Geometry and Control in Cortona

In honor of Andrei Agrachev's 70th+1 birthday

27–31 March 2023

Program

27 March 2023		
Time	Speaker	Title
9h30 – 10h20	Nicola Garofalo	Heat asymptotics in sub-Riemannian geometry
10h20 – 10h50	*Coffee Break*	
10h50 – 11h40	Katrin Fässler	On a duality principle for horizontal lines in the Heisenberg group
11h40 – 12h30	Sergei Kuksin	Random perturbations of linear and integrable systems: averaging and mixing
12h30 – 14h30	*Lunch Break*	
14h30 – 15h20	Lev Lokutsievskiy	On L_p-Hölder continuity of sub-Riemannian optimal control
15h20 – 16h10	Antonio Lerario	Optimal transport between algebraic hypersurfaces
16h10 – 16h40	*Coffee Break & Poster Session*	
16h40 – 17h30	Gianna Stefani	TBA

28 March 2023		
Time	Speaker	Title
9h30 – 10h20	Roberto Monti	Higher order Goh conditions
10h20 – 10h50	*Coffee Break*	
10h50 – 11h40	Isabelle Gallagher	Fourier analysis on the Heisenberg and Engel groups, and applications
11h40 – 12h30	Andrea Mondino	Unified synthetic Ricci curvature lower bounds for Riemannian and sub-Riemannian structures
12h30 – 14h30	*Lunch Break*	
14h30 – 15h20	Ivan Beschastnyi	Symplectic geometry and second variation
15h20 – 16h10	Frédéric Jean	Inverse optimal control problem, general methodology and non-autonomous LQ case
16h10 – 16h40	*Coffee Break & Poster Session*	
16h40 – 17h30	Andrei Agrachev	"Good Lie Brackets" for Control-affine Systems
20h00	*Social dinner (Osteria del Teatro)*	

29 March 2023		
Time	Speaker	Title
9h30 – 10h20	Andrea Malchiodi	On the role of embeddability in conformal sub-Riemannian geometry
10h20 – 10h50	*Coffee Break*	
10h50 – 11h40	Velimir Jurdjevic	Rolling geodesics and affine quadratic problems
11h40 – 12h30	Zoltan Balogh	Rigidity of Wasserstein spaces over Heisenberg groups
12h30 – 14h30	*Lunch Break*	

30 March 2023		
Time	Speaker	Title
9h30 – 10h20	Igor Zelenko	Morse inequalities for eigenvalue branches of generic families of self-adjoint matrices
10h20 – 10h50	*Coffee Break*	
10h50 – 11h40	Enrico Le Donne	Looking for geodesic loops
11h40 – 12h30	André Belotto da Silva	On the minimal rank Sard Conjecture in analytic manifolds
12h30 – 14h30	*Lunch Break*	
14h30 – 15h20	Karen Habermann	Intrinsic sub-Laplacian for hypersurface in a contact sub-Riemannian manifold
15h20 – 16h10	Thomas Chambrion	Averaging techniques for bilinear quantum systems with mixed spectrum
16h10 – 16h40	*Coffee Break & Poster Session*	
16h40 – 17h30	*Free Discussions*	

31 March 2023		
Time	Speaker	Title
9h30 – 10h20	Alexey Davydov	Cyclic exploitation of distributed renewable resource with diffusion and its optimization
10h20 – 10h50	*Coffee Break*	
10h50 – 11h40	Armen Shirikyan	Exponential stability of the Burgers flow
11h40 – 12h30	Yuri Sachkov	Lorentzian geometry in the Lobachevsky plane
12h30 – 14h30	*Lunch Break*	

Abstracts

"Good Lie Brackets" for Control-affine Systems

Andrei Agrachev (SISSA)

We consider smooth systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{k} u_i f_i(x),\ x \in M,$$

and study controllability issues both on M and on the group $\mathrm{Diff}\, M$.

There exists a satisfactory theory for control-linear systems $\dot{x} = \sum_{i=0}^{k} u_i f_i(x)$. Namely, the system can arbitrarily well approximate the movement in the direction of any Lie bracket polynomial of $f_0, \dots, f_k$. Moreover, the controls that approximate the movement in the desired direction depend only on the structure of the Lie bracket polynomial and not on f_i. Any Lie bracket polynomial is "good" in this sense.

In the case of a control-affine system, we are not able to go (approximately) in the direction of any Lie bracket polynomial but some Lie bracket polynomials are still fine. Hunting for "good brackets" for control-affine systems leads to interesting objects in the free Lie algebras, which I am going to discuss. I am grateful to Alexander Zuev and Khazhgali Kozhasov for very useful stimulating conversations.

Rigidity of Wasserstein spaces over Heisenberg groups

Zoltan Balogh (University of Bern)

The space of probability measures over a metric space can be endowed with the Wasserstein metric via optimal mass transportation. In this way we obtain an infinite dimensional space containing a copy of the underlying space itself. Isometries of the underlying space induce, by push forward, isometries of the Wasserstein space in a natural way. The question is: can one find other isometries of the Wasserstein space different than these? This question has been recently studied in the settings of Euclidean spaces and Hadamard manifolds. In this talk I will present related results in the case of Heisenberg groups. This is a joint work with Daniel Virosztek and Tamas Titkos.

On the minimal rank Sard Conjecture in analytic manifolds

André Belotto da Silva (Université Paris Cité)

We will present our recent result in collaboration with Parusinski and Rifford, concerning the Sard Conjecture in analytic sub-Riemannian manifolds (M, D). When D is a corank 1 distribution, the minimal rank Sard Conjecture is equivalent to the Sard Conjecture.

More precisely, we will explain how the Conjecture can be interpreted as a geometrical problem concerning the behavior of a characteristic singular foliation in the cotangent bundle. Under the hypothesis of analyticity of M and D, we can study this singular foliation via methods of singularity theory, subanalytic geometry and control measure theory. Under an additional qualitative property of the foliation which we call "splittable", we provide a proof of the minimal rank Sard conjecture in the analytic category.

Symplectic geometry and second variation

Ivan Beschastnyi (University of Aveiro)

Symplectic geometry is a very natural language for writing second order minimality conditions in optimal control. A typical example is given by the Morse index theorem in the classical calculus of variations, which states that the Morse index of the second derivative of the minimized functional is equal to the Maslov index of the Jacobi curve. This allows us to encode information about an infinite-dimensional object in a geometric quantity. Broadly speaking symplectic geometry gives a framework for unification of various results concerning local minimality of extremal curves. In this talk I will discuss the symplectic approach in a broad context and some results obtained in joint works with A. Agrachev and S. Baranzini.

Averaging techniques for bilinear quantum systems with mixed spectrum

Thomas Chambrion (Université de Bourgogne Franche-Comté)

Dynamics of closed quantum systems submitted to slowly varying external excitation (e.g., a laser) can be described using bilinear conservative control systems (i.e., a linear drift and a bilinear control component) in complex Hilbert spaces. When the underlying Hilbert space is finite dimensional, geometric control theory provides powerful tools to answer most of the natural controllability questions. The problem is way more intricate when the ambient Hilbert space is infinite dimensional. The case where the ambient space admits a Hilbert basis of eigenfunctions of the drift has been thoroughly studied in the last two decades and is now reasonably understood. By contrast, much less efforts have been devoted to the study of the "mixed spectrum" case where the eigenvectors of the drift do not span a dense subset of the ambient space.

The aim of this talk is to present some new results about the use of averaging techniques (inspired by finite dimensional methods) for the control of such bilinear quantum systems with mixed spectrum.

Cyclic exploitation of distributed renewable resource with diffusion and its optimization

Alexey Davydov (Lomonosov Moscow State University)

We consider a renewable resource distributed in a periodic environment, which is taken as n-dimensional torus. The dynamics of the resource is described by Kolmogorov-Piskunov-Petrovsky-Fisher equation [1], [7], [8] in divergent form

$$p_t = (\alpha(x)p_x)_x + a(x)p - b(x)p^2.$$

Here $p = p(x,t)$ is the density of the resource at the point x of its distribution area at the time t, and functions α, a and b characterize the diffusion of the resource, the rates of its renewal and saturation of the environment with it, respectively. It is assumed that the functions a and b are measurable and depend on a point of this area, but do not depend on time, and, in addition, the function b is positive and separated from zero by some positive constant. The matrix α is positive definite, and its elements have derivatives satisfying the Holder condition with some positive exponent. Such conditions are imposed on the saturation and diffusion exponents in [1], some of the results of which are used in our proofs.

The resource is exploited by either the permanent harvesting, or periodic impulse harvesting, or else in the case of the circle (when $n = 1$) by harvesting control machine, which periodically

moves along the circle and at each moment collect a part of resource density that depends of the current machine position and difficulties to search or extract the resource from this position.

We show that for each of considered harvesting modes there exists admissible strategy which provides the maximum time averaged income in kind, and the respective state of the resource is periodic in time or stationary [2], [3], [4], [5], [6].

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[8] Kolmogorov A. N., Petrovskii I. G., Piskunov N. S. *A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem,* Bull. Moscow Univ., 1937, Math. Mech. 1, pp. 1–25.

On a duality principle for horizontal lines in the Heisenberg group

Katrin Fässler (University of Jyväskylä)

I will discuss a duality principle that associates to points and horizontal lines in the Heisenberg group $\mathbb{H}^1$ 'conical' lines and points in $\mathbb{R}^3$. This principle allows to solve certain problems about Hausdorff dimension, horizontal lines, and vertical Heisenberg projections by relating them to recent results in Euclidean geometric measure theory and harmonic analysis (notably about restricted families of orthogonal projections to planes in $\mathbb{R}^3$ due to Gan, Guo, Guth, Harris, Maldague, and Wang, and about point-plate incidences in $\mathbb{R}^3$ due to Guth, Wang, and Zhang, respectively). This talk is based on collaboration with Tuomas Orponen.

Fourier analysis on the Heisenberg and Engel groups, and applications

Isabelle Gallagher (ENS Paris)

The Heisenberg and Engel groups are prototypes of Carnot groups of steps 2 and 3 respectively. We shall present properties of the sublaplacian on these groups, which are related via the Fourier transform to the quadratic and quartic oscillators. The spectral analysis of these operators leads to estimates on the convolution kernel of operators of the type $F(-\Delta)$, which in turn allow to recover classical functional embeddings, and estimates on propagators, via Fourier techniques. This corresponds to joint works with Hajer Bahouri, Davide Barilari and Matthieu Léautaud.

Heat asymptotics in sub-Riemannian geometry

Nicola Garofalo (Università degli Studi di Padova)

I will discuss some heat asymptotics in Carnot groups. The content of my talk is joint work with G. Tralli.

Intrinsic sub-Laplacian for hypersurface in a contact sub-Riemannian manifold

Karen Habermann (University of Warwick)

We construct and study the intrinsic sub-Laplacian, defined outside the set of characteristic points, for a smooth hypersurface embedded in a contact sub-Riemannian manifold. We prove that, away from characteristic points, the intrinsic sub-Laplacian arises as the limit of Laplace-Beltrami operators built by means of Riemannian approximations to the sub-Riemannian structure using the Reeb vector field. We carefully analyse three families of model cases for this setting obtained by considering canonical hypersurfaces embedded in model spaces for contact sub-Riemannian manifolds. In these model cases, we show that the intrinsic sub-Laplacian is stochastically complete and in particular, that the stochastic process induced by the intrinsic sub-Laplacian almost surely does not hit characteristic points.

Inverse optimal control problem, general methodology and non-autonomous LQ case

Frédéric Jean (ENSTA Paris)

Given a control system and a set of optimal trajectories, is it possible to recover the cost for which the trajectories are minimized? In this talk, we present a general approach to answering this question in the widely used class of optimal control problems defined by control-affine systems and quadratic costs. The basic notion is that of cost equivalence – a generalization of affine equivalence of metrics in the Riemannian and sub-Riemannian case - and the key element of our approach is the use of ample geodesics. We will also explore the non-autonomous setting with a comprehensive study of the linear-quadratic case based on a new characterization of optimal syntheses.

Rolling geodesics and affine quadratic problems

Velimir Jurdjevic (University of Toronto)

In this talk we will reveal several remarkable connections between affine quadratic optimal control problems and rolling problems. Both problems are defined in terms of a semi-simple Lie group G and a compact subgroup K. For such pairs of Lie groups the Lie algebra $\mathfrak{g}$ of G admits an orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ relative to the bilinear form $\langle\ ,\ \rangle$ a scalar multiple of the Killing form. Additionally we assume that the Killing form is of definite sign on $\mathfrak{p}$ and that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

The rolling problem is induced by the rollings of $M = G/K$ on its tangent space $\hat{M} = T_oM$, $\pi(e) = 0$, where π is the natural projection $\pi(g) = gK$. We show that there is natural metric on M relative to which the natural action $\tau_g(\pi(h)) = \pi(gh)$ is an isometry. Then a curve $\alpha(t)$ is rolled on a curve $\hat{\alpha}(t)$ in $\hat{M}$ if and only if $\alpha(t) = \tau_{g(t)}(o)$ and $(g(t), \hat{\alpha}(t))$ is a solution curve of the distribution

$$\frac{dg}{dt} = g(t)U(t), \frac{d\hat{\alpha}}{dt} = \vec{U}(t)o, U(t) \in \mathfrak{p},$$

where $\vec{U}$ is the Killing vector field induced by $U \in \mathfrak{p}$. These solutions are called rolling motions. Each rolling motion has natural length $\int_0^T \sqrt{\langle U(t), U(t)\rangle}\, dt$. Rolling motions of minimal length that connect two given points in $G \times \hat{M}$ are called rolling geodesics. We will show that the associated extremal curves are the integral curves of the Hamiltonian vector field $\vec{\mathbf{H}}$ associated with the Hamiltonian $\mathbf{H} = \frac{\mathbf{1}}{\mathbf{2}}\langle \mathbf{A} + \mathbf{L}_\mathfrak{p}, \mathbf{A} + \mathbf{L}_\mathfrak{p}\rangle, \mathbf{L}_\mathfrak{p} \in \mathfrak{p}$ for some constant $\mathbf{A}$ in $\mathfrak{p}$. The main results are that $\vec{\mathbf{H}}$ admits an isospectral representation and hence is completely integrable. Secondly, we show that the constant $\mathbf{A}$ identifies an affine-quadratic Hamiltonian $H = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle A, L_\mathfrak{p}\rangle, L_\mathfrak{p} \in \mathfrak{p}, L_\mathfrak{k} \in \mathfrak{k}$, whose Poisson equations are diffeomorphic with the Poisson equations of $\mathbf{H}$ on the coadjoint orbit that contains $\mathbf{A}$. We will also show the relevance of the above results for the equations of mechanical tops and elastic curves.

Random perturbations of linear and integrable systems: averaging and mixing

Sergei Kuksin (Université Paris Cité)

I will discuss stochastic ε-perturbations of deterministic integrable Hamiltonian systems in $\mathbb{R}^{2n}$, linear and non-linear. I will show that, firstly, on time intervals of order $1/\varepsilon$ the actions of solutions for perturbed equations are close to those of solutions for specially constructed effective stochastic equations. Secondly, if the effective equation is mixing, then the approximation of the actions of solutions for perturbed equations, provided by it, is uniform in time. The mixing assumption admits an easy sufficient condition, as well as a more complicated sufficient condition, close to necessary, given in terms of control theory.

Looking for geodesic loops

Enrico Le Donne (University of Fribourg)

In several geometric and algebraic contexts one wonders whether some specific metric space admits geometric copies of the standard circle equipped with its arc distance. We shall start reviewing why in finite-dimensional normed spaces there are none, while there are in larger Banach spaces. Next we discuss the situation in Carnot groups with subRiemannian or subFinsler distances. While the general conjectures are still unsolved, mostly because of the presence of abnormal curves, we shall prove a strong quantitative result for normal curves, which implies that every non-constant normal curve into a subFinsler Carnot group escapes from every compact set. Joint work with Nicola Paddeu.

Optimal transport between algebraic hypersurfaces

Antonio Lerario (SISSA)

Optimal transport is the general problem of moving one distribution of mass to another one as efficiently as possible, typically keeping track of the ambient geometry. In this seminar I will present recent results on the optimal transport problem between algebraic hypersurfaces of the same degree in complex projective space – integration on an algebraic hypersurface defines a measure on projective space, and all these measures have the same mass if the degree of the hypersurface is fixed. I will discuss how this problem is equivalent to a Riemannian geodesic problem away from the discriminant and connect to the theory of regularity of roots of families of polynomials and to the condition number of polynomial system solving. This is joint work with P. Antonini and F. Cavalletti

On L_p-Hölder continuity of sub-Riemannian optimal control

Lev Lokutsievskiy (Steklov Mathematical Institute, Russian Academy of Sciences)

The talk is devoted to the long-standing open problem on the smoothness of abnormal sub-Riemannian geodesics. I will show that abnormal sub-Riemannian geodesics have an L_p-Hölder continuous derivative and try to explain the main reason for that. This result has a number of interesting implications concerning (i) the Fourier coefficients decay on optimal controls, (ii) the rate of control approximation by smooth functions, (iii) the corresponding generalization of the Poincare inequality, and (iv) the compact embedding of the set of shortest paths into the space of Bessel potentials. These results are obtained in the joint work with Mikhail I. Zelikin.

On the role of embeddability in conformal sub-Riemannian geometry

Andrea Malchiodi (Scuola Normale Superiore)

We consider the CR-Yamabe problem, consisting in prescribing the Tanaka-Webster curvature under conformal changes of contact form. We will see the role of embeddability in the three-dimensional case, concerning in particular a positive mass theorem, extremality of Sobolev quotients and the second variation of the Einstein-Hilbert action in this context. This is joint work with C. Afeltra, J.H.Cheng and P.Yang.

Unified synthetic Ricci curvature lower bounds for Riemannian and sub-Riemannian structures

Andrea Mondino (University of Oxford)

Recent advances in the theory of metric measures spaces on the one hand, and of sub-Riemannian ones on the other hand, suggest the possibility of a "great unification" of Riemannian and sub-Riemannian geometries in a comprehensive framework of synthetic Ricci curvature lower bounds, as put forth by Villani in his 2017 Bourbaki Seminar. With the aim of achieving such a unification program, in joint work with Barilari and Rizzi, we initiate the study of gauge metric measure spaces. The seminar aims to introduce the problem, the abstract setting of gauge metric measure spaces and synthetic Ricci curvature lower bounds thereof and, time permitting, an overview of the main results: comparison theorems, stability under convergence, classes of examples entering the framework.

Higher order Goh conditions

Roberto Monti (Università degli Studi di Padova)

Starting from an example, we deduce necessary conditions of order $n \in \mathbb{N}$ for strictly singular length minimizing curves in sub-Riemannian manifolds. The key tools are new open mapping theorems and the analysis of the end-point map at any order. This is a joint work with F. Boarotto and A. Socionovo.

Lorentzian geometry in the Lobachevsky plane

Yuri Sachkov (Program Systems Institute of the Russian Academy of Sciences)

For these problems, the attainability set is described, the existence of optimal trajectories is studied, a parameterization of extremal trajectories is obtained, their optimality is proved, Lorentzian distance and spheres are described.

Exponential stability of the Burgers flow

Armen Shirikyan (CY Cergy Paris Université)

We shall give an overview of some results on controllability of the 1D Burgers equation on a bounded interval and discuss the global exponential stability of its flow. The latter is true for both periodic and Dirichlet boundary conditions, even though the proofs are rather different. We shall outline those proofs and present a new idea due to A. Djurdjevac and T. Rosati that enables one to treat the case of Dirichlet's condition. Various applications of these results will also be discussed.

TBA

Gianna Stefani (Università degli Studi di Firenze)

TBA

Morse inequalities for eigenvalue branches of generic families of self-adjoint matrices

Igor Zelenko (Texas A&M University)

The topic of the talk was originally motivated by the Floquet-Bloch theory of Schrödinger equations with periodic potential and other problems in Mathematical Physics but in course of working on it, we realized that it is strongly related and uses Andrei Agrachev's results on topology of vector-vector valued maps motivated by second-order conditions in optimal control. The eigenvalue branches of families of self-adjoint matrices are not smooth at points corresponding to repeated eigenvalues (called diabolic points or Dirac points). Generalizing the notion of critical points as points for which the homotopical type of (local) sub-level set changes after the passage through the corresponding value, in the case of the generic family we give an effective criterion for a diabolic point to be critical for those branches and compute the contribution of each such critical point to the Morse polynomial of each branch, getting the appropriate Morse inequalities as a byproduct of the theory. These contributions are expressed in terms of the homologies of Grassmannians. The talk is based on the joint work with Gregory Berkolaiko.