Distance, strong convexity, flagness, and associahedra

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joint work with
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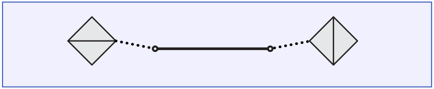
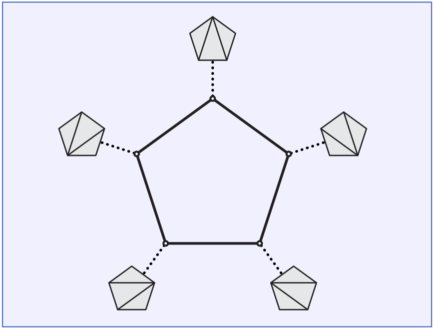
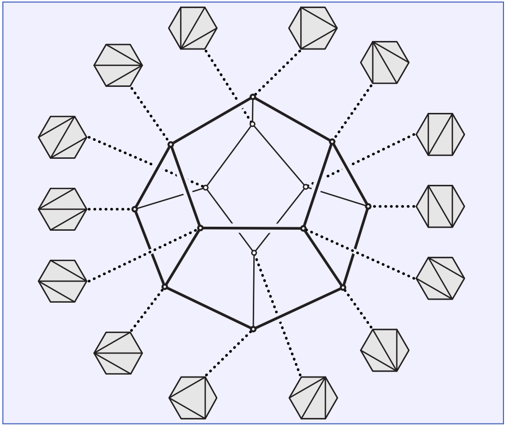






A bit of history

Associahedra were discovered

→ by Dov Tamari (1951),

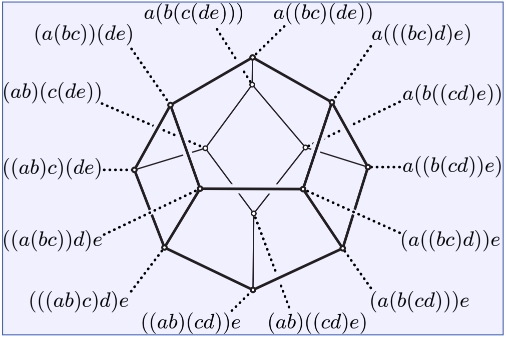
→ by Jim Stasheff (1963),

... constructed as polytopes

→ by John Milnor,

→ by Mark Haiman (1984),

→ by Carl Lee (1989),



... and generalized (or related polytopes discovered/constructed)

→ by Mikhail Kapranov (permutoassociahedron, 1993),

→ by Raoul Bott and Clifford Taubes (cyclohedron, 1994),

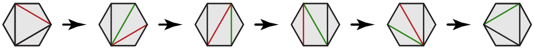
→ by Sergey Fomin and Andrei Zelevinsky (2001),

→ by Alexander Postnikov (2009),

→ and *many* more!

Metric properties

Remark. The graph of the associahedron provides a geometry (in terms of arc flips) to the set of the triangulations of a convex polygon.



Question (open). Given two triangulations of a convex n-gon, what is the distance between them in the graph of the associahedron?

Question (solved). What is the largest possible distance between any two triangulations in the graph of the associahedron? In other words, what is the diameter (of the graph) of the associahedron?

Theorem (Sleator–Tarjan–Thurston 1988). The $(n-3)$-dimensional associahedron has diameter $2n-10$ *when n is large enough*.

Theorem (P. 2014). *Large enough* means $n \geq 13$.

Metric properties: why? (1)

Distances in the graph of the associahedron are also rotation distances between binary trees. Rotations are used in computer science to re-balance binary trees in order to improve data storage efficiency.

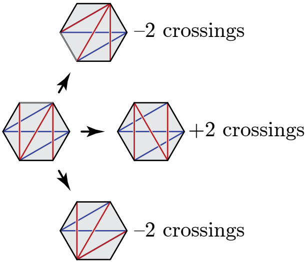
Distances in the graph of the associahedron are used to measure dissimilarity between two binary (phylogenetic) trees in computational biology. Here exact distances need to be computed (but how?)

Algorithm to estimate $d(T_1, T_2)$

(1) Flip an arc in T_1 such that the number of arc crossings with T_2 decreases,

(2) Repeat until T_2 is reached.

Theorem (Hanke–Ottman–Schuierer 1997). One can always flip some arc in T_1 such that the number of arc crossings with T_2 decreases after the flip.



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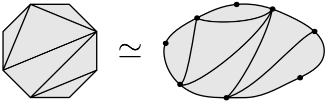
Unfortunately...

Theorem (Cleary–Maio 2018). The distance estimation computed from this arc crossings-based method is sometimes *one off* $d(T_1, T_2)$.

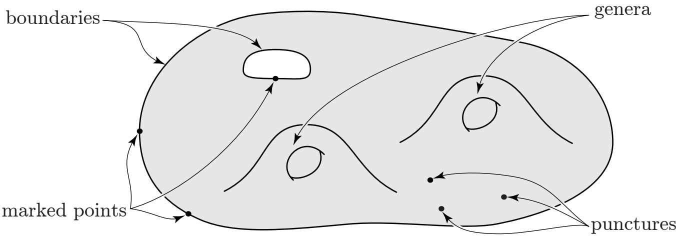
Can this get worse? Can the estimation be larger than $\alpha d(T_1, T_2)$ where $\alpha > 1$?

Metric properties: why? (2)

Remark. The (geometric) case of a convex n-gon is *the same* as a topological disk with n marked points in its boundary.



Question. Instead of a disk, can we pick a topological surface, possibly with punctures and at least one marked point in each boundary component?



Lemma (Švarc 1955, Milnor 1968). The graph whose vertices are the triangulations of the surface Σ and whose edges correspond to flipping arcs is quasi-isometric to any Cayley graph of the mapping class group of Σ.

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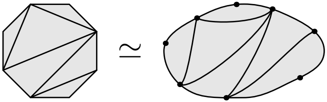

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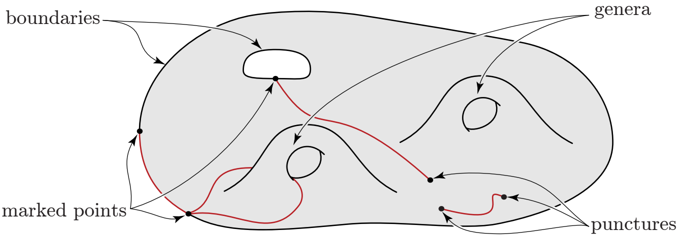

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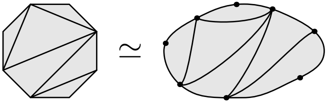

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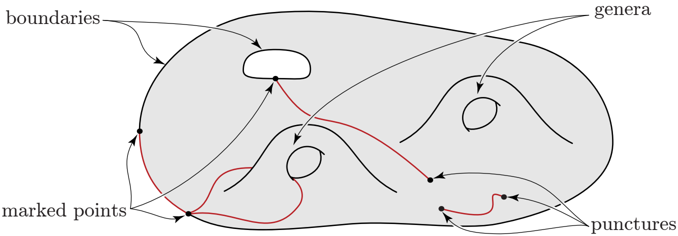
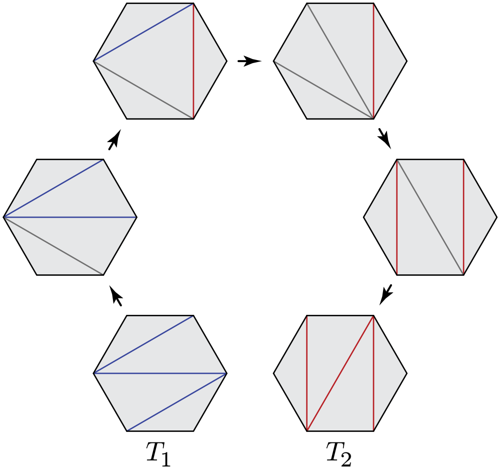


Theorem (Disarlo–Parlier 2019). Given two triangulations T_1 and T_2 of the surface Σ, one can always flip some arc in T_1 such that the number of arc crossings with T_2 decreases after the flip.

Metric properties: why? (3)

Remark. A path between triangulations T_1 and T_2 in the graph of the associahedron can be thought of as a certain type of 3-dimensional triangulation.



Remark. A *blow-up triangulation* is not a usual triangulation:

(1) Multiple arcs are allowed,

(2) Two tetrahedra can be glued along the union of two triangles.

Remark. $d(T_1, T_2)$ is also the number of tetrahedra required to fill a triangulated sphere S.

...and an upper bound on the L^1 norm of the 3-chains whose boundary is a 2-cycle corresponding to the triangulation of S.

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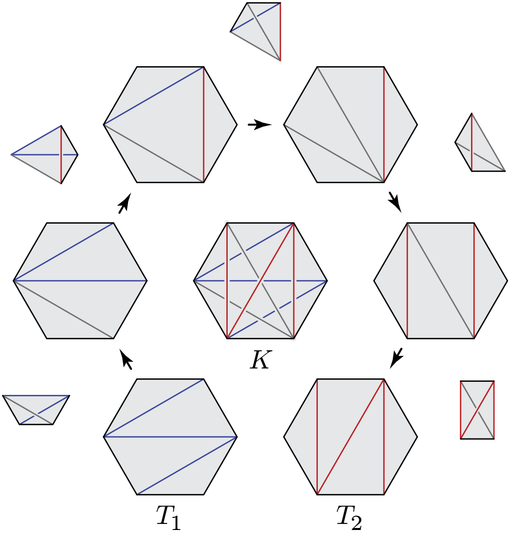

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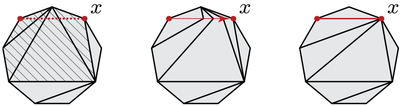
Strong convexity

Lemma (Sleator–Tarjan–Thurston 1988). The triangulations that contain a given arc ε induce a strongly convex subgraph in the graph of the associahedron: ε is not removed along any geodesic between two such triangulations.

Proof. Consider a path between two triangulations that contain ε and project each triangulation T in that path as follows.



Consecutive triangulations in the path are projected to either

(1) two triangulations related by a flip or

(2) the same triangulation.

A flip that removes ε is of the second kind. If there is such a flip in the considered path, the projected path is shorter. □

Theorem (Disarlo–Parlier 2019). The same holds for topological surfaces.

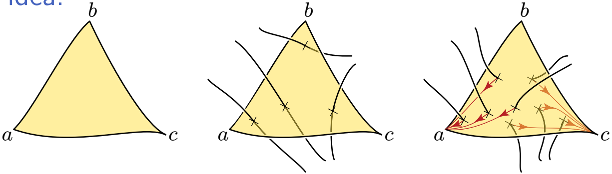
Geodesicity ⇒ flagness

Theorem (P.-Wang 2021). The blow-up triangulations that correspond to a geodesic path in the graph of the associahedron are *flag*.

A blow-up triangulation K is flag when three arcs in K that form a cycle always bound a triangle of K.

In other words: if the three edges of a triangle *abc* appear in possibly distinct triangulations along a geodesic path in the graph of the associahedron, then *abc* itself appears along that path.

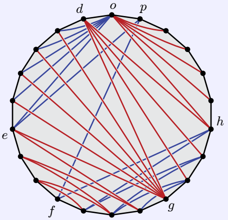
Proof *rough* idea.

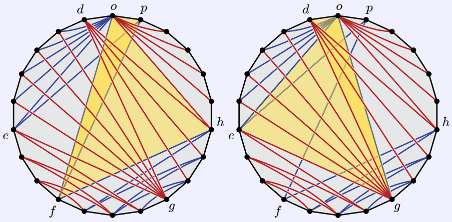


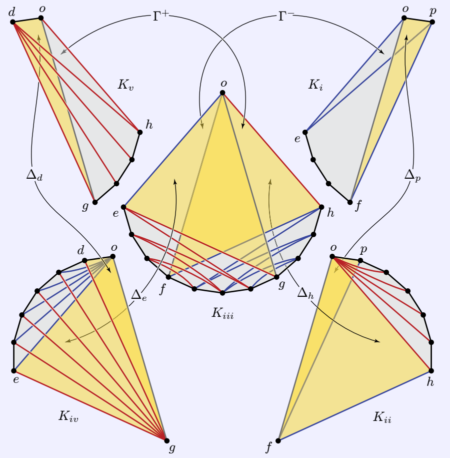
Theorem (P.-Wang 2021). Blow-up triangulations corresponding to geodesic paths remain flag when the initial convex polygon (or topological disk) is replaced by a convex polygon with well-placed flat vertices.

Some consequences (1)

Theorem (P.-Wang 2021). The arc crossings based distance estimate method sometimes overestimates distances by a factor that can get arbitrarily close to 3/2 both in the cases of associahedra and topological surfaces.







Some consequences (2)

Theorem (P.-Wang 2021). The subgraph induced by the triangulations that contain a given arc is not always strongly convex in the limit case of a geometric convex polygon with as few as two flat vertices or punctures.