A tropical version of Hilbert polynomial

Dima Grigoriev

CNRS

02/12/2021, Bures-sur-Yvette

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.

If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;

• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;

• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;

$x = (x_1, \dots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus, \otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;
$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;
$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;
$x = (x_1, \dots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;
$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j \{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus, \otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;
$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus, \otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \leq j \leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;
$x = (x_1, \dots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus, \otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring:
$(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j \{Q_j\}$;
$x = (x_1, \dots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.
If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.
If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.
Examples • $\mathbb{Z}^+ := \{0 \leq a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;
• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;
• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1\leq j\leq n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.
Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;
$x = (x_1, \ldots, x_n)$ is a **tropical zero** of f if minimum $\min_j\{Q_j\}$ is attained for at least two different values of j.

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\dots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n}(t_{j,i} + s_i)X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \dots, k_n) : 0 \le k_1, \dots, k_n < N\}$. A **linearization** of $f_{s_1,\dots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \dots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \dots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \dots, x_n) \in V(f)$ then point

$$\{u(k_1, \dots, k_n) = k_1x_1 + \dots + k_nx_n : 0 \le k_1, \dots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\dots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\dots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \dots, k_n) : 0 \le k_1, \dots, k_n < N\}$. A **linearization** of $f_{s_1,\dots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \dots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \dots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \dots, x_n) \in V(f)$ then point

$$\{u(k_1, \dots, k_n) = k_1 x_1 + \dots + k_n x_n : 0 \le k_1, \dots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\dots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \ldots, f_k$ in n variables a **tropical prevariety** $V(f_1, \ldots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \ldots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \ldots, f_k)$ are zeros of the semiring ideal $I(f_1, \ldots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\ldots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \ldots, k_n) \,:\, 0 \le k_1, \ldots, k_n < N\}$. A **linearization** of $f_{s_1,\ldots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \ldots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \ldots, x_n) \in V(f)$ then point

$$\{u(k_1, \ldots, k_n) = k_1 x_1 + \cdots + k_n x_n \,:\, 0 \le k_1, \ldots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\ldots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\dots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \dots, k_n) : 0 \le k_1, \dots, k_n < N\}$. A **linearization** of $f_{s_1,\dots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \dots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \dots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \dots, x_n) \in V(f)$ then point

$$\{u(k_1, \dots, k_n) = k_1 x_1 + \dots + k_n x_n : 0 \le k_1, \dots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\dots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1\le j\le m}\{a_j + \sum_{1\le i\le n} t_{j,i}X_i\}$ its shift

$$f_{s_1,\dots,s_n} := \min_{1\le j\le m}\{a_j + \sum_{1\le i\le n}(t_{j,i}+s_i)X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1,\dots,k_n) : 0 \le k_1,\dots,k_n < N\}$. A **linearization** of $f_{s_1,\dots,s_n}$ is a tropical linear polynomial $\min_{1\le j\le m}\{a_j + u(t_{j,1}+s_1,\dots,t_{j,n}+s_n)\}$, provided that $0 \le t_{j,1}+s_1,\dots,t_{j,n}+s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1,\dots,x_n) \in V(f)$ then point

$$\{u(k_1,\dots,k_n) = k_1x_1 + \dots + k_nx_n : 0 \le k_1,\dots,k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\dots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \ldots, f_k$ in n variables a **tropical prevariety** $V(f_1, \ldots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \ldots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \ldots, f_k)$ are zeros of the semiring ideal $I(f_1, \ldots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\ldots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \ldots, k_n) : 0 \le k_1, \ldots, k_n < N\}$. A **linearization** of $f_{s_1,\ldots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \ldots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \ldots, x_n) \in V(f)$ then point

$$\{u(k_1, \ldots, k_n) = k_1 x_1 + \cdots + k_n x_n : 0 \le k_1, \ldots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\ldots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\dots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \dots, k_n) \, : \, 0 \le k_1, \dots, k_n < N\}$. A **linearization** of $f_{s_1,\dots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \dots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \dots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \dots, x_n) \in V(f)$ then point

$$\{u(k_1, \dots, k_n) = k_1 x_1 + \cdots + k_n x_n \, : \, 0 \le k_1, \dots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\dots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \dots, f_k$ in n variables a **tropical prevariety** $V(f_1, \dots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \dots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \dots, f_k)$ are zeros of the semiring ideal $I(f_1, \dots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1, \dots, s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \dots, k_n) \,:\, 0 \le k_1, \dots, k_n < N\}$. A **linearization** of $f_{s_1, \dots, s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \dots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \dots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \dots, x_n) \in V(f)$ then point

$$\{u(k_1, \dots, k_n) = k_1 x_1 + \cdots + k_n x_n \,:\, 0 \le k_1, \dots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1, \dots, s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \ldots, f_k$ in n variables a **tropical prevariety** $V(f_1, \ldots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \ldots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \ldots, f_k)$ are zeros of the semiring ideal $I(f_1, \ldots, f_k)$.

For a tropical polynomial $f = \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} t_{j,i} X_i\}$ its shift

$$f_{s_1,\ldots,s_n} := \min_{1 \le j \le m}\{a_j + \sum_{1 \le i \le n} (t_{j,i} + s_i) X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \ldots, k_n) \,:\, 0 \le k_1, \ldots, k_n < N\}$. A **linearization** of $f_{s_1,\ldots,s_n}$ is a tropical linear polynomial $\min_{1 \le j \le m}\{a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \ldots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \ldots, x_n) \in V(f)$ then point

$$\{u(k_1, \ldots, k_n) = k_1 x_1 + \cdots + k_n x_n \,:\, 0 \le k_1, \ldots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\ldots,s_n}$

Tropical prevarieties

For tropical polynomials $f_1, \ldots, f_k$ in n variables a **tropical prevariety** $V(f_1, \ldots, f_k) \subset \mathbb{R}^n$ is the set of tropical zeros of $f_1, \ldots, f_k$. A tropical prevariety is a finite union of polyhedra. $V(f_1, \ldots, f_k)$ are zeros of the semiring ideal $I(f_1, \ldots, f_k)$.

For a tropical polynomial $f = \min_{1\le j\le m}\{a_j + \sum_{1\le i\le n} t_{j,i}X_i\}$ its shift

$$f_{s_1,\ldots,s_n} := \min_{1\le j\le m}\{a_j + \sum_{1\le i\le n}(t_{j,i} + s_i)X_i\} \in I(f).$$

Fix an integer N and introduce N^n variables $\{u(k_1, \ldots, k_n) \,:\, 0 \le k_1, \ldots, k_n < N\}$. A **linearization** of $f_{s_1,\ldots,s_n}$ is a tropical linear polynomial $\min_{1\le j\le m}\{a_j + u(t_{j,1} + s_1, \ldots, t_{j,n} + s_n)\}$, provided that $0 \le t_{j,1} + s_1, \ldots, t_{j,n} + s_n < N$.

Linearizations appear as the rows of Macauley matrix of f.

If $(x_1, \ldots, x_n) \in V(f)$ then point

$$\{u(k_1, \ldots, k_n) = k_1x_1 + \cdots + k_nx_n \,:\, 0 \le k_1, \ldots, k_n < N\} \in \mathbb{R}^{N^n}$$

is a tropical zero of any linearization $f_{s_1,\ldots,s_n}$

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \leq H \leq 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \leq H \leq 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Entropy of a tropical polynomial/ideal

Denote by $U_N \subset \mathbb{R}^{N^n}$ the tropical prevariety of all the points being tropical zeroes of all the linearizations $f_{s_1,\dots,s_n}$. A **tropical Hilbert function** of f is $T_N(f) := \dim(U_N)$.

There exists the limit

$$H := H(f) := \lim_{N\to\infty} \dim(U_N)/N^n$$

which we call the **(tropical) entropy** of f. Clearly, $0 \le H \le 1$.

One can literally generalize the entropy $H(I)$ to semiring ideals I.

Relation to Hilbert polynomial

The tropical prevariety U_N plays a role similar to the quotient ring $F[X_1,\dots,X_n]/(g)$ with respect to the filtration degree N in classical algebra.

Therefore, the entropy H plays a role of the coefficient at n-th power Z^n of Hilbert polynomial of a polynomial g. This coefficient always vanishes in classical algebra (the degree of Hilbert polynomial equals $n-1$). It is not the case in the tropical setting: H can be positive.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \ldots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \dots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \ldots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \ldots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \ldots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Tropical entropy of a univariate polynomial

Example

i) For a tropical polynomial $f = \min\{2X, 1+X, 0\}$ its entropy $H(f) = 1/4$;
ii) for a tropical polynomial $f = \min\{sX, (s-1)X, \dots, X, 0\}$ its entropy $H(f) = 1 - 2/(s+1)$.

Sharp bounds on the entropy.

Theorem

i) If $H(f) > 0$ then $H(f) \geq 1/4$;
ii) If f is a tropical univariate polynomial of degree s then $H(f) \leq 1 - 2/(s+1)$.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \leq k \leq s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \geq a_k\}, 0 \leq k \leq s$.

Theorem

The entropy $H(f) = 0$ iff all the points (k, a_k), $a_k < \infty$, $0 \leq k \leq s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \ge a_k\}$, $0 \le k \le s$.

Theorem

The entropy $H(f) = 0$ iff all the points (k, a_k), $a_k < \infty$, $0 \le k \le s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \geq a_k\}$, $0 \leq k \leq s$.

Theorem

The entropy $H(f) = 0$ iff all the points (k, a_k), $a_k < \infty$, $0 \leq k \leq s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \ge a_k\}, 0 \le k \le s$.

Theorem

The entropy $H(f) = 0$ iff all the points (k, a_k), $a_k < \infty$, $0 \le k \le s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \ge a_k\}, 0 \le k \le s$.

Theorem

The entropy $H(f) = 0$ iff all the points $(k, a_k), a_k < \infty, 0 \le k \le s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Newton polygon of a tropical polynomial

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ its **Newton polygon** $P(f) \subset \mathbb{R}^2$ is the convex hull of the vertical rays $\{(k, y) : y \ge a_k\}, 0 \le k \le s$.

Theorem

The entropy $H(f) = 0$ iff all the points $(k, a_k), a_k < \infty, 0 \le k \le s$ are the vertices of Newton polygon $P(f)$, and the indices k such that $a_k < \infty$ form an arthmetic progression.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$)*. The period does not exceed* $\exp((ms)^{O(s)})$*. In addition, the entropy $H(f)$ is a rational number.*

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \ldots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;
ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \dots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \ldots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;

ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Tropical Hilbert function for univariate polynomials

For a tropical polynomial $f = \min_{0 \le k \le s}\{a_k + kX\}$ with finite coefficients $0 \le a_k \le m$, $0 \le k \le s$ consider a tropical linear prevariety $U_N(f) \subset \mathbb{R}^N$ of the points $(u_1, \ldots, u_N)$ such that for each $1 \le j \le N - s$ the minimum in

$$\min_{0 \le k \le s}\{u_{j+k} + a_k\}$$

is attained at least twice. The tropical Hilbert function $T_N(f) = \dim(U_N(f))$.

Theorem

$T_N(f)$ is the sum of the linear function $H(f) \cdot N$ and a periodic function with an integer period (for $N > (ms)^{O(s)}$). The period does not exceed $\exp((ms)^{O(s)})$. In addition, the entropy $H(f)$ is a rational number.

Example

i) For a tropical polynomial $f = \min\{2X, 1 + X, 0\}$ we have $T_N(f) = \lfloor N/4 \rfloor$;
ii) For a tropical polynomial $f = \min\{2X, X, 0\}$ we have $T_N(f) = \lfloor N/3 \rfloor$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes.

For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n - 1$ (for sufficiently large N).

Theorem

• *If V consists of a finite number of points then $H(rad(V)) = 0$;*

• *the strong conjecture holds for univariate polynomials;*

• *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X + Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \ge 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n - 1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X + Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \ge 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \leq j \leq r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n - 1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X + Y\}$. Then $H(f) \ge 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \geq 1/2$.

Radical of a tropical prevariety

For a tropical prevariety $V \subset \mathbb{R}^n$ its **radical** $rad(V)$ is the semiring ideal of all the tropical polynomials for which points of V are tropical zeroes. For a semiring ideal I its radical is $rad(V(I))$.

Conjecture. For any semiring ideal I it holds $H(rad(I)) = 0$.

Strong conjecture. For a semiring ideal I of tropical polynomials in n variables its tropical Hilbert function $T_N(rad(I))$ is a polynomial of degree at most $n-1$ (for sufficiently large N).

Theorem

- *If V consists of a finite number of points then $H(rad(V)) = 0$;*
- *the strong conjecture holds for univariate polynomials;*
- *if $f = \min_{1 \le j \le r}\{t_{j,1}X + t_{j,2}Y\}$ is a tropical polynomial in 2 variables then $H(rad(f)) = 0$.*

Example

Consider a tropical quadratic polynomial $f = \min\{0, X, Y, X+Y\}$. Then $H(f) \ge 1/2$.