qquad n^a \in \Gamma\left[T\mathscr{I}\right], \quad h_{ab} n^b = 0$$

Carrollian manifold and BMS

L. Leblond (1965), G. Gibbons et al. (2014)

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A *Carrollian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, h_{ab}, n^a\right)$ is the data of

- a nowhere vanishing vector field $n^a \in \Gamma\left[T\mathscr{I}\right]$
- whose integral lines form the fibres of a (trivial) bundle $\mathscr{I} \xrightarrow{\pi} \Sigma$

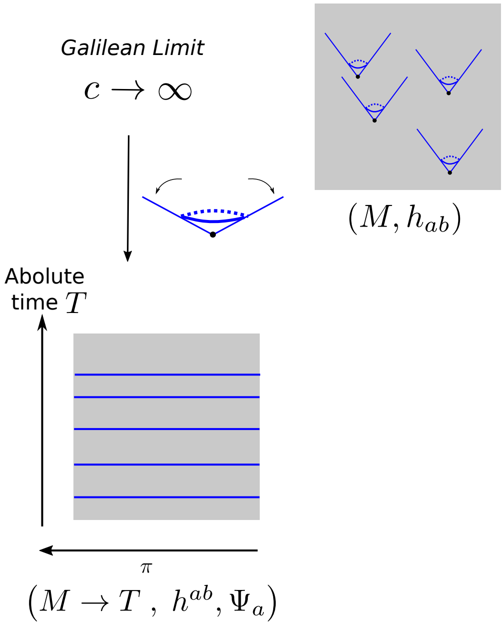
$$\mathscr{I}^{(n)} \simeq \mathbb{R} \times \Sigma^{(n-1)}$$

- where the base Σ is a Riemannian manifold (Σ, h_{AB})

In particular $h_{ab} := \pi^* h_{AB}$ is degenerate with kernel spanned by n^a:

$$h_{ab} n^b = 0, \qquad \mathcal{L}_n h_{ab} = 0.$$

Carrollian manifolds mirror Galilean manifolds



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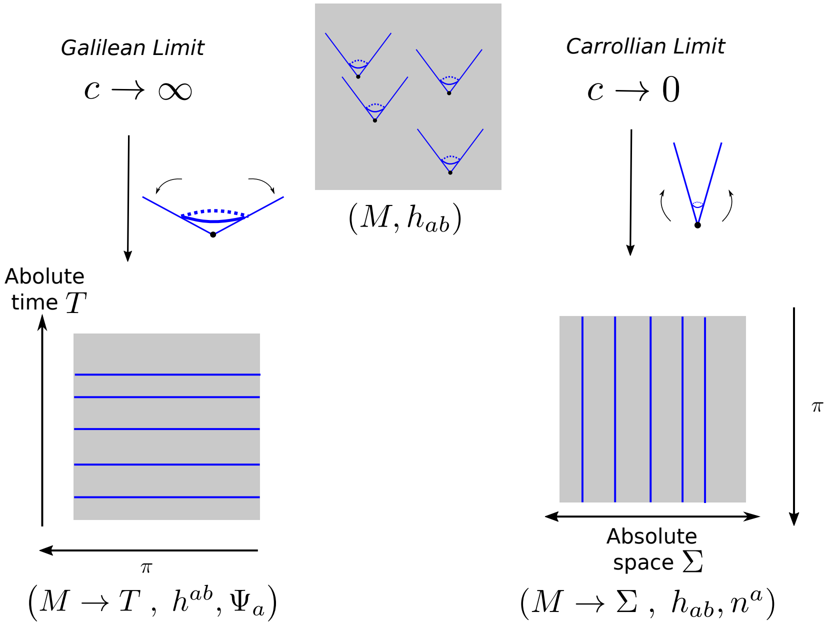

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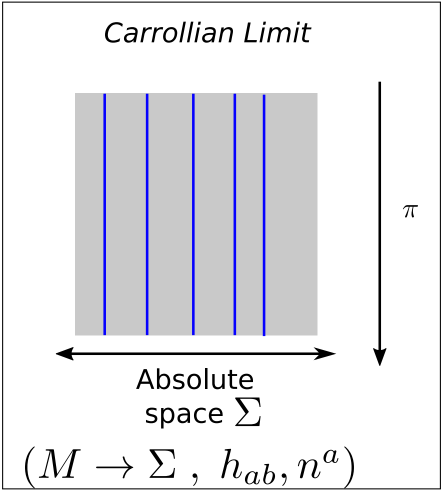

"since absence of causality as well as arbitrarinesses in the length of time intervals is especially clear in Alice's adventures (in particular in the Mad Tea-Party) this did not seem out of place to associate Lewis Carroll's name"

Levy-Leblond (1965)

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G. Gibbons et al. (2014)

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A *conformal Carrolian manifold* $\left(\mathscr{I} \xrightarrow{\pi} \Sigma, [h_{ab}, n^a]\right)$ is the data of

- a 1-dimensional fibre bundle $\mathscr{I} \to \Sigma$ over a conformal Riemannian manifold $(\Sigma, [h_{AB}])$
- with an equivalence class of nowhere vanishing vector field $[h_{ab}, n^a]$ generating the vertical direction $\pi_* (n) = 0$

$$(h_{ab}, n^a) \sim \left((\lambda_0)^2 h_{ab}, (\lambda_0)^{-1} n^a\right), \qquad \lambda_0 \in \mathcal{C}^\infty (\mathscr{I}) .$$

In particular $[h_{ab}] := [\pi^* h_{AB}]$ is degenerate with kernel spanned by n^a:

$$h_{ab} n^b = 0, \qquad \mathcal{L}_n h_{ab} \propto h_{ab}.$$

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The subgroup of automorphisms of $\left(\mathscr{I} \to S^{d-2}, [h_{ab}, n^a]\right)$, i.e diffeomorphisms $\phi\colon \mathscr{I} \to \mathscr{I}$ satisfying

$$\phi^* h_{ab} = \lambda^2 h_{ab}, \qquad \phi_* n^a = \lambda^{-1} n^a,$$

is the BMS group

$$\boxed{BMS_d \simeq \mathcal{C}^{\infty}\left(S^{d-2}\right) \rtimes \mathrm{SO}(d-1,1)}$$

What is more if $(\mathscr{I}, [h_{ab}, n^a])$ form the conformal boundary of $(M, [g_{\mu\nu}, \Omega])$ this group of automorphism is canonically identified with the group of asymptotic symmetries.

Gravity Vacua (1')

Let $\left(\mathscr{I} \simeq S^2 \times \mathbb{R}, [h_{ab}, n^a]\right)$ be a conformal Carrollian (CCarr) manifold, and let us consider the space

$$\Gamma' := \left\{ \phi \colon \mathscr{I} \to {}^{ISO\,(3,1)}\!\big/_{(\mathbb{R} \times \mathrm{ISO}(2)) \ltimes \mathbb{R}^3} \;\middle|\; \phi \text{ CCarr isomorphism} \right\}$$

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This gives an alternative definition for Gravity vacua,

$$\text{(Gravity Vacua)'} := {}^{\Gamma'}\!\diagup_{ISO\,(3,1)}.$$

In this case, the symmetry group for gravity vacua is just BMS_4 ,

$$\text{(Gravity Vacua)'} := {}^{BMS_4}\!\diagup_{ISO\,(3,1)}.$$

From Gravity Vacua to Gravitational Radiation

Let $(M, [g_{\mu\nu}, \Omega])$ be an asymptotically flat space-time
and let $(\mathscr{I}, [h_{ab}, n^a])$ be its conformal boundary,

Q: Does it comes with a preferred gravity vacua,

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In fact, in a (mathematically) very precise sense,

gravitational radiation is the obstruction

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If this was the case, we would have a preferred Poincaré action on null-infinity.

In fact, in a (mathematically) very precise sense,

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$\Rightarrow$ one needs the machinery of Cartan geometry.

Cartan Geometry

A Cartan geometry $(\mathcal{G} \to M, \omega)$ modelled on G/H is the data of

- a H-principal bundle $\mathcal{G} \to M$
- a "$\mathfrak{g}$-valued Cartan connection" ω, i.e a section ω of $\Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying
  1. $\omega\left(X^{\#}\right) = X$ for all $X \in \mathfrak{h}$
  2. $R_h^* \omega = Ad_{h^{-1}}(\omega)$
  3. s.t. $\omega \colon T\mathcal{G} \to \mathfrak{g}$ is an isomorphism.

The main example is the flat model :

$(G \to G/H, \omega_G)$ whith ω_G the Maurer-Cartan form on G.

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Fundamental theorem (E. Cartan)

A Cartan geometry $(\mathcal{G} \to M, \omega)$ is locally isomorphic to the flat model $(G \to G/H, \omega_G)$ if and only if the curvature $F = d\omega + \frac{1}{2}[\omega, \omega]$ vanishes.

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If M and G/H have the same topology and ω has no holonomy then this globalizes:

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It follows that

$$\text{Gravity Vacua} := \left\{ \begin{array}{c} \text{Flat cartan geometry } (\mathcal{G} \to \mathscr{I}, \omega) \\ \text{modelled on } {}^{ISO(3,1)}\!\big/_{(\mathbb{R} \times ISO(2)) \ltimes \mathbb{R}^3} \end{array} \right\}$$

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Cartan Geometry of null-infinity (YH 2020)

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Let $(\mathscr{I}, [h_{ab}, n^a])$ be a conformal Carrollian manifold,

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By the previous discussion these will be "curved" version of the gravity vacua.

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Null-tractors frames (YH 2020)

Let $(\mathscr{I}, [h_{ab}, n^a])$ be a n-dimensional conformal Carrollian manifold,
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Normal conformal Carrollian Cartan connection (YH 2020)

Let $(\mathscr{I}, [h_{ab}, n^a])$ be a n-dimensional conformal Carrollian manifold. Compatible normal Cartan connections

$$\omega \in \Omega^1(\mathcal{G}, \mathfrak{iso}(n,1))$$

form an affine space isomorphic to

- ($n \geq 4$) "zero modes for the asymptotic shear" $C_{AB}(u=0,y)$,
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The previous results are crucially different from the situation for usual (i.e non-degenerate) conformal manifold :

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Let $(\mathscr{I}, [h_{ab}])$ be a n-dimensional conformal Lorentzian manifold then

it defines a *unique* normal cartan geometry $(\mathcal{G} \to \mathscr{I}, \omega)$ modelled on

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In particular,

$$\text{(Gravity Vacua)'} := \left\{ \begin{array}{c} \text{Flat cartan geometry } (\mathcal{G} \to \mathscr{I}, \omega) \\ \text{compatible with a given } (\mathscr{I}, [h_{ab}, n^a]) \end{array} \right\}$$

Trivialisations

Let $(\mathscr{I} \to \Sigma, [h_{ab}, n^a])$ be a conformal Carrollian manifold, in particular recall that

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A choice of trivialisation, is a choice of function $u \in \mathcal{C}^\infty(\mathscr{I})$ realizing a global trivialisation of $\mathscr{I} \to \Sigma$ i.e such that

$$\phi_u \left| \begin{array}{ccc} \mathscr{I} & \to & \mathbb{R} \times \Sigma \\ x & \to & (u(x), \pi(x)) \end{array} \right.$$

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A trivialisation $u \in \mathcal{C}^\infty(\mathscr{I})$ defines preferred representatives

$$(h_{ab}, n^a) \in [h_{ab}, n^a]$$

by requiring the condition

$$n^a \partial_a u = 1 \qquad \Leftrightarrow \qquad n^a = \partial_u$$

The null-tractor bundle (YH, 2020)

Let $(\mathscr{I} \to \Sigma, [h_{ab}, n^a])$ be a conformal Carrollian geometry.

The null-tractor bundle $\mathcal{T} \to \mathscr{I}$ is the canonical bundle obtained as the associated bundle to $\mathcal{G} \to \mathscr{I}$ for the fundamental representation.

Practically, a choice of trivialisation $u \in \mathcal{C}^\infty(\mathscr{I})$ gives an isomorphism

$$\mathcal{T} \overset{u}{\simeq} \mathbb{R} \oplus T\mathscr{I} \oplus \mathbb{R}$$

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In a trivialisation, a tractor $Y^I \in \mathcal{C}^\infty(\mathcal{T})$ can be written as:

$$Y^I \overset{u}{=} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^- \end{pmatrix}$$

with Y^+, $Y^- \in \mathcal{C}^\infty(\mathscr{I})$
$Y^A \partial_A + Y^u \partial_u \in \mathcal{C}^\infty(T\mathscr{I})$

This is a 5-dimensional vector bundle, canonically defined from $([h_{ab}, n^a])$ and equipped with a degenerate metric :

$$Y^2 = 2Y^+Y^- + Y^A Y^B h_{AB}$$

and a preferred degenerate direction $I^I = \left(0, 0^A, 1, 0\right)$.

Tractor "transformation rules"

A trivialisation $u \in \mathcal{C}^{\infty}(\mathscr{I})$ gives an isomorphism

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Any other trivialisation $\hat{u}$ will give another isomorphism and we have the transformation rules

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$$Y^I \overset{u}{=} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^-. \end{pmatrix} \mapsto Y^I \overset{\hat{u}}{=} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda^{-1} U^A & \lambda^{-1} \delta^A{}_B & 0 & 0 \\ \beta & \lambda^{-1} \nabla_B u & 1 & 0 \\ -\frac{\lambda^{-1}}{2} U^2 & -\lambda^{-1} U_B & 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} Y^+ \\ Y^A \\ Y^u \\ Y^-. \end{pmatrix}$$

here $\lambda := \mathcal{L}_n \hat{u}$,
$U_A := \lambda^{-1} \left(\nabla_A \lambda - \left(\frac{\dot{\lambda}}{\lambda} + \frac{1}{2(n-1)} h^{CD} \dot{h}_{CD} \right) \nabla_A \hat{u} \right)$,
and $\beta := fct \left(\nabla_A \nabla_B \hat{u}, \dot{\lambda}, \nabla_A \lambda, \nabla_A \hat{u}, \lambda \right)$

Proposition (YH 2020)

Let $([h_{ab}, n^a])$ be a n-dimensional conformal Carrollian geometry. Compatible normal tractor connections form an affine space isomorphic to

- ($n \geq 4$) "zero modes for the asymptotic shear" $C_{AB}\,(u=0,y)$,
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Sketch of Proof

The coordinate of a "compatible tractor connection" are

$$A^I{}_J \overset{u}{=} \begin{pmatrix} 0 & -h_{BC}\,dy^C & 0 & 0 \\ -\xi^A & \omega^A{}_B & 0 & dy^A \\ -\psi & -\frac{1}{2}C_B & 0 & du \\ 0 & \xi_B & 0 & 0 \end{pmatrix}$$

- $F^I{}_J X^J = 0 \;\Rightarrow\; \omega^A{}_B = fct\,(h),\, \xi_B = \xi_{(AB)}dy^A,\, C_B = C_{(AB)}dy^A.$
- $F^a{}_{bcd}n^c = 0 \;\Rightarrow\; \xi_{AB}\big|_{TF} := \frac{1}{2}\dot{C}_{AB},\; (n-2)Tr\xi = -\frac{1}{2}R(h),\; \psi_b n^b \propto Tr\xi.$
- $h^{bc}F^a{}_{bcd} = 0 \;\Rightarrow\; (n-3)\frac{1}{2}\dot{C}_{AB} = -R_{AB}\big|_{TF},\, (n-2)\psi_A = -\frac{1}{2}\nabla^C C_{CA}.$

Back to asymptotically flat space-time

Asymptotically dS/flat/AdS Space-Times (revisited)

Almost-Einstein Space-Times (Gover 10)

Asymptotically ds/flat/AdS Space-Times $(M, [g_{\mu\nu}, \Omega])$ are equivalent to conformal manifolds $(M, [g_{\mu\nu}])$ together with a covariantly constant tractor:

$$D_\mu I^I = 0, \qquad\qquad I^I \stackrel{g}{=} \begin{pmatrix} \Omega \\ \mu^\mu \\ \rho \end{pmatrix}$$

such that $X_I I^I\big|_{\mathscr{I}} = \Omega\big|_{\mathscr{I}} = 0$ ($d\Omega\big|_{\mathscr{I}} \neq 0$).
What is more the cosmological constant Λ is invariantly given by $I^2 = -\Lambda$.

- D_μ is the ($\mathrm{SO}(d,2)$-valued) normal-Tractor connection.
- Manifestly invariant under Weyl rescaling.
- All fields are smooth everywhere, *including at* $\mathscr{I}$.

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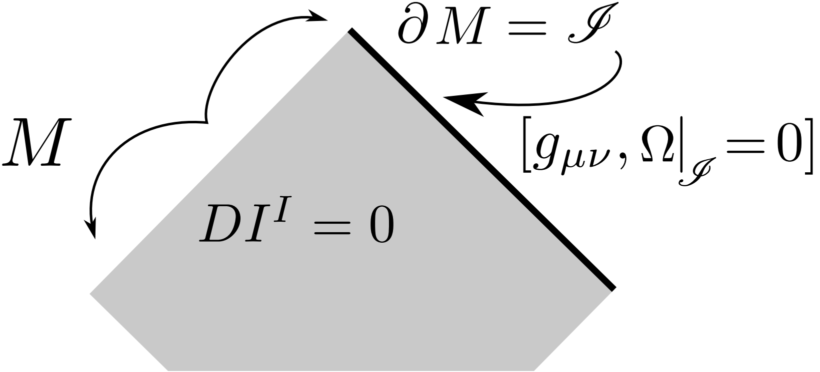
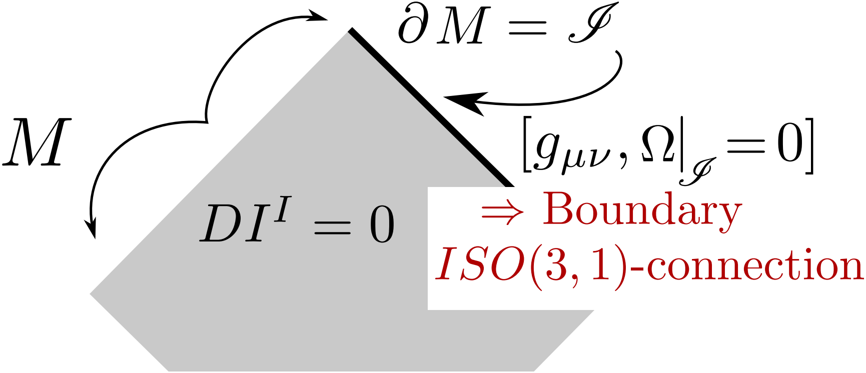
- For asymptotically flat space-times ($\Lambda = 0$), Einstein's equations $D_\mu I^I = 0$ effectively reduces the $\mathrm{SO}(d,2)$-valued connection D_μ to a $\mathrm{ISO}(d,1)$-valued connection.



$$[g_{\mu\nu}, \Omega] = [\Omega^{-2} g_{\mu\nu}, 1]$$

$$\Rightarrow ISO(3,1) \text{ MacDowell-Mansouri}$$



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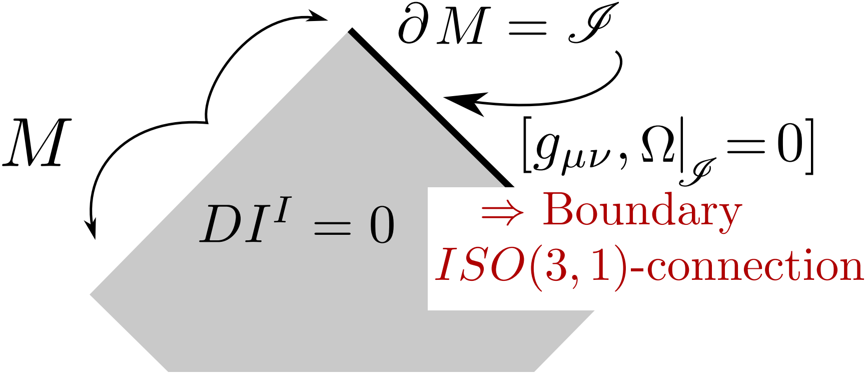

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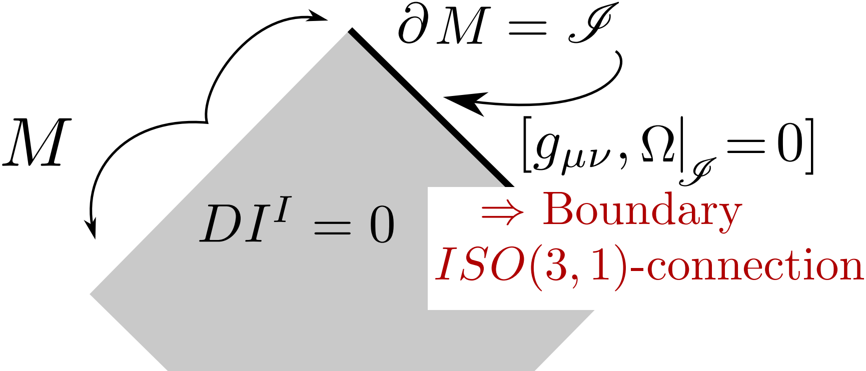
(particular case of the "orbit decomposition of Cartan geometry" of Cap-Gover-Hammerl 2014)



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An asymptotically flat space-time induces at $\mathscr{I}$ more than a conformal Carrollian geometry $([h_{ab}, n^a])$:

It induces a full Cartan geometry modelled on $\mathrm{ISO}(3,1)\,/P$!

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The curvature of the induced Cartan geometry coincides with the pull-back of the Weyl tensor

$$\iota^*\left(\Omega^{-1} n^\mu W_{\mu\nu\rho\sigma}\right)$$

equivalently with Ψ_4^0, Ψ_3^0, $Im\left(\Psi_2^0\right)$.

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Disclaimer: The connection itself is known to twistor experts since Penrose–MacCallum 1973 as "local twistor connection" (and needed to define asymptotic twistor space). New aspects here are: 1) Its appearance from the intrinsic geometry of conformal Carrollian mfld, 2) The Cartan geometry interpretation at $\mathscr{I}$.

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BMS coordinates, Asymptotic shear and gravitational waves

Let $(M, [g_{\mu\nu}, \Omega])$ be an asymptotically flat space-time.

BMS coordinates (Bondi–VanDerBurg–Metzner–Sachs 1962)

Trivialisation $u \in \mathcal{C}^{\infty}(\mathscr{I})$ of $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g_{\mu\nu}, \Omega])$ i.e with local coordinates

$$(u, \Omega, \pi) \left| \begin{array}{ccc} M & \to & \mathbb{R} \times \mathbb{R} \times S^2 \\ x & \to & (u(x), \Omega(x), y^A(x)) \end{array} \right.$$

on a neighbourhood of $\mathscr{I}$ in M such that

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} \left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u, y) + \mathcal{O}\left(\Omega^2\right) \right)$$

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The “asymptotic shear” (i.e of the null-geodesic congruence defined by the BMS coordinates) C_{AB} is known (BMS 1962) to encode the gravitational radiation reaching null-infinity.

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The "asymptotic shear" (i.e of the null-geodesic congruence defined by the BMS coordinates) C_{AB} is known (BMS 1962) to encode the gravitational radiation reaching null-infinity.

$\Rightarrow$ this is however nothing like a tensor on $\mathscr{I}$!

BMS coordinates, Asymptotic shear and gravitational waves

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Had we chosen another trivialisation $\hat{u} \in \mathcal{C}^\infty(\mathscr{I})$ (with $\lambda := \mathcal{L}_n \hat{u}$), we would have

$$\begin{aligned} h_{AB} &\mapsto \hat{h}_{AB} = \lambda^2 h_{AB} \\ n^a &\mapsto \hat{n}^a = \lambda^{-1} n^a \\ C_{AB} &\mapsto \hat{C}_{AB} = \lambda C_{AB} - 2\nabla_A \nabla_B\big|_0 \hat{u} - 4\Upsilon_{(A}\nabla_{B)}\big|_0 \hat{u} \\ &\qquad + \lambda^{-1}\left(2\frac{\dot{\lambda}}{\lambda} - \frac{h^{CD}\dot{h}_{CD}}{d-2}\right)\nabla_A \hat{u} \nabla_B \hat{u}\big|_0 \end{aligned}$$

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A Cartan connection on $\mathscr{I}$ modelled on $ISO(3,1)/H$

Ambient vs Null tractors (YH 21)

Let $\left(M, [g_{\mu\nu}, I^I]\right)$ be an almost Einstein manifold with $I^2 = 0$ (equivalently, asymptotically flat). Let $(\mathscr{I}, [h_{ab}, n^a])$ be the conformal Carrollian manifold induced by the boundary.

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- The null-tractor bundle $\mathcal{T}_{\mathscr{I}} \to \mathscr{I}$ is canonically identified with
$$I^{\perp}\big|_{\mathscr{I}} \subset \mathcal{T}\big|_{\mathscr{I}}.$$
- The normal tractor connection D on the ambient tractors $\mathcal{T}$ induces a normal tractor connection $D_{\mathscr{I}}$ on null-tractors $\mathcal{T}_{\mathscr{I}}$.
- A choice of trivialisation $u \colon \mathscr{I} \to \mathbb{R}$ defines simultaneously
  - a splitting isomorphism $\mathcal{T}_{\mathscr{I}} \overset{u}{=} \mathbb{R} \oplus T\Sigma \oplus \mathbb{R} \oplus \mathbb{R}$,
  - a preferred scale $g^{(u)}_{\mu\nu} \in [g_{\mu\nu}]$ (in a neighbourhood of $\mathscr{I}$) and accordingly a splitting isomorphism $\mathcal{T} \overset{g^{(u)}}{=} \mathbb{R} \oplus TM \oplus \mathbb{R}$,
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One can then explicitly check that the asymptotic shear "C_{AB}" of the null geodesic congruence and the freedom "C_{AB}" appearing in the coordinate expression $D_{\mathscr{I}}\overset{u}{=}d+A^I{}_J$ are the same.

Conclusion

- There is well-defined tractor-bundle for conformal Carrollian geometry.
- Asymptotic symmetries are related to the non-uniqueness of the normal tractor connection / Cartan geometry.
- For a given asymptotically flat space-time one obtains a preferred connection at the boundary induced by the ambient tractor connection.
- In all dimensions but $d = 4$ these induced connections are flat and define a “gravity vacua”; i.e a map

$$\phi \colon \mathscr{I} \to {}^{\mathrm{ISO}(d-1,1)}\!\diagup_{P}.$$

- In dimension $d = 4$ the tractor curvature invariantly encodes the presence of gravitational radiation (and is the obstruction to finding such isomorphism).
- There is in fact a precise sense in which “gravitational radiation induces transition between gravity vacua”.

What's next?

- Can be used to write invariant functionals at null-infinity as

$$\int_{\mathscr{I}} CS\,(A)$$

  (see Nguyen-Salzer 2021).

- Tractors are meant to "proliferate invariants". Can we use this machinery to produce new (physical!) invariants for asymptotically flat space-times?
- Rephrasing the geometry of null-infinity as a Cartan geometry immediately suggests generalisation

Gauging $\simeq$ Cartan geometry

The idea of gauging a symmetry group has been a fruitful tool to construct

- super gravity theories (by gauging the super-Poincaré group)
- higher spin field equations (by gauging the higher-spin algebra)

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These are not formal example: these should be the (very poorly understood for now) dual geometry to the 4D bulk super- or higher- symmetries !

Thank you for your attention