Functional Equations in LIMoges (FELIM) 2022

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Functional Equations in LIMoges (FELIM) 2022 is the thirteenth in a series of annual international gatherings for researchers in functional equations. This conference, held annually at the University of Limoges since 2008, aims to present recent advances in symbolic or symbolic-numeric algorithms which treat systems of linear or nonlinear, ordinary or partial, differential equations, (q-)difference equations,... Additionally, FELIM emphasizes on the development state of related software implementations and the publicity of such codes. Topics include, but are not limited to, ordinary differential equations, difference equations, partial differential equations; integration of dynamical systems; methods for local solving (formal, symbolic-numeric, modular); methods for global solving or simplification (e.g., decomposition, factorisation); applications and software applications.

The conference website with links to the previous editions can be found at:

https://indico.math.cnrs.fr/event/7221/

1 Invited talks abstracts

Juan José Morales Ruiz (Universidad Politecnica de Madrid, Spain)
Differential Galois theory and quantum physics.

`Monday, March 28, 10:45-11:45`

We survey my 2020 paper, where it is observed how a well-known object of classical mechanics, the variational equation, is the key to solve the semiclassical (or WKB) approximation in the corresponding problem in quantum mechanics. Then using a joint result with Jean-Pierre Ramis obtained more than 22 years ago, it is easy to prove that if the classical system is integrable then, the semiclassical approach it must be also integrable: given by Liouvillian functions.

If time permits, I would like to give some idea of the many open problems arising of the above simple result: problems from quantum mechanics to quantum field theory.

Lucia Di Vizio (CNRS, Université de Versailles Saint-Quentin-En-Yvelines, France)
Differential transcendence for the Bell numbers and their relatives.

`Tuesday, March 29, 9:30-10:30`

Martin Klazar proved in 2003 that the ordinary generating function of the Bell numbers is differentially transcendental over the germs of meromorphic functions at 0. We show that this result is an instance of a general phenomenon: on one hand we prove a general result, in a compact way, using difference Galois theory; on the other hand, we obtain as a consequence the differential transcendence of the generating functions of many other combinatorial sequences, including Bernoulli, Euler and Genocchi numbers. These results bring concrete evidence in support to the Pak-Yeliussizov conjecture. This is joint work with A. Bostan and K. Raschel.

Gwladys Fernandes (Laboratoire de Mathématiques de Versailles, France)
Hypertranscendence of solutions of linear difference equations.

`Wednesday, March 30, 9:30-10:30`

The general question of this talk is the classification of differentially algebraic solutions of linear difference equations of the following type :

$$a_0(z)f(z) + a_1(z)f(R(z)) + \cdots + a_n(z)f(R^n(z)) = 0, \tag{1}$$

where, for every i, $a_i(z) \in \mathbb{C}(z)$, $R(z) \in \mathbb{C}(z)$ and $R^n(z)$ is the n-th composition of $R(z)$ with itself.

We say that such a function is *differentially algebraic* over $\mathbb{C}(z)$ if there exist an integer $n \geq 0$ and a non-zero polynomial $P \in \mathbb{C}(z)[X_0, \ldots, X_n]$ such that $P(f(z), \ldots, f^{(n)}(z)) = 0$, where $f^{(i)}$ is the i-th derivative of f with respect to z. Otherwise, it is *hypertranscendental* over $\mathbb{C}(z)$.

The classification of differentially algebraic solutions is known for three types of non-linear difference equations : the Schröder's, Böttcher's and Abel's equations : $f(qz) = R(f(z))$, $f\left(z^d\right) = R(f(z))$, $f(R(z)) = f(z) + 1$, respectively, where $q \in \mathbb{C}^*$, $d \in \mathbb{N}, d \geq 2$. A classification of the differential algebraicity of solutions of linear difference equations of type (1) is made in an article of B. Adamczewski, T. Dreyfus, C. Hardouin, for these same operators : q-differences $z \to qz$, mahlerian $z \to z^d$, and shift $z \to z + 1$, by the means of an adapted difference Galois theory.

In this talk, we discuss the generalisation of these results to any function R (rational or algebraic over $\mathbb{C}(z)$), in the case where (1) is of order 1. This is a work in progress with L. Di Vizio. Natural applications appear in examples of gererating series of random walks, which satisfy this kind of equation of order 1.

2 Contributed talks abstracts

Thierry Combot (Institut mathématique de Bourgogne, France)
Computing rational first integrals on surfaces.

`Monday, March 28, 14:00-14:40`

Given an algebraic surface S and a rational vector field X on its tangent, we define the notion of degree and indecomposability of a rational first integral of X. An important novel property of these integrals is the genus of the image curve of the first integral, and we will prove that first integrals of high genus are of bounded degree. We will then present an algorithm for computing rational first integrals of X up to degree N in time $O(N^{\omega+1})$, and some examples in the delicate case of genus 1.

François Ollivier (LIX, CNRS, École polytechnique, France)
Singularities in differential algebra and diffiety theory.

`Monday, March 28, 14:45-15:25`

Investigating some practical issues in control systems, one may face quickly difficult problems, related to many different notions of singularities, that may be encountered in differential algebra.

A prime differential polynomial may define a prime main component and possibly also singular components [9]. Among the components returned by algorithms such as Diffalg [1], some are singular components, which may be certified by Ritt's low power theorem and others could be singular points of the main components or of the singular components. The envelop of the curves defined by the main component is defined by a singular component.

See Hubert [3]. We propose a general definition of singular components, that uses Jacobi's bound [7].

The intrinsic singular points of a single main component are difficult to characterize. Any point of the prime component P can be defined by a maximal ideal $m \supset P$, that is not in general a differential ideal. Then, the point is regular if there exist a characteristic set A of P, such that the product S_A of separants of A does not belong to m. But this is only a sufficient condition. A possible definition of singular point would be that S_B vanish for all characteristic sets B, considering all possible orderings, but also all possible changes of coordinates.

Johnson [4] proposed an abstract definition, that does not rely on characteristic sets. The equivalence of the two definitions remains to be investigated, and an effective criterion is missing. A necessary condition for regularity is $\bigcap_{r\in N} m^r = 0$. The $k[[t]]$-module M defined by the linearized system atifies then an equivalent property: $\bigcap_{r\in N} t^r M = 0$.

We also need to mention regular representations of differential ideals, according to Boulier (see [1]).

Flat systems [2] involve a different kind of singularities. Such systems are characterized by the existence of *flat outputs*, that are functions z such that their solution may be locally parametrized by z and their derivatives up to a finite order, in a dense open space of flat points. As some systems admit flat outputs but no algebraic flat outputs, diffiety theory [10] that allows to work with C^∞ unctions is more convenient here, even if for algebraic systems.

The complementary set correspond to *intrinsic flat singularities* [5,6]. A necessary condition of flat regularity is that the $k[[t]]$-module M defined by the linearized system is a free module [8].

Working with a reasonable control system, the regularity of all points of our trajectories is in general granted, but one may have to change charts. *E.g.*, if the state space is a sphere, we may have to use different sets of Euler angles. It seems also reasonable to assume that one works in a single main component, defined by a prime differential ideal. Anyway, considering systems with first integrals, one may wish to take the value of this first integral as a new parameter, which make often appear apparent singularities and singular components.

Intrinsic flat singularities are of a different nature and include points where the controllability of the linearized system is lost.

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[2] Fliess (Michel), Lévine (Jean), Martin (Philippe), and Rouchon (Pierre), « Flatness and defect of nonlinear systems: introductory theory and examples », International Journal of Control, Vol. 61, No. 6, p.1327–1361, 1995.

[3] Hubert (Évelyne), « Essential components of an algebraic differential equation », J. Symbolic Computation, 21, 1–24, 1999.

[4] Johnson (Joseph), « A Notion of Regularity for Differential Local Algebras », in Contributions to Algebra, a collection of papers dedicated to Ellis Kolchin, Academic Press,

211–232, 1977.

[5] Kaminski (Yirmeyahu J.), Lévine (Jean), and Ollivier (François), « Intrinsic and apparent singularities in flat differential systems and application to global motion planning », Systems & Control Letters, Volume 113, 117–124, 2018.

[6] Kaminski (Yirmeyahu J.), Lévine (Jean), and Ollivier (François), « On Singularities of Flat Affine Systems With n States and $n-1$ Controls », International Journal of Robust and Nonlinear Control, vol. 30, issue 9, 3547–3565, Wiley, 2020.

[7] Ollivier (François), « Jacobi's bound. Jacobi's results translated in König's, Egerváry's and Ritt's mathematical languages », to appear in AAECC, 2022.

[8] Ollivier (François), « Extending Flat Motion Planning to Non-flat Systems. Experiments on Aircraft Models Using Maple », 2022.

[9] Ritt, (Joseph Fels), « Differential Algebra », Amer. Math. Soc. Colloq. Publ., vol. xxxiii, A.M.S., New-York, 1950.

[10] Krasil'shchik (I.S.), Lychagin (V.V.), and Vinogradov (A.M.), « Geometry of Jet Spaces and Nonlinear Partial Differential Equations », Gordon and Breach, New York, 1986.

Hadrien Notarantonio (Inria Saclay, France)
Algorithms for solving fixed point equations of order 1.

`Monday, March 28, 16:00-16:40`

Fixed point equations of order 1 are functional equations relating polynomially $F(t,u)$, a power series in t with polynomial coefficients in a "catalytic" variable u, and one of its specializations, say $F(t,1)$. Such equations are ubiquitous in combinatorics, notably in the enumeration of maps and walks. When the solution F is unique, a celebrated result by Bousquet-Mélou, reminiscent of Popescu's theorem in commutative algebra, states that F is *algebraic*. We address algorithmic and complexity questions related to this result. In *generic* situations, we first revisit and analyze known algorithms, based either on polynomial elimination or on the guess-and-prove paradigm. We then design two new algorithms: the first has a geometric flavor, the second blends elimination and guess-and-prove. In the *general* case (no genericity assumptions), we prove that the total arithmetic size of the algebraic equations for $F(t,1)$ is bounded polynomially in the size of the input discrete differential equation, and that one can compute such equations in polynomial time. This is a joint work with Alin Bostan, Frédéric Chyzak and Mohab Safey El Din.

Raphaël Pagès (Université de Bordeaux, France)
Computing characteristic polynomials of p-curvatures in average polynomial time.

`Monday, March 28, 16:45-17:25`

p-curvatures are linear maps related to linear differential operators in characteristic p which provides a wealth of informations about those operators (such as the algebraic nature of their solutions for example) while being somewhat easily computable as the naive algorithm (sometimes refered to as Katz's algorithm) can compute it in $\tilde{O}(p^2)$ bit operations.

In 2014, Alin Bostan, Xavier Caruso, and Éric Schost presented an algorithm [BCS14] to compute the characteristic polynomial of the p-curvature of a differential operator with coefficients in $\mathbb{F}_p(x)$ in $\tilde{O}(\sqrt{p})$ bit operations. We expand on their idea to design a fast algorithm that computes, for a given linear differential operator with coefficients in $\mathbb{Z}[x]$, all the characteristic polynomials of its p-curvatures, for all primes $p < N$, in asymptotically quasi-linear bit complexity in N. We discuss implementations and applications of our algorithm.

[BCS14] Alin Bostan, Xavier Caruso, and Éric Schost. *A fast algorithm for computing the characteristic polynomial of the p-curvature.* In ISSAC 2014. Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation, pages 59-66. ACM, New York, 2014.

Marina Poulet (Institut Fourier, Université Grenoble Alpes, France)
An algorithm to recognize regular singularities of Mahler systems.

`Tuesday, March 29, 11:00-11:40`

One of the main interests in studying difference or differential equations with regular singularities is the good analytical properties of their solutions. For example, let $Ly = 0$ be a linear differential equation whose coefficients are convergent Laurent series. If 0 is a regular singularity of $Ly = 0$, the solutions of this equation have moderate growth at 0, that is, at most a polynomial growth. In this case, Fuchs' criterion allows to easily know if 0 is a regular singularity : we just have to look at the valuation at 0 of each coefficient of L. From the point of view of systems, general algorithms to recognize regular singularities of differential systems and difference systems have been given and they apply to many systems such as differential and q-difference systems. However, they do not apply to the Mahler case. The aim of this talk is to explain the difficulties of the Mahler case and to present an algorithm which permits to recognize regular singularities of Mahler systems. It is a joint work with Colin Faverjon.

Thomas Dreyfus (CNRS, Université de Strasbourg, France)
Computing the Galois group of order three difference equations.

`Tuesday, March 29, 14:00-14:40`

In this talk we will explain how to compute the difference Galois group an of order 3 difference operator. This is a joint work with Marina Poulet.

Mercedes Haiech (Université de Limoges, XLIM, France)
Deformations of solutions of differential equations.

`Tuesday, March 29, 14:45-15:25`

Given an algebraic differential equation (or system) we can be interested in the geometrical object defined by the set of its solutions. In algebraic geometry, the study of the geometry of the solutions of a polynomial equation gives information about the singularities of this equation. If we transpose some of these algebraic invariants into the differential world, we can wonder what information they give about the differential equation. In this presentation, we will study the behavior of the formal neighborhood of a fixed solution of the differential equation. In our framework, this object can also be understood as the space of deformations of this solution.

Sergey Yurkevich (Inria Saclay and University of Vienna, France and Austria)
Conjecturing and proving that the generating function of the Yang-Zagier numbers is algebraic.

`Tuesday, March 29, 16:00-16:40`

In a recent paper Don Zagier mentions a mysterious integer sequence $(a_n)_{n\geq 0}$ which arises from a solution of a topological ODE discovered by Marco Bertola, Boris Dubrovin and Di Yang. In my talk I show how to conjecture, prove and even quantify that $(a_n)_{n\geq 0}$ actually admits an algebraic generating function which is therefore a very particular period. The methods are based on experimental mathematics and algorithmic ideas in differential Galois theory, which I will show in the interactive part of the talk. The presentation is based on joint work with A. Bostan and J.-A. Weil.

François Ollivier (LIX, CNRS, École polytechnique, France)
Extending flat motion planning to non-flat systems. Experiments on aircraft models using Maple.

`Tuesday, March 29, 16:45-17:25`

Aircraft models may be considered as flat if one neglects some terms associated to aerodynamics. However some maneuvers may be hard or even impossible to achieve with this flat approximation. Computational experiments in Maple show that in some cases a suitably designed feed-back allows to follow such trajectories, when applied to the non-flat model.

We investigate an iterated process to compute a more achievable trajectory, starting from the flat reference trajectory. More precisely, the unknown neglected terms in the flat model are iteratively re-evaluated using the values obtained at the previous step. This process may be interpreted as a new trajectory parametrization using an infinite number of derivatives, a property that may be called *generalized flatness*.

The pertinence of this approach is illustrated with trajectories that reproduce flight conditions of increasing difficulties. *E.g.*, we consider power-off gliding flight, as used by the "Gimli glider". Air Canada Flight 143 ran out of fuel on July 23, 1983 due to ground staff mistaking pounds for kilograms. It was able to land safely at a former Royal Canadian Air Force base in Gimli, Manitoba, using "slip forward" maneuver.

Alexandre Goyer (Inria Saclay, France)
Symbolic-numeric factorization of differential operators.

`Wednesday, March 30, 11:00-11:40`

I will present a symbolic-numeric Las Vegas algorithm for factoring Fuchsian ordinary differential operators with rational function coefficients. The new algorithm combines ideas of van Hoeij's "local-to-globa" method and of the "analytic" approach proposed by van der Hoeven. It essentially reduces to the former in "easy" cases where the local-to-global method succeeds, and to an optimized variant of the latter in the "hardest" cases, while handling intermediate cases more efficiently than both. Joint work with Frédéric Chyzak and Marc Mezzarobba.