WORKSHOP ON GEOMETRY AND COMPLEXITY THEORY

25-29 April 2022, Toulouse Mathematics Institute

Program

Monday	8:30-9:00	*Registration* (Schwartz Amphitheater)
	9:00-10:15	Alman
	10:15-10:45	*Coffee break*
	10:45-12:00	Conner-Ikenmeyer
	12:00-14:00	*Lunch*
	14-00:15:15	Derksen-Makam
	15:15-15:45	*Coffee break*
	15:45-17:00	Buczynski
Tuesday	9:00-9:40	Michalek
	9:45-10:25	Blaeser
	10:55-12:20	Working groups
	14:10-14:50	Chiantini
	15:00-16:00	Working groups
	16:30-17:30	Short summaries from each group
	19:30	Dinner at *Aux Pieds sous la Table*
Wednesday	9:00-9:40	Ressayre
	9:45-10:25	Efremenko
	10:55-12:20	Working groups
	14:10-14:50	Ottaviani
	15:00-16:00	Working groups
	16:30-17:30	Short summaries from each group
Thursday	9:00-9:40	Draisma
	9:45-10:25	Buergisser
	10:55-11h35	Chang
	11h40-12:20	Huang
	14:10-14:50	Koiran
	15:00-16:00	Working groups
	16:30-17:30	Short summaries from each group
Friday	9:00-9:40	Gesmundo
	9:45-10:25	Oeding
	10:55-12:20	Working groups
	14:10-14:50	Lysikov
	15:00-16:00	Summaries from working groups 1,2
	16:30-17:30	Summaries from working groups 3,4

Tutorials

Josh Alman, Laser method and geometry. All new upper bounds on the exponent of matrix multiplication since 1987 have been obtained using Strassen's laser method. One starts with an "easy" tensor whose "cost" (border rank) is known (in practice either minimal or next to minimal border rank) and then shows it has high "value" in the sense that a large Kronecker power of it degenerates to a large matrix multiplication tensor. However the upper bound on the exponent has only advanced about .008 since 1988 (when it was shown $\omega \leq 2.38$) and starting with work of Ambainis-Filmus-LeGall in 2014 we now understand there are limits of the utility of the laser method for "easy" tensors of minimal border rank and more generally (the "irreversibility" which is essentially the ratio of the asymptotic rank to the asymptotic subrank). Nonetheless, it is possible that one could prove $\omega < 2.3$ with tensors of minimal border rank. For tensors of next to minimal border rank, in order to advance the laser method, one would need to show that the border rank of their Kronecker powers are strictly submultiplicative.

For the workshop tutorial, I'll give an overview of the Laser method, the tool used to show that tensors have high "value", and how it is used to design matrix multiplication algorithms. I'll aim to introduce enough so that we can work on the following goals for the workshop.

(1) When applying the Laser method to a tensor, one needs to give it as input a partitioning of the variables defining the tensor. Currently these partitions are found manually. Can we formally prove when we have found the best partitioning of our tensor, or in other words, the best application of the Laser method to our tensor? This could help us to search for better algorithms, and to prove further limitations on this method.

(2) How can we compute the asymptotic subrank of a tensor? In particular, what minimal border rank tensors are not subject to a 2.3 barrier? (We may have a presentation of explicit computations for tensors of interest.)

Jarek Buczynski, Border Apolarity. Border apolarity is an elementary method to study the border rank of polynomials and tensors, analogous to the classical apolarity lemma. It can be used to describe the border rank of all cases uniformly, including those very special ones that resisted a systematic approach. We also define a border rank version of the variety of sums of powers and analyse its usefulness in studying tensors and polynomials with large symmetries. In particular, it can be applied to provide lower bounds for the border rank of some interesting tensors, such as the matrix multiplication tensor. A critical ingredient of the border apolarity is an irreducible component of a multigraded Hilbert scheme.

Conner-Ikenmeyer, Computational methods. We present recent progress and open problems in the area of computational methods for geometry and complexity theory.

Visu Makam, Computational invariant theory and complexity. Invariant theory (and representation theory) is the study of symmetries and equivalences that are often captured in the setting of group actions. From it's very beginnings in the 19th century, computation has played a fundamental role in the progress of invariant theory. The theory of computation itself has witnessed humongous advances in the last century, starting from the ideas of Godel and Turing to modern-day supercomputers and the theory of computational complexity. This evolution in our understanding of computation has at many points in history revolutionized invariant theory directly or indirectly. Over the last two decades, we have come to realize that there are deeper connections between invariant theory and computational complexity – the lives of some of the central problems in complexity such as identity testing, graph isomorphism and even the celebrated P vs NP problem have become inextricably intertwined with some new research directions in invariant theory.

In this talk, we will give a quick overview of the goals of computational invariant theory (in particular orbit problems) taking a complexity theoretic view. We will discuss the recent developments on matrices, quivers and tensors and how they inform us of the techniques, challenges, and possibilities in understanding orbit problems and related problems in greater generality.

Some concrete open problems that we suggest for the workshop are the following:

(1) Tensors: Perhaps focus on the action of $SL(n) \times SL(n) \times SL(n)$ on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$.
- What is the computational power needed to define and compute invariants with a view towards orbit problems
- Efficient algorithms for orbit problems
- Tensor product semistability in positive characteristic: Given two semi-stable tensors T_1 and T_2, is their vertical product semistable? In characteristic zero, this is true thanks to the existence of the moment map.

(2) Quivers: For quivers with a constant number of vertices, we have poly-time algorithms for orbit problems (formulated precisely in the talk). What happens beyond that? For finite and tame quivers, we can get poly-time algebraic algorithms using foundational structural theorems in the theory of quivers.
- Are there settings beyond finite and tame quivers where we can get poly-time algorithms with current techniques? The answer is definitely a yes, but to what extent has not been explored enough
- For finite and tame quivers, how can one use the analytic techniques of BF-GOWW based on geodesic descent? The main quantity which is used in the analysis of the speed of their algorithms is called the gap (which they bound by the weight margin as gaps are difficult to calculate). In the case of finite quivers, weight margins are too bad an approximation to ensure efficient algorithms (Franks and Reichenbach). So, can we find good bounds on the gap itself? In the counterexample of Franks and Reichenbach, the gap is much better than the weight margin (not publicly available yet). Understanding the gaps boils down to giving bounds on the solutions to a special class of Linear programs.
- Given a quiver, a dimension vector alpha and a weight sigma, give an efficient algorithm to determine if a generic alpha-dimensional representation is sigma-semistable/polystable/stable. This is motivated by applications and connections to algebraic statistics.

(3) Others:
- For degree two repns of $SL(n)$, do we have polynomial degree bounds (for a minimal generating set of invarianrs) or not? This comes down to a single action, i.e., the action of $SL(n)$ on n^2-tuples of matrices given by $g \cdot (A_1, ..., A_{n^2}) = (gA_1g^t, ..., gA_{n^2}g^t)$. Note that for degree 3 and higher, we have exponential lower bounds.
- Extend the geodesic descent algorithms in BFGOWW to be useful for orbit closure intersection as well
- Orbit problems in NP vs co-NP
- Orbit problems for finite group actions (captures for e.g., the graph isomorphism problem)
- Analogs of moment polytopes and Kempf-Ness theory in positive characteristic.

Research talks

Markus Bläser, On the complexity of evaluating highest weight vectors. Geometric complexity theory (GCT) is an approach towards separating algebraic complexity classes through algebraic geometry and representation theory. Fundamental theorems of algebraic geometry and representation theory grant that every lower bound in GCT can be proved by the use of so-called highest weight vectors (HWVs). In the setting of interest in GCT (namely in the setting of polynomials) we prove the NP-hardness of the evaluation of HWVs in general, and we give efficient algorithms if the treewidth of the corresponding Young-diagram is small, where the point of evaluation is concisely encoded as a noncommutative algebraic branching program! In particular, this gives a large new class of separating functions that can be efficiently evaluated at points with low (border) Waring rank. Joint work with Julian Dörfler and Christian Ikenmeyer.

Peter Burgisser, Algorithms in invariant theory for torus actions. Recently, polynomial time algorithms for natural isomorphism problems for torus actions were found: these problems are orbit equality, orbit closure intersection, and orbit closure containment. These are exact algebraic algorithms that do not tolerate errors. The question of finding polynomial time numerical algorithms for robustly solving approximate versions of the above problems is open and challenging. For the null cone membership problem, a special case of the above problems, such algorithms are known: they rely on the sophisticated technology of self-concordant functions for interior-point methods. In talk we try to give an overview of the state of the art.

This is joint work with Levant Dogan, Yinan Li, Visu Makam, Harold Nieuwboer, Michael Walter, and Avi Wigderson.

Chia-Yu Chang, Maximal border subrank tensors. It is unknown if the subrank and border subrank of a generic tensor are the same. So I want to look at the set of maximal border subrank tensors. In this talk, I will construct some maximal border subrank tensors and will use this to compute a lower bound of the dimension of the closure of the set of maximal border subrank tensors.

Luca Chiantini, Singularities of secant varieties, and computation of minimal decompositions. Given one decomposition of a tensor T, viewed as a configuration of points in a projective space, it is relevant to understand if the decomposition is minimal, so that it computes the rank, and even if it is unique. The problem is related with the inclusion of T in the singular locus of the corresponding secant variety, whose complete description is still widely unknown. I will illustrate how, at least in the symmetric case, the study of the geometry of the configuration of points corresponding to a decomposition can give concrete answers to the previous questions.

Jan Draisma, Implicitisation and parameterisation for polynomial functors. Given (direct sums of) Schur functors P and Q, there is a natural notion of (polynomial) morphisms $f : P \to Q$: they are given by polynomial maps $f_V : P(V) \to Q(V)$ for each finite-dimensional vector space V, such that $f_V \circ P(\phi) = Q(\phi) \circ f_U$ for each linear map $\phi : U \to V$. Examples of such morphisms are the natural parameterisations of varieties of tensors of bounded border rank, or of bounded slice rank, etc.

I will report on joint work with Andreas Blatter and Emanuele Ventura in which we show that there exists an algorithm that, on input P, Q, and f, computes a finite set of equations that defines the image closure of f_V for all V set-theoretically. Conversely, there exists an algorithm that computes a parameterisation for any closed subvariety of P given by equations.

Our work builds on further joint work with Bik, Snowden, and Eggermont in which we show that points in such image closures can be approximated by curves of the form $f_V(\gamma(t))$, where $\gamma(t)$ is a Laurent-series valued point of $P(V)$ in which the negative exponents of t are uniformly bounded independently of V.

Klim Efremenko, Barriers for Rank Methods in Arithmetic Complexity. In this talk, we will study rank methods, which were long recognized as encompassing and abstracting almost all known arithmetic lower bounds to-date, including the most recent impressive successes. Rank methods (under the name of flattenings) are also in wide use in algebraic geometry for proving tensor rank and symmetric tensor rank lower bounds. Our main results are barriers to these methods. In particular,

(1) Rank methods *cannot* prove better than $\Omega_d(n^{\lfloor d/2 \rfloor})$ lower bound on the tensor rank of *any* d-dimensional tensor of side n. (In particular, they cannot prove super-linear, indeed even $> 6n$ tensor rank lower bounds for *any* 3-dimensional tensors.)
(2) Rank methods *cannot* prove $\Omega_d(n^{\lfloor d/2 \rfloor})$ on the *Waring rank* of any n-variate polynomial of degree d. (In particular, they cannot prove such lower bounds on stronger models, including depth-3 circuits.)

The proofs of these bounds use simple linear-algebraic arguments, leveraging connections between the *symbolic* rank of matrix polynomials and the usual rank of their evaluations. These techniques can perhaps be extended to barriers for other arithmetic models on which progress has halted. For this purpose, we develop a general framework for barriers to lower bounds via "sub-additive" complexity measures, of which rank measures are a special case.

Fulvio Gesmundo, Subrank of tensors and homomorphism duality. The subrank of a tensor is a value encoding to what extent a tensor is "stronger" than any tensor of a given rank. For this reason, tensors with large subrank play the role of universal objects for tensor rank, and find applications in numerous areas such as quantum physics and computational complexity. In this seminar, I will show that mild genericity properties give strong lower bounds for the (asymptotic) subrank of a tensor. I will emphasize connections with the notion of homomorphism duality, originated in graph theory, as well as the role of classical algebraic geometry and invariant theory. This is based on joint work with Matthias Christandl and Jeroen Zuiddam.

Amy Huang, Difference between Tensor Ranks and Border Rank of 3×3 Permanent. Border apolarity is a new and powerful tool that gives us many information about the border rank of a lot of tensors of interest. I will talk about two results about applications of it involving different notions of tensor rank and the border rank of 3×3 permanent. The first result relates different notions of tensor rank to polynomials of vanishing Hessian. The second one computes the border rank of 3×3 permanent, which is isomorphic to second Kronecker power of a small Coppersmith-Winograd tensor. This tensor is of interest in the study of exponent of matrix multiplication complexity. These are joint works with Emanuele Ventura, JM Landsberg, Austin Conner and Mateusz Michalek.

Pascal Koiran, Black Box Absolute Reconstruction for Sums of Powers of Linear Forms. We give the first (randomized) polynomial time algorithm for the following problem: given a homogeneous polynomial of degree d, decide if it can be written as a sum of d-th powers of linearly independent complex linear forms. This is joint work with Subhayan Saha. A preprint is available at https://arxiv.org/pdf/2110.05305.

Vladimir Lysikov, Lower bounds for degree-restricted strength decompositions and homogeneous algebraic branching programs. We relate the algebraic branching programs computing a polynomial to certain subschemes of the projective variety cut out by this polynomial. This allows us to refine Kumar's quadratic lower bound for algebraic branching programs. The talk is based on a joint work with Purnata Ghosal, Fulvio Gesmundo and Christian Ikenmeyer.

Mateusz Michalek, Invariants of tensors from algebraic geometry + sweet parts of tensors in the laser method. In the first part of the talk we will briefly present new invariants of tensors (that are integer numbers). They generalize some of central, classcal notions in different areas of mathematics, including chromatic polynomial of a graph, maximum likelihood degree of a linear concentration model, algebraic degree of semi-definite programming, Euler characteristic of determinental hypersurface, Betti numbers of hyperplane arrangements and characteristic numbers for linear systems of quadrics. In the second part, we would like to briefly report on on-going work with Joachim Jelisiejew, were among others, we show that parts of the tensors used in the laser method have smaller border rank and rank then currently used estimates.

Luke Oeding, A Jordan Decomposition for Tensors. The Jordan decomposition of linear operators is a powerful conjugation invariant which allows, in particular, for classification of orbits. When a group G acts on a space of tensors M we can construct a graded algebra, on which elements of both G and M act as linear operators (the adjoint action). This generalizes an idea of Vinberg, and allows us to construct adjoint operators, and hence Jordan decompositions, of tensors for many more cases than were previously thought possible. I will explain this construction and how we are using it to separate orbits that are relevant for quantum information. I will also explain how we may use this construction to study tensors relevant for computational complexity.

This is joint work in progress with Frederic Holweck.

Giorgio Ottaviani Euclidean distance degree of orthogonally invariant varieties. Let X be an affine variety embedded in V, invariant for the action of the orthogonal group $SO(V)$. Meaningful examples are the secant varieties of flag varieties, they include the spaces of rank one tensors.

The number of critical points on X of the distance function from a general f in V is called the EDdegree of X, and it is a good measure of the complexity to compute the best approximation lying in X. Our main result is that all critical points lie in the subspace orthogonal to $(LieSO(V)).f$.

The proof is quite simple, nevertheless this result allows to compute the EDdegree of complete flag varieties and of qubit spaces, in any of their complete embeddings. We will discuss some consequences for tensor decomposition and the open case of Grassmann varieties.

Nicolas Ressayre, On bidilations of small Littlewood-Richardson coefficients. In this talk, we are interested in small Littlewood-Richardson coefficients, more precisely equal to one or two. We describe their behavior after dilatation of the three partitions in both the horizontal and vertical direction. (Joint work with Pierre-Emmanuel Chaput.)