Categorified DT invariants II: Perverse sheaves of vanishing cycles

Workshop on Donaldson–Thomas invariants, talk 7

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We have seen in Tristan's talk that the local critical chart of a critical virtual manifold can be used to define locally a semi-perfect obstruction theory, and its virtual fundamental class, giving a DT invariant. Continuing the story from yesterday in the context of critical virtual manifolds, we will focus here on the other (equivalent) way of defining DT-type invariants, through the Behrend function. It turns out that the way to perform it on critical virtual manifolds leads to a categorification of the Behrend function and its invariants.

We mainly follow [KL16, Chapter 2], as well as [Dim04] and[Max20] for further explanations on vanishing cycles.

Conventions: We use the notational convention that "everything is derived", that is, the inverse and direct image functors (including global sections) are implicitly derived, and $\mathbb{Q}_V\text{-}\mathfrak{Mod}^{\mathrm{b}}_{\mathrm{c}}$ denotes the (bounded) derived ∞-category[1] of (constructible) $\mathbb{Q}_V$-modules, $\mathbb{Q}_V$ being the constant sheaf with value $\mathbb{Q}$ on V.

1 Complexes of vanishing cycles

In this section, we only work in one local critical chart of a critical virtual manifold, that is in a Landau–Ginzburg pair (recall that this includes the hypothesis that the "potential" only has 0 as critical value). We are interested in defining and studying its sheaves of nearby and vanishing cycles, which contain information about the (infinitesimal) neighbourhood of the critical locus, and so for simplicity we will consider that the potential is only defined over such a neighbourhood[2].

Let $D \subset \mathbb{A}^1_{\mathbb{C}}$ denote a (possibly very small, so not algebraic) disk centred around 0.

[1] Working with the derived ∞-category rather than just its homotopy category will allow us to obtain for free the functoriality of vanising cycles sheaves, since we define them as a fibre in a stable ∞-category rather than a cone in a triangulated category.

[2] The point of this is simply to not introduce additional notation every time we have to restrict to the small neighbourhood for local arguments.

Let V be a complex manifold, and $f\colon V \to D$ be a holomorphic function which only has the critical value 0, so that we refer to the pair (V, f) as a Landau–Ginzburg pair.

1.1 Construction

Construction 1.1.1 (Nearby cycles functor). We let $D^\times = D \setminus \{0\}$, and $\varpi\colon \widetilde{D^\times} \to D^\times$ be its universal cover. Consider the fibre product

$$\begin{tikzcd}
& \widetilde{V^\times} \arrow[r] \arrow[d, "\varpi_f"] \arrow[dd, bend right, "\overline{\varpi_f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{D^\times} \arrow[d, "\varpi"] \\
& V^\times \arrow[r, "f|_{V^\times}"'] \arrow[d, hook, "\jmath"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^\times \arrow[d, hook] \\
V_0 := f^{-1}(0) \arrow[r, hook, "\iota"] & V \arrow[r, "f"'] & D
\end{tikzcd}$$

The **nearby cycles functor** is the functor

$$\Psi_f := \iota^* \overline{\varpi_f}_* \overline{\varpi_f}^* \colon \mathbb{Q}_V\text{-}\mathfrak{Mod}_c^b \to \mathbb{Q}_{V_0}\text{-}\mathfrak{Mod}_c^b.$$

Proposition 1.1.2. *At any $x \in V_0$, for any $\mathscr{M} \in \mathbb{Q}_V\text{-}\mathfrak{Mod}_c^b$ and any $k \in \mathbb{Z}$, there is an ismorphism of $\mathbb{Q}$-modules*

$$\mathscr{H}^k(\Psi_f \mathscr{M})_x \simeq \mathbb{R}^k\Gamma(\mathrm{MF}_{x,f}, \mathscr{M}),$$

where $\mathrm{MF}_{x,f} = V^\times \cap B_\varepsilon(x)$ is the Milnor fibre of f at x, where $B_\varepsilon(x)$ is an open ball[3] *centered at x of radius $\varepsilon \ll 1$ such that the radius of $D^\times$ is $\ll \varepsilon$.*

Globally, when f is proper, the nearby cycles functor contains information about specialising from the generic fibre V_s $(s \in D^\times)$ to the special fibre V_0. More precisely:

Proposition 1.1.3 ([GMP95, Part II, Section 6.13])**.** *There exist a retraction $\mathrm{sp}\colon V \to V_0$ and an equivalence*

$$\Psi_f \mathscr{M} \simeq \mathrm{sp}_*(\mathscr{M}|_{V_s}),$$

inducing

$$\mathbb{R}^k\Gamma(V_0; \Psi_f \mathscr{M}) \simeq \mathbb{R}^k\Gamma(V_s; \mathscr{M}|_{V_s}).$$

Remark 1.1.4. The morphism sp is difficult to construct (the construction is explained in [MP91, §5.8.1]), but its cohomological pullback admits a much easier description: suppose D is small enough so that $V \sim V_0$, and write $i_s\colon V_s \hookrightarrow V \sim V_0$. Then

$$\mathrm{sp}_{\mathbb{Q}}^*\colon H^\bullet(V_0; \mathbb{Q}) \simeq H^\bullet(V; \mathbb{Q}) \xrightarrow{i_s^*} H^\bullet(V_s; \mathbb{Q}).$$

Note further that the natural transformation

$$\mathrm{sp}^* := \iota^*\eta\colon \iota^* \Rightarrow \iota^* \overline{\varpi_f}_* \overline{\varpi_f}^*$$

induced by the unit η of the adjunction $\overline{\varpi_f}^* \dashv \overline{\varpi_f}_*$ induces $\mathrm{sp}_{\mathbb{Q}}^*$ after applying it to $\mathbb{Q}_V$ and taking the cohomology of global sections.

[3] If V is singular, such a ball is defined by using an embedding of the germ (V, x) in a complex affine space.

Definition 1.1.5 (Vanishing cycles)**.** *The **vanishing cycles functor** Φ_f is the cofibre of the natural transformation*

$$\mathrm{sp}^*\colon \iota^* \Rightarrow \iota^*\overline{\varpi_{f*}}\,\overline{\varpi_f}^*.$$

It follows that

$$\mathscr{H}^k(\Phi_f\mathscr{M})_x \simeq \mathbb{R}^{k+1}\Gamma(B_\varepsilon(x), \mathrm{MF}_{x,f}; \mathscr{M}).$$

In particular, with $\mathscr{M} = \mathbb{Q}_V$, we get

$$\mathscr{H}^k(\Phi_f\mathbb{Q}_V)_x \simeq \widetilde{H}^k(\mathrm{MF}_{x,f}; \mathbb{Q})$$

since $B_\varepsilon(x) \cap V_0$ is contractible.

As a consequence, we see that its support is contained in $\mathrm{Crit}(f)$.

Remark 1.1.6 (Alternate construction). Let $V_{\geq 0} = \{x \in V \mid \Re[f(x)] \geq 0\}$ (it is a closed real semi-analytic subset), and $\jmath_<\colon V_{<0} \coloneqq V \setminus V_{\geq 0} \hookrightarrow V$ the inclusion. We define Γ_Z as the fibre

$$\Gamma_Z \to \mathrm{id} \to \jmath_{<,*}\jmath_<^*,$$

so that it is the right derived functor of

$$\mathbb{R}^0\Gamma_Z\mathscr{M}\colon U \mapsto \ker\big(\mathbb{R}^0\Gamma(U, \mathscr{M}) \to \mathbb{R}^0\Gamma(U \cap (V_{<0}), \mathscr{M})\big).$$

Then there is a natural equivalence $\Phi_f[-1] \simeq \iota^{-1}\Gamma_Z$.

We shall introduce the notations

$${}^p\Psi_f = \Psi_f[-1] \qquad \text{and} \qquad {}^p\Phi_f = \Phi_f[-1],$$

called the perverse nearby and vanishing cycles. We also set $\mathscr{P}_f \coloneqq {}^p\Phi_f\mathbb{Q}_V[\dim V]$.

Corollary 1.1.7. *At any $x \in \mathrm{Crit}(f)$, we have*

$$\chi(\mathscr{P}_f)_x \coloneqq \sum_n (-1)^m \dim \mathscr{H}^m(\mathscr{P}_f)_x = (-1)^{\dim V}(1 - \chi(\mathrm{MF}_{x,f})) = \nu_{\mathrm{Crit}(f)}(x),$$

where $\nu_{\mathrm{Crit}(f)}$ is the Behrend function of $\mathrm{Crit}(f)$.

A more involved computation shows that, globally,

$$\chi\big(\Gamma_c(\mathrm{Crit}(f), \mathscr{P}_f)\big) = \chi(\mathrm{Crit}(f), \nu_{\mathrm{Crit}(f)}).$$

The upshot of this is that the perverse sheaf $\mathscr{P}_f$ categorifies the Behrend function on $\mathrm{Crit}(f)$ along with its DT-type invariant, and thus the DT invariant when $\mathrm{Crit}(f)$ is compact.

1.2 First properties of vanishing cycles as a perverse sheaf

Proposition 1.2.1. *The functors ${}^p\Psi_f$ and ${}^p\Phi_f$ both commute with Verdier duality (up to natural equivalences), that is $\mathbb{D}({}^p\Psi_f) \simeq {}^p\Psi_f(\mathbb{D}-)$ and $\mathbb{D}({}^p\Phi_f) \simeq {}^p\Phi_f(\mathbb{D}-)$.*

In particular, $\mathscr{P}_f$ is Verdier self-dual.

Lemma 1.2.2. *Let $\pi\colon W \to V$ be a proper analytic morphism, and set $g = f \circ \pi$. Let $\pi_0\colon W_0 \to V_0$ denote the restriction. There are equivalences*

$$\pi_{0,*}{}^p\Psi_g \simeq {}^p\Psi_f\pi_{0,*} \text{ and } \pi_{0,*}{}^p\Phi_g \simeq {}^p\Phi_f\pi_{0,*}.$$

By adjunction, we also get

$${}^p\Phi_g\pi_0^* \simeq \pi_0^{*p}\Phi_f.$$

Proof. By diagram chasing. □

When π is a homeomorphism (so that in particular $\dim Y = \dim V$), plugging in $\mathbb{Q}_V[\dim V]$, this gives $\Sigma\colon \mathscr{P}_g \simeq \pi_0^*\mathscr{P}_f$.

Corollary 1.2.3. *Let X be a critical virtual manifold, with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. For any α, let $\mathscr{P}_\alpha \coloneqq \varphi_\alpha^*\mathscr{P}_{f_\alpha}$. For each pair (α, β), there is an equivalence*

$$\sigma_{\alpha\beta}\colon \mathscr{P}_\alpha|_{V_{\alpha\beta}} \xrightarrow{\simeq} \mathscr{P}_\beta|_{V_{\alpha\beta}}$$

of $\mathbb{Q}_{V_{\alpha\beta}}$-modules.

Proof. The biholomorphic change-of-chart maps $\varphi_{\alpha\beta}\colon V_{\alpha\beta} \to V_{\beta\alpha}$ are compatible with the potentials, so we get an isomorphism $\Sigma\colon \mathscr{P}_{f_\alpha} \xrightarrow{\simeq} \varphi_{\alpha\beta}^*\mathscr{P}_{f_\beta}$. Applying φ_α^* and using $\varphi_\beta = \varphi_{\alpha\beta} \circ \varphi_\alpha$, we get

$$\mathscr{P}_\alpha \coloneqq \varphi_\alpha^* \xrightarrow{\simeq} \varphi_\alpha^*\varphi_{\alpha\beta}^*\mathscr{P}_{f_\beta} \simeq \varphi_\beta^*\mathscr{P}_{f_\beta} \eqqcolon \mathscr{P}_\beta.$$

□

2 Gluing the perverse sheaves of vanishing cycles

2.1 Perverse sheaves and their locality properties

Definition 2.1.1 (Perverse sheaf)**.** *An object $\mathscr{M} \in \mathbb{Q}_V\text{-}\mathfrak{Mod}_c^b$ is a (**middle-**)**perverse sheaf** of $\mathbb{Q}_X$-modules if*

(***support condition***) $\dim \operatorname{supp} \mathscr{H}^i\mathscr{M} \leq -i$

(***cosupport condition***) $\dim \operatorname{supp} \mathscr{H}^i(\mathbb{D}\mathscr{M}) \leq i$.

Fact 2.1.2. The two conditions defining perverse sheaves determine a t-structure, of which the category of perverse sheaves is then the heart.

Corollary 2.1.3. *The category* $\mathfrak{Perv}(X)$ *is abelian.*

Proposition 2.1.4. *On a LG pair* (V, f)*, the functors* ${}^p\Psi_f \coloneqq \Psi_f[-1], {}^p\Phi_f \coloneqq \Phi_f[-1] \colon \mathbb{Q}_V\text{-}\mathfrak{Mod}_c^b \to \mathbb{Q}_{V_0}\text{-}\mathfrak{Mod}_c^b$ *are perverse t-exact, so induce functors* $\mathfrak{Perv}(V) \to \mathfrak{Perv}(V_0)$.

Sketch of proof. We will only focus on the case of ${}^p\Psi_f$. By Verdier duality, it is enough to show right t-exactness. For simplicity, we will assume that $f \colon V \to D$ is algebraic.

In the definition of ${}^p\Psi_f$, the generator of $\pi_1(D^\times) \simeq \mathbb{Z}$ acts on $\widetilde{V^\times}$ through deck transformations on $\widetilde{D^\times}$, inducing a monodromy operator h on ${}^p\Psi_f$. Since we have assumed algebraicity over a curve, this monodromy is quasi-unipotent, meaning that some power of h is unipotent (equivalently, its eigenvalues are roots of unity), and can further be assumed unipotent by taking a ramified cover of D.

Now consider the fibre sequence

$$ {}^p\Psi_f \xrightarrow{h-1} {}^p\Psi_f \to \imath^* \jmath_* \jmath^* $$

where $\jmath_*$ and $\jmath^*$ are t-exact and $\imath^*$ is right t-exact. So, for any $\mathscr{P} \in \mathfrak{Perv}(V)$, taking perverse cohomology produces the long exact sequence

$$ {}^p\mathscr{H}^k({}^p\Psi_f\mathscr{P}) \xrightarrow{h-1} {}^p\mathscr{H}^k({}^p\Psi_f\mathscr{P}) \to {}^p\mathscr{H}^k(\imath^*\jmath_*\jmath^*), $$

with the last group vanishing for any $k > 0$ by right t-exactness. This means that the nilpotent endomorphism $h-1$ is also surjective, and thus ${}^p\mathscr{H}^k({}^p\Psi_f\mathscr{P}) = 0$ for any $k > 0$, showing that ${}^p\Psi_f\mathscr{P}$ is in the connective (*i.e.* ≤ 0) part of the t-structure. $\square$

Theorem 2.1.5 ([BBD82, Corollaire 2.1.23])**.** *The assignment* $U \mapsto \mathfrak{Perv}(U)$ *defines* (*the objects part of*) *a stack on* X*. That is:*

Prestack condition: *For* $\mathscr{P}, \mathscr{Q} \in \mathfrak{Perv}(X)$ *suppose given morphisms* $\sigma_\alpha \colon \mathscr{P}|_{X_\alpha} \to \mathscr{Q}|_{X_\alpha}$ *along a covering* $(X_\alpha)_{\alpha\in A}$*, such that* $\sigma_\alpha|_{X_{\alpha\beta}} = \sigma_\beta|_{X_{\alpha\beta}}$*. Then there is a unique morphism* $\sigma \colon \mathscr{P} \to \mathscr{Q}$ *such that* $\sigma_\alpha = \sigma|_{X_\alpha}$ *for all* $\alpha \in A$.

Effective descent: *Suppose, for a covering* $(X_\alpha)_{\alpha\in A}$*, we have*

- $\forall \alpha \in A$*, a perverse sheaf* $\mathscr{P}_\alpha \in \mathfrak{Perv}(X_\alpha)$*;*
- $\forall (\alpha, \beta) \in A^2$*, an isomorphism* $\sigma_{\alpha\beta} \colon \mathscr{P}_\alpha|_{X_{\alpha\beta}} \xrightarrow{\simeq} \mathscr{P}_\beta|_{X_{\alpha\beta}}$*;*
- $\forall (\alpha, \beta, \gamma) \in A^3$*, a cocycle equality* $\sigma_{\gamma\alpha}\sigma_{\beta\gamma}\sigma_{\alpha\beta} = \mathrm{id}_{\mathscr{P}_\alpha|_{X_{\alpha\beta\gamma}}}$.

Then there is a perverse sheaf $\mathscr{P} \in \mathfrak{Perv}(\mathfrak{X})$ *and isomorphisms* $\sigma_\alpha \colon \mathscr{P}|_{X_\alpha} \xrightarrow{\simeq} \mathscr{P}_\alpha$ *such that* $\sigma_{\alpha\beta} = \sigma_\beta\sigma_\alpha^{-1}$.

Remark 2.1.6. The gluing properties of perverse sheaves (and their morphisms) are directly related to the cohomological vanishing properties defining them. Indeed the prestack property is true for the heart of any (locally defined) t-structure, as it comes from the following more general statement ([BBD82, Proposition 3.2.2]): given $\mathscr{K}$ bounded below and $\mathscr{L}$ bounded above such that $\mathscr{E}xt^i(\mathscr{K}, \mathscr{L}) = 0$ for $i < 0$, then the presheaf $U \mapsto \mathbb{Q}_U\text{-}\mathfrak{Mod}(\mathscr{K}|_U, \mathscr{L}|_U)$ is a sheaf.

Meanwhile, effective descent comes from the following gluing property ([BBD82, Théorème 3.2.4]). Let $(\mathscr{K}_U)$ be a family of locally defined objects ($\mathbb{Q}_U$-modules) indexed by a cover, and assume there exists some interval $I \subset \mathbb{Z}$ (independent of U) such that for all U, $\mathscr{H}^i \mathscr{K}_U = 0$ whenever $i \notin I$. Assume further that for any U, $\mathscr{E}xt^i(\mathscr{K}_U, \mathscr{K}_U) = 0$ whenever $i < 0$. Then there is a (necessarily unique) $\mathbb{Q}_X$-module $\mathscr{K}$ such that each $\mathscr{K}_U$ is obtained from $\mathscr{K}$.

2.2 Geometric gluing for virtual critical manifolds

Let X be a critical virtual manifold, with charts $(X_\alpha \xrightarrow{\varphi_\alpha} V_\alpha \xrightarrow{f_\alpha} \mathbb{C})$. Recall that the perverse sheaves $\mathscr{P}_\alpha$ come naturally equipped with isomorphisms $\sigma_{\alpha\beta}$.

Definition 2.2.1 (Isomorphism of geometric origin)**.** *Let* (X_1, f_1) *and* (X_2, f_2) *be two LG pairs, and let* $\zeta\colon \mathrm{Crit}(f_1) \xrightarrow{\simeq} \mathrm{Crit}(f_2)$ *be an isomorphism of analytic spaces. An isomorphism of perverse sheaves* $\sigma\colon \mathscr{P}_{f_1} \to \zeta^* \mathscr{P}_{f_2}$ *is **of geometric origin** if there is an open neighbourhood* $\mathrm{Crit}(f_1) \subset U_1 \subset X_1$ *and a holomorphic map* $\varphi\colon U_1 \to X_2$ *biholomorphic onto its image such that* $\varphi|_{\mathrm{Crit}(f_1)} = \zeta$ *and* $f_2 \circ \varphi = f_1|_{U_1}$*, and* σ *is the isomorphism* $\Sigma\colon \mathscr{P}_{f_1} \xrightarrow{\simeq} \varphi^* \mathscr{P}_{f_2}$*.*

*A **geometric gluing** is a gluing* P *such that the* $\sigma_\beta \circ \sigma_\alpha^{-1}$ *are of geometric origin.*

Corollary 2.2.2. *There exists a geometric gluing of the* $\mathscr{P}_\alpha$ *if one can find (possibly after refining the covering* (X_α)*) a* $\mathbb{Z}/(2)$*-valued* 1*-cochain* $\{\mu_{\alpha\beta}\}$ *such that, writing* $\overline{\sigma}_{\alpha\beta} = \mu_{\alpha\beta} \cdot \sigma_{\alpha\beta}$*, we have*

$$\overline{\sigma}_{\alpha\beta\gamma} \coloneqq \overline{\sigma}_{\gamma\alpha} \circ \overline{\sigma}_{\beta\gamma} \circ \overline{\sigma}_{\alpha\beta} = 1 \text{ over } X_{\alpha\beta\gamma}.$$

Proposition 2.2.3. *Let* $\mathscr{P}$ *and* $\mathscr{P}'$ *be two geometric gluings of the local vanishing cycles. There is a* $\mathbb{Z}/(2)$*-local system* $\rho \in H^1(X, \mathbb{Z}/(2))$*, with an isomorphism* $\mathscr{P}' \simeq \mathscr{P} \otimes \rho$*.*

In other words, geometric gluings are unique up to twist with a $\mathbb{Z}/(2)$-local system.

We now turn to existence. Note that the corollary implies that the 2-cocycle $\{\sigma_{\alpha\beta\gamma}\}$ be locally constant with values $\mu_{\alpha\beta\gamma} \coloneqq \mu_{\gamma\alpha}\mu_{\beta\gamma}\mu_{\alpha\beta}$.

Proposition 2.2.4. *Let* $\xi = \{\xi_{\alpha\beta\gamma}\}$ *be the* $\mathbb{Z}/(2)$*-valued* 2*-cocycle obstructing the gluing of the anticanonical line bundles* $\mathscr{K}_\alpha^\vee = \varphi_\alpha^* \det T_{V_\alpha}|_{X_\alpha^{\mathrm{red}}}$*. Then* ξ *coincides with the gluing* 2*-cocycle* $\{\sigma_{\alpha\beta\gamma}\}$*.*

These results come from the following central lemma, also proven through different methods by [BBDJS15]:

Lemma 2.2.5. *Let* (V, f) *be a LG pair. Let* U *be an open subset such that* $\mathrm{Crit}(f) \subset U \subset V$ *and let* $\varphi\colon U \to V$ *be a holomorphic map, biholomorphic onto its image, such that* $f \circ \varphi = f|_U$ *and* $\varphi|_{\mathrm{Crit}(f)} = \mathrm{id}_{\mathrm{Crit}(f)}$*. Then the isomorphism of compatibility* Σ *is equal to* $\det(d\varphi|_{\mathrm{Crit}(f)}) \cdot \mathrm{id}\colon \mathscr{P}_f \to \mathscr{P}_f$*, where* $\det(d\varphi|_{\mathrm{Crit}(f)})$ *is locally constant with values in* $\mathbb{Z}/(2)$*.*

Putting the results together, we obtain the main result.

Theorem 2.2.6. *Suppose* X *is orientable. Then there exists a geometric gluing of the local vanishing cycles sheaves, unique up to twisting by a* $\mathbb{Z}/(2)$*-local system.*

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