Preliminary remark: there is a large overlap between these notes and Juerg's slides. These notes are based on some joint works with J. Fröhlich, M. Fraas and M. Ballesteros.

In these notes, we focus on two problems in QM:

1: The characterization of empirical properties in Q.M. We try to answer the question: Given a quantum mechanical system S, can Q.M. tells us what observables on S are measured first?

2: Effective dynamics for a quantum mechanical system under repetitive measurements. We try to answer the question: What can be inferred about a QM system by carrying out repetitive indirect measurements (i.e. projective measurements on (a) probe(s) coupled to it)?

1 Introduction: Algebraic data characterizing a (closed) physical system

We first present a (well-known) algebraic formalism that can be used to describe classical as well as quantum physical systems.

- Closed system S: specified by a set of 4 data $(\mathscr{O}_S, \mathscr{A}_S, \{\tau_{t,s}\}_{t,s\in\mathbb{R}}, \mathscr{S}_S)$,

  (I) $\mathscr{O}_S = \{A_i = A_i^*\}_{i\in I} \subset \mathscr{A}_S$: set of observables that can be measured in projective measurements,

  (II) $\mathscr{A}_S$: dynamical (C^*-)algebra, large enough such that $\tau_{t,s} : \mathscr{A}_S \to \mathscr{A}_S$ (S is "closed"),

  (III) $\{\tau_{t,s}\}_{t,s\in\mathbb{R}}$: groupoid of *automorphisms on $\mathscr{A}_S$, $\tau_{t,s} = \tau_{t,u} \circ \tau_{u,s}$, for arbitrary times t, u and s in $\mathbb{R}$. Encodes time evolution.

  (IV) $\mathscr{S}_S$: set of states $\mathscr{A}_S$ (positive linear functionals $\omega : \mathscr{A}_S \to \mathbb{C}$): used to predict probabilities of measurement outcomes/events

- Classical system: $\mathscr{A}_S$ abelian (ex: $\mathscr{C}_0(M_S)$: continuous functions on compact Hausdorff space M_S) $\Rightarrow$ determinism: If system in a *pure* state, probability of outcomes given by $0-1$ law (c.f. tabular below).

- Quantum system: $\mathscr{A}_S$ non-commutative (ex: $\mathscr{B}(\mathscr{H}_S)$, $\dim(\mathscr{H}_S) \geq 2$: algebra of bounded operators on a Hilbert space) $\Rightarrow$ theory intrinsically probabilistic: even if system in a pure state, one can *only* predict the probabilities of measurement outcomes: no $0-1$ law in general!

	Classical	Quantum
Typical examples	$\mathscr{A}_S = \mathscr{C}_0(M_S)$	$\mathscr{A}_S = \mathscr{B}(\mathscr{H}_S)$
States	prob. measures on M_S	pos. trace class op. ρ
Pure states	δ_x (Dirac mass), $x \in M_S$	Unit rays $\lvert\psi\rangle \in \mathscr{H}_S$
Time evolution (autonomous)	$\tau_{t,s}(f) = f \circ \phi_{t,s}$ $\phi_{t,s} = \phi_{t-s}$	$\tau_{t,s}(A) = U^*_{t,s} A U_{t,s}$ $U_{t,s} = U_{t-s} = e^{-i(t-s)H_S}$.

Remarks:

- In general, $\mathscr{O}_S$ contains very few elements (c.f. Section 3 and Juerg's talk).

- Gelfand-Naimark's theorem shows that every commutative abelian C^*-algebra with unit is isometrically isomorphic to $\mathscr{C}_0(M_S)$ (continuous functions on compact Hausdorff topological space M_S).

- Every C^*-algebra is isometrically isomorphic to a norm closed self-adjoint algebra of operators on a Hilbert space (c.f. GNS construction)

2 Empirical properties in Quantum Mechanics

Even if QM non-deterministic, one can predict what observables in $\mathscr{O}_S$ will be measured first. In this Section, we therefore give an algebraic characterization of the observables that are measured around some time t, and we try to answer the question:

Question: Can we predict what observables in $\mathscr{O}_S$ are measured first by knowing initial state ω at $t = 0$ and time evolution $\{\tau_{t,s}\}$?

- One first introduces the C^*-algebra of potential properties observable after time t:

$$\mathscr{E}_{\geq t} := \langle \{ \sum_{i \in I} \prod_{j_i} \tau_{t_{j_i},0}(A_{j_i}) \mid I \text{ finite}, \ A_{j_i} \in \mathscr{O}_S, \ t_{j_i} \geq t \} \rangle \subseteq \mathscr{A}_S.$$

  ($\langle . \rangle$ means that $\mathscr{E}_{\geq t}$ is the smallest self-adjoint and norm closed subalgebra of $\mathscr{A}_S$ containing the set of arbitrary finite sums and products of observables translated after time t. For "small" sets $\mathscr{O}_S$, one generally has that $\mathscr{E}_{\geq t} \subsetneq \mathscr{A}_S$)

- Two important mechanisms :

  1. If $\mathscr{E}_{\geq t} \subsetneq \mathscr{E}_{\geq t'}$ $(t > t')$, then $\omega_{|\mathscr{E}_{\geq t}}$ is generally mixed even if $\omega_{|\mathscr{E}_{\geq t'}}$ were pure (see e.g. Schmidt decomposition for bipartite quantum systems) $\Rightarrow$ mechanism called information loss.

2. If $A = \sum_j \alpha_j \Pi_j \in \mathscr{O}_S$ is observed/measured around time t, then

$$\omega(B) = \sum_{i,j} \omega(\Pi_i(t) B \Pi_j(t)) \approx \sum_j \omega(\Pi_j(t) B \Pi_j(t))$$

for all $B \in \mathscr{E}_{\geq t} \Rightarrow$ decoherence.

Remark: $\approx$ means that there is equality up to a very tiny error. Requiring decoherence and commutativity of empirical properties, one arrives at characterization:

- Empirical property around time t (if $\mathscr{E}_{\geq t}$ is type I von Neumann algebra) : Define $\omega_{|\mathscr{E}_{\geq t}} =: \mathrm{Tr}(\rho_t \ \cdot \ )$ (ρ_t density matrix). Then $A \in \mathscr{O}_S$ is empirical around t if

$$A(t)_{\mathrm{ran}(\rho_t)} := (\tau_{t,0}(A))_{\mathrm{ran}(\rho_t)} \approx f(\rho_t)$$

for some real valued measurable function f (c.f. spectral calculus for definition of $f(\rho_t)$). (Here again, $\approx$ means that there is equality up to a very tiny error).

Q.M. is random since it does not tell us which outcome of A is measured, but only predicts its probability:

- Axiom: Projective measurement
If A empirical around time t, the probability to observe α_j around t is given by $\omega(\Pi_j(t))$. If α_j observed, then replacement

$$\omega \mapsto \omega_j := \frac{\omega(\Pi_j(t) \cdot \Pi_j(t))}{\omega(\Pi_j(t))}$$

to improve predictions.

Remarks:

- Even if everything above might seem too abstract to be really useful, one can provide examples where the algebras $\mathscr{E}_{\geq t}$ and the empirical properties can be calculated more or less explicitly. Typically: examples of non-relativistic QED, where S is made of atom/finite level system coupled to a quantized field.

- Empirical properties given here only for type I von Neumann algebras. For more general settings, c.f. [3].

3 The emergence of an effective stochastic dynamics from quantum evolution and repeated indirect measurements

3.1 Motivation

Consider system

$$S = P \vee E$$

E: a probe used to study P. Projective measurements (in the sense of Section 2) are made on E. For simplicity, consider system for which $\mathscr{A}_S = \mathscr{B}(\mathscr{H}_P \otimes \mathscr{H}_E)$.

Question: What can be inferred about P by carrying out repetitive projective measurements on E? In particular, is it possible to measure $\mathscr{N} = \mathscr{N}^* \in \mathscr{B}(\mathscr{H}_P)$ indirectly?

Examples:
-Ex1: Serge Haroche's (and collaborators) experiment: indirect measurement of number of photons in pure optical cavity by sending Rydberg atoms (or similarly, the ideal experiment drawn on Juerg's slides concerning the measurement of electrons trapped in a quantum dot)
-Ex2: Indirect measurement of the position of a particle/atom by means of light scattering

In Ex1: E is itself made of ∞-many independent probes (the Rydberg atoms/ the e- in Juerg's talk)
In Ex2: only one probe: E is the quantized e.m. field. Even if Ex2 much easier to realize experimentally, it is much more difficult to treat with rigorous maths (work in progress in this direction)!

In this note, we only give some (we hope a little new) insights concerning a "standard" model used to describe Ex1 (see e.g. [5, 1] for previous results).

3.2 Specific model

Let $k \in \mathbb{N}$. Assume $S = P \vee E_1 \vee ... \vee E_k$. E_j's: identical probes (e.g. Rydberg atoms/or e^- in Juerg's talk).

- Hilbert space
$$\mathscr{H}_S := \mathscr{H}_P \bigotimes_{j=1}^{k} \mathscr{H}_j, \tag{1}$$
where $\mathscr{H}_j = \mathscr{H}_{\text{prob}}$.

- Initial state
$$\rho := \rho^{(0)} \bigotimes_{j=1}^{k} |\psi_j\rangle\langle\psi_j|, \qquad \psi_j = \psi, \tag{2}$$
where $\rho^{(0)}$ density matrix on $\mathscr{H}_P$.

- Unitary time evolution between j and $j+1$ acts non-trivially only on the factors $\mathscr{H}_P$ and $\mathscr{H}_{j+1}$.

- Von Neumann measurement of $X = X^* = \sum_\xi \xi \pi_\xi \in \mathscr{B}(\mathscr{H}_{\text{pro}})$ carried out on probe E_j at time $\approx j$, after it has interacted with P.

We are willing to measure $\mathscr{N} = \sum_{\nu=1}^{N} \nu \Pi_\nu \in \mathscr{B}(\mathscr{H}_P)$ indirectly. Two types of unitary time evolution:

(I) **Non-demolition measurement** :

$$U(j, j-1) = U_{\mathrm{ND}}(j, j-1) := \sum_{\nu \in \sigma(\mathscr{N})} \Pi_\nu \otimes \mathbb{1} \otimes ... \otimes \underbrace{U_\nu}_{\mathscr{H}_j} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \tag{3}$$

for all $j = 1, ..., k$. U_ν: unitary operators.

(II) **Indirect measurements**:

$$U(j, j-1) := U_{\mathrm{ND}}(j, j-1) W(j, j-1) \tag{4}$$

for all $j = 1, ..., k$, W unitary. Operator norm of $W(j, j-1) - \mathbb{1}$ uniformly bounded in j by small constant $\eta > 0$.

3.3 Results under Hypothesis (I) (QND)

Already treated in [1]; see also [4] and [5]. In [1], authors statements based on Martingale's convergence theorem. Here, we use large deviation tools. Consider

$$\Xi := \{\underline{\xi} = (\xi_i)_{i \in \mathbb{N}} \mid \xi_i \in \sigma(X)\}, \tag{5}$$

space of infinitely long measurement protocols: sequence of probe measurement outcomes. Ξ equipped with σ-algebra generated by cylinder sets, and probability measure μ_ρ defined inductively via

$$\mu_\rho(\underbrace{\xi_1, ..., \xi_k}_{:=\underline{\xi}_k}) := \mathrm{Tr}(\rho \pi^{(1)}_{\xi_1}(t_1) ... \pi^{(k)}_{\xi_k}(t_k) ... \pi^{(1)}_{\xi_1}(t_1)). \tag{6}$$

If time evolution as in (I),

$$\mu_\rho(\underline{\xi}^{(k)}) = \sum_{\nu \in \sigma(\mathscr{N})} \mathrm{Tr}(\Pi_\nu \rho^{(0)} \Pi_\nu) \mu^{(k)}_\nu(\underline{\xi}^{(k)}). \tag{7}$$

with

$$\mu^{(k)}_\nu(\underline{\xi}^{(k)}) = \prod_{m=1}^{k} |\langle \xi_m | U_\nu \psi \rangle|^2. \tag{8}$$

Moreover,

$$\Pi_\nu \rho^{(k)}(\underline{\xi}^{(k)}) \Pi_\nu = \frac{\mu^{(k)}_\nu(\underline{\xi}^{(k)})}{\mu_\rho(\underline{\xi}^{(k)})} \, \Pi_\nu \rho^{(0)} \Pi_\nu, \tag{9}$$

where $\rho^{(k)}$: reduced state of P after k measurements on probes.

- $\Rightarrow \mu_\rho$ in (7) weighted sum of product measures.

- $\Rightarrow$ strong analogy with ideal lattice gas in statistical mechanics: $\nu \in \sigma(\mathscr{N}) \leftrightarrow$ thermodynamic parameter (like fugacity), $\xi \in \sigma(X) \leftrightarrow$ label for distinct particles, $|\langle \xi | U_\nu \psi \rangle|^2 \leftrightarrow$ prob to have ξ if thermo param $= \nu$, $\lim k \to \infty \leftrightarrow$ thermodynamic limit.

- Many well known results (without appealing to MCT convergence)! Use tools of large deviation theory (related to equivalence of ensembles). In particular, we know that the equilibrium measures μ_ν and $\mu_{\nu'}$ mutually singular if $(|\langle \xi | U_\nu \psi \rangle|^2)_{\xi \in \sigma(X)} \neq (|\langle \xi | U_{\nu'} \psi \rangle|^2)_{\xi \in \sigma(X)}$.

- Using Sanov's theorem, it is easy to show that the reduced state of P-μ_ρ almost surely converges (in L^1 norm) to $\frac{\Pi_\Theta \rho^{(0)} \Pi_\Theta}{\mathrm{Tr}(\Pi_\Theta \rho^{(0)} \Pi_\Theta)}$, where $\Theta : \Xi \to \sigma(\mathscr{N})$, if the above non-degeneracy criterion is satisfied; see [1] for proof with MCT

  $\Rightarrow \mathscr{N}$ is measured at infinity.

Using tools of large deviation, one gets convergence estimates: show that the measures $\mu_\nu^{(k)}$ concentrate exponentially fast, and that the states converge (almost) exponentially fast.

Lemma 3.3.1. *Let $b \in (0,1)$. Assume $\mathbf{p}_\nu := (|\langle \xi | U_\nu \psi \rangle|^2)_{\xi \in \sigma(X)} \neq \mathbf{p}_{\nu'} := (|\langle \xi | U_{\nu'} \psi \rangle|^2)_{\xi \in \sigma(X)}$ for all $\nu, \nu' \in \sigma(\mathscr{N})$ with $\nu \neq \nu'$. Assume $\sigma(X)$, $\sigma(\mathscr{N})$ finite. Then, there is $K \in \mathbb{N}$ and a family of sets $\Xi_\nu^{(k)} \subset \sigma(X)^k$, $\nu \in \sigma(\mathscr{N})$, such that the following holds* ***uniformly*** *in $\rho^{(0)}$ for all $k \geq K$:*

(i) $\Xi_\nu^{(k)} \cap \Xi_{\nu'}^{(k)} = \emptyset$ if $\nu \neq \nu'$.

(ii) For all $\nu, \nu' \in \sigma(\mathscr{N})$ with $\nu \neq \nu'$,

$$|\mu_\nu^{(k)}(\Xi_{\nu'}^{(k)})|, \quad |\mu_\rho(\Xi_\nu^{(k)}) - \mathrm{Tr}(\Pi_\nu \rho^{(0)})|, \quad 1 - \mu_\rho\Big(\bigcup_{\nu \in \sigma(\mathscr{N})} \Xi_\nu^{(k)}\Big) \leq \delta_k(b), \tag{10}$$

where $\delta_k(b) := (k+1)^{|\sigma(X)|} \exp(-k^{1-b})$.

(iii) If $\underline{\xi}^{(k)} \in \Xi_\nu^{(k)}$ and if $\mathrm{Tr}(\rho^{(0)} \Pi_\nu) > 0$, then

$$\sum_{\nu' \neq \nu} \mathrm{Tr}(\Pi_{\nu'} \rho^{(k)}(\underline{\xi}^{(k)})) \leq \sum_{\nu' \neq \nu} \mathrm{Tr}(\Pi_{\nu'} \rho^{(0)}) \frac{\exp\big(-kI_\nu(\mathbf{p}_{\nu'}) + Ck^{1-b/2}\big)}{\mathrm{Tr}(\Pi_\nu \rho^{(0)})}, \tag{11}$$

for some constant $C > 0$ (that only depends on the $\mathbf{p}_\nu$'s).

In (11), $I_\nu(\gamma)$ relative entropy

$$I_\nu(\gamma) := \sum_{\xi \in \sigma(X)} \gamma_\xi \log\left(\frac{\gamma_\xi}{|\langle \xi | U_\nu \psi \rangle|^2}\right).$$

These estimates are useful to treat perturbations of the QND hypothesis (I).

3.4 Results for indirect measurements: Hypothesis (II)

Assume now time evolution (II). A typical situation where (3.4) is satisfied occurs if evolution between j and $j+1$ generated by Hamiltonian

$$H_{\text{jump}} := H_P \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \sum_{\nu} \Pi_\nu \otimes \mathbb{1} \otimes \ldots \otimes \underbrace{H_\nu}_{j+1} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}, \tag{12}$$

where $[H_P, \mathscr{N}] \neq 0$. Using standard perturbative arguments and Lemma 3.3.1, one gets in that case (under the same non-degeneracy hypotheses as before):

Lemma 3.4.1. *Let $\varepsilon \in (0, 1/|\sigma(\mathscr{N})|)$. There is $\eta_\varepsilon > 0$ and $k_\varepsilon \in \mathbb{N}$, such that, if $\|W(j, j-1) - \mathbb{1}\| < \eta_\varepsilon$ for all j, then, for all $k \geq k_\varepsilon$, there is a set $\Lambda^{(k)} \subseteq \sigma(X)^k$ with $\mu_\rho(\Lambda^{(k)}) \geq 1 - \varepsilon$ and such that*

$$\|\rho^{(k)}(\underline{\xi}^{(k)}) - \Pi_{\nu(\underline{\xi}^{(k)})}\rho^{(k)}(\underline{\xi}^{(k)})\Pi_{\nu(\underline{\xi}^{(k)})}\|_1 \leq \varepsilon \tag{13}$$

for every $\underline{\xi}^{(k)} \in \Lambda^{(k)}$ and for some $\nu(\underline{\xi}^{(k)}) \in \sigma(\mathscr{N})$. $\rho^{(k)}$ in (13) is the reduced state of P at step k.

$\Rightarrow$ Lemma 3.4.1 shows that the reduced state of P experiences a jump process on the spectrum of $\mathscr{N}$ (up to a set of protocols with very small measure).

$\Rightarrow$ Jump rates can be extracted for simple model, or in some limiting Markovian regime.

3.5 Outlook

The model treated in 3.2 is very simple since, as we have seen, it is analogous to the ideal gas in Stat Mech. However, tools of large deviations can be used for much more complicated settings (in analogy with statistical mechanics where particles of the gas interact in non-trivial models, we can introduce here interactions between the probes, and we expect that the result still holds). $\Rightarrow$ Some hope to treat more natural phenomena such as indirect position measurements via light scattering, using large deviation tools (e.g. Gartner-Ellis th.) for easy (but somewhat realistic) QED models.

References

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