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Fibering out Calabi-Yau motives

joint work with Don Zagier
joint work in progress with Kilian Bönisch and Albrecht Klemm

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We find modular fibering-outs of conifold periods in one–parameter Calabi-Yau differential equations in a number of hypergeometric cases by using a technique described in VG's recent joint paper with Don Zagier.

An identity. Define the numbers a_n by the expansion

$$\sum_{n=1}^{\infty} a_n q^n = \frac{\eta(5\tau)^{10}}{\eta(\tau)\eta(25\tau)} + 5\eta(\tau)^2\eta(5\tau)^4\eta(25\tau)^2.$$

One numerically observes the identity

$$\sum_{k=0}^{\infty}\left(\log(1/5^5)+5\left(\sum_{i=1}^{5k}\frac{1}{i}-\sum_{i=1}^{k}\frac{1}{i}\right)\right)\frac{(5k)!}{k!^5}\frac{1}{5^{5k}}=-\frac{125}{2\pi^2}\sum_{n=1}^{\infty}\frac{a_n}{n^3},$$

the first 50000 terms summing up to $-8.12776\cdots$ on both sides.

Significance. The rational curve count on the generic quintic hypersurface in $\mathbb{P}^4$ has been famously related by Candelas et al. to the hypergeometric variation of periods arising in a family of special Calabi–Yau threefolds, called the mirror family

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0.$$

The Picard-Fuchs equation arising in this family is the pullback of the hypergeometric differential equation

$$[D^4 - 5^5\lambda(D+1/5)(D+2/5)(D+3/5)(D+4/5)]\Phi(\lambda) = 0$$

w.r. to the Kummer map $\lambda \mapsto 1/(5\psi)^5$; $D = \lambda\frac{d}{d\lambda}$.

The numbers of rational curves of any given degree d are finite and can be extracted from the expansion of its Wronskian (the ‘Yukawa coupling’) in terms of the natural parameter (the exponential of the ratio of the logarithmic to the analytic solution) .

Singular fibers.

- Being hypergeometric, this variation has singularities at 0, ∞ and the so-called *conifold* point $\lambda = 5^{-5}$.
- In accordance with the gamma conjecture, the behavior of the variation near 0 encodes the Chern numbers of the quintic, or the ambient $\mathbb{P}^4$, such as the Euler characteristic and $c_1 c_2$.
- The fiber at infinity appears to be highly singular; nevertheless, the interesting motive behind it is the one that occurs in the pullback family with respect to the Kummer map $\lambda = 1/(5\psi)^5$ and is given by $\sum_{i=0}^{4} x_i^5 = 0$.

  The periods of this Fermat quintic, according to Weil, can be related to the values of $\Gamma(i/5)$, $i = 1 \dots 4$.

- This so-called Calabi-Yau differential equation underlies a variation of polarized Hodge structure of type $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$.
- There is a single vanishing cycle at the conifold point $\lambda_0 = 5^{-5}$. The symplectic polarization arising in the family causes the splitting of the Hodge structure in the singular fiber into a sum of a rank 2 piece and a rank 1 piece.
- The motive of the conifold fiber in this [or a similar family] underlying the rank 2 Hodge structure is expected to be modular.

  This means that it also arises in a Kuga–Sato threefold.

In particular, the periods of the conifold motive sitting in the threefold $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 5x_0x_1x_2x_3x_4$ should be expressible in terms of certain values of the Eichler integral of some weight 4 modular form $f(\tau)$.

The conifold modular form in the quintic case. In our case of the quintic, it is the form

$$f(\tau) = \frac{\eta(5\tau)^{10}}{\eta(\tau)\eta(25\tau)} + 5\,\eta(\tau)^2\eta(5\tau)^4\eta(25\tau)^2;$$

the particular presentation in terms of the eta products is in fact inessential.

The fourteen one parameter Calabi-Yau families with fourth order hypergeometric differential Picard Fuchs operators are a direct generalisation:

$$[D^4 - \mu\,\lambda(D + a_1)(D + a_2)(D + a_3)(D + a_4)]\Phi(\lambda) = 0$$

N	AESZ	a_1, a_2, a_3, a_4	μ	Mirror M	κ	$c_2.D$	χ
8	3	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	2^8	$X_{2,2,2,2}(1^8)$	16	64	-128
9	11	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	$2^6 3^3$	$X_{4,3}(1^5 2)$	6	48	-156
16	6	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	2^{10}	$X_{4,2}(1^6)$	8	56	-176
25	1	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	5^5	$X_5(1^5)$	5	50	-200
27	4	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	3^6	$X_{3,3}(1^6)$	9	54	-144
32	10	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	2^{12}	$X_{4,4}(1^4 2^2)$	4	40	-144
36	5	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	$2^4 3^3$	$X_{3,2,2}(1^7)$	12	60	-144
72	14	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	$2^8 3^3$	$X_{6,2}(1^5 3)$	4	52	-256
108	8	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	$2^4 3^6$	$X_6(1^4 2)$	3	42	-204
128	7	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2^{16}	$X_8(1^4 4)$	2	44	-296
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864	9	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	$2^{12} 3^6$	$X_{2,12}(1^4 4, 6)$	1	46	-484

Problem

In each of these cases, find a period identity relating the conifold Hodge structure to a modular Hodge structure. If possible, exhibit an explicit correspondence between the conifold fiber and a Kuga-Sato variety.

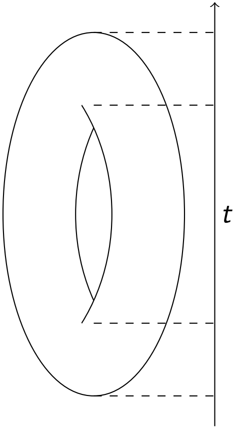
- In practice, this means that we are looking for a certain codimension 3 cycle in a 6-dimensional cartesian product of the conifold fiber and the Kuga-Sato variety.
- Compare our situation with where the theory of elliptic curves was in the pre-Heegner [Birch–Stephens] period. In the range of low conductors, the search over small boxes is quite efficient at producing the Mordell-Weil generators of analytic rank 1 curves. The limitations of this method are clear.
- In the absence of a general theory, we have to resort to exactly this: search over small–coordinate boxes ...

37a1	(0, 0)
43a1	(0, 0)
53a1	(0, 0)
57a1	(2, -2)
58a1	(0, 1)
61a1	(1, -1)
65a1	(-1, 1)
77a1	(2, 3)
79a1	(0, 0)
82a1	(-1, 1)
83a1	(0, 0)
88a1	(2, -2)
89a1	(0, 0)
91a1	(0, 0)
91b1	(-1, 3)
92b1	(1, -1)
99a1	(0, 0)

. . . except that ‘embedding varieties as differential ideals may be more economical than embedding them as polynomial ideals’.

Fibered motives and fibered periods.

Morse's method to approach the topology of a manifold is to equip it with a function and try to pass to level hypersurfaces. If we put a usual 2–torus T vertically on its top, i.e. on a point on the equator, there are 4 critical levels where the signatures are, successively, $(2,0),(1,1),(1,1),(0,2)$. A cycle is acquired in the homology of the subcomplex $\{x \in T \mid h(x) \leq t\}$ whenever we reach next critical t; the dimension of the cycle is given by the signature.



As one passes to more rigid theories where cohomology groups bear more structure, one seeks to refine this crude information while still keeping to the original Morse–Lefschetz idea.

For instance, if V is a variety over $\mathbb{F}_q$ and h is a rational function on it, then

$$\#V(\mathbb{F}_q) = \sum_{t \in \mathbb{P}^1(\mathbb{F}_q)} \#\{x \in V(\mathbb{F}_q) \mid h(x) = t\}.$$

Let now V be an affine variety over $\mathbb{Q}$, and h an invertible function on it. Assume we are given a regular differential k–form Ω and a Betti k–cycle C such that $\int_C \Omega$ makes sense. Define c_t to be the result of intersecting C with the level $h = t$, and ω_t to be the residue form $\mathrm{Res}\frac{\Omega}{h-t}$. Put $\Phi(t) = \int_{c_t} \omega_t$. We will assume for a while that $\dim h(C) = 1$.

By a version of Fubini, one may expect to find

$$\int_C \Omega = \int_\gamma \Phi(t)dt,$$

[perhaps up to a simple correction term],

where γ is a collection of paths in $h(C)$, either closed, or connecting interesting points such as the critical values of h on V (or on C), 0 and ∞.

Observations:

- tautologically: in the presence of a unit $h\,(= t)$, the period $\int_C \Omega$ becomes a special value (namely, at 0) of the *motivic gamma function*
$$\Gamma_{C,\Omega,h}(s) = \int_C h^s\,\Omega$$
which satisfies an ordinary linear recursion with polynomial coefficients;
- secondly, the *derivatives* of Γ at 0 (or other integer arguments) are periods as well.

Fibered periods. We will say that a period Π is fibered out by a differential equation $L[\cdot] = 0$ on $\mathbf{G}_m$ if the DE underlies a variation of mixed $\mathbb{Z}$–Hodge structures of geometric nature and there exists a solution $\Phi(t)$ in the integral structure and a path γ on $\mathbf{G}_m(\mathbb{C})$ with $\operatorname{supp} \partial\gamma \subset \{0, \infty, \text{singularities of } L\}$ such that $\Pi = \int_\gamma \Phi(t) \frac{dt}{t}$.

Two core practical questions about fibered motives:

(A) Given a differential operator L, how to compute in practical terms the motive (e.g. the period matrix and the L–function) of $H^1_?(\mathbf{G}_m, \mathcal{D}/L\mathcal{D})$?

(B) Given a motive, how to construct effectively its fibering(s)?

Philosophically, (A) and (B) should be seen as aiming to connect four facets of periods, namely, 1) integrals of forms on domains, 2) entries of the matrices of the monodromy transformation between the spaces of local solutions to Picard–Fuchs equations at different singularities, 3) higher derivatives of motivic gammas as above and 4) the coefficients of the universal series that explicitly solve the Riemann–Hilbert problem on varieties over $\overline{\mathbb{Q}}$.

A believer in all standard motivic conjectures may define the rank of a period to be the minimal dimension of a period matrix (a Betti–de Rham structure) over $\mathbb{Q}$ (or K) where it appears. The optimality question is then: for a period of rank r, is it possible to fiber it out in a DE whose degree in t — call it J — is exactly r?

If so, what is the minimal possible *order* of L? If not, what is the minimal J (and therefore the order of the respective recursion)?

For $r = 1$ we are essentially dealing with the periods of the grossencharacter motives. By the Lerch–Chowla–Selberg theorem, sufficiently high powers of these can be fibered out optimally in certain hypergeometric ($J = 1$) D–modules which we will call the Chowla–Selberg objects. For $r = 2$, modular periods constitute an important class; the question of which modular periods correspond to order 2 (= three–term) recursions seems to be completely open in weights ≥ 2.

B: Hypergeometric fibering-out (Golyshev–Zagier).

Suppose we are given a hypergeometric variation of Hodge structures $\mathcal{V}$ on the torus $\mathbf{G}_m = \mathsf{Spec}\,\mathbb{C}[\lambda, \lambda^{-1}]$. Fix $\lambda = \lambda_0$ and write the period(s) of $H = \mathcal{V}_{\lambda_0}$ generically as $\sum \mathbf{\Gamma}(k)\lambda_0^k$, where the fact that $\mathcal{V}$ is hypergeometric simply means that $\mathbf{\Gamma}(k) = \prod_i \Gamma(l_i(k))^{n_i}$ for some linear functions $l_i(k) \in \mathbb{Z}k + \mathbb{Q}$ and some exponents $n_i \in \mathbb{Z}$.

Choose a *lift* of $\mathbf{\Gamma}(k)$ to $\widetilde{\mathbf{\Gamma}}(n,k)$ given by the formula $\widetilde{\mathbf{\Gamma}}(n,k) = \prod_i \Gamma(\widetilde{l}_i(n,k))^{n_i}$, where $\widetilde{l}_i(n,k)$ are now linear functions in *two* variables (i.e. belonging to $\mathbb{Z}n + \mathbb{Z}k + \mathbb{Q}$) such that $\widetilde{l}_i(0,k) = l_i(k)$, and set formally

$$C_n = \sum_k \widetilde{\mathbf{\Gamma}}(n,k)\lambda_0^k \quad (n \in \mathbb{C}).$$

Remarks.

- The formal infinite sum $\Phi(t) = \sum_n C_n t^n$ satisfies a differential equation $L\Phi(t) = 0$.
- Cauchy's formula says $C_n = \int \Phi(t) t^{-n} \frac{dt}{t}$.
- The same quantity $C_0 = \int \Phi(t) \frac{dt}{t}$ can be interpreted, depending on one's optic,
  - as a period of the Hodge structure H
  - or as a period in the Hodge structure arising in the cohomology of the t-torus $\mathsf{Spec}\,\mathbb{C}\left[t, t^{-1}\right]$ with coefficients in the Hodge module $\mathcal{H}$ given by L.

A proto-definition. We will say that the Hodge structure H is *fibered out* by t into a Hodge module $\mathcal{H}$ on $\mathbf{G}_m(t)$.

One can imagine an invertible function, or a "unit", on the motive M that underlies H which turns it into a pencil of motives $\mathcal{M}$ over the t-torus.

Why fibered motives rather than functions on varieties?

No need to compactify and resolve.

Remark. One anticipates a relation between the value of the *motivic gamma function*

$$\Gamma_{\mathcal{H}}^{\mathrm{mot}}(s) := \int \Phi(t) t^s \frac{dt}{t}$$

at $s = 0$ and the entries of the period matrix of H — provided, of course, that one can make the meaning of the integral precise. See Bloch–Vlasenko for the fundamentals of motivic gammas.

Search-in-a-box strategy revisited.

- The rank 2 conifold motive M that underlies the conifold Hodge structure H can be fibered out "in different directions" that correspond to the choice of the lifts.
- We are looking for special choices that turn $\mathcal{H}$ into a *modular* variation of Hodge structure.
- One should look for those directions of fibering–out that correspond to short lift vectors.

Modular variations of Hodge structure. Modularity means in this context that:

- in the differential equation $\mathcal{L}\Phi(t) = 0$ that controls $\mathcal{H}$, the solution $\Phi(t)$ can be chosen to be a [weak] weight 2 modular form, and t a Hauptmodul;
- the modular parameter τ can be interpreted as the ratio $\frac{\Phi^*(t)}{\Phi(t)}$ of the normalized log to the analytic solution around $t = 0$;
- among the entries of the conifold period matrix of H is the *Eichler integral* of the weight 4 form $\Phi(t(\tau))t'(\tau)$.

Results.

1. The lift

$$\frac{(5k)!}{k!^5} \rightsquigarrow \frac{(5k+n)!}{k!^4(k+n)!}$$

is modular (and eventually leads to the identity that appeared at the beginning). Define E as the unique normalized level 2 Eisenstein series of weight 2 and f_{50} as the normalized Hecke eigenform in $S_2(\Gamma_0(50))^{\mathrm{new}}$ with Hecke eigenvalue $a_3 = -1$. One then finds that

$$5t(\tau) = 5 + \frac{E(5\tau)}{f_{50}(\tau)}$$

$$\Phi(t(\tau)) = f_{50}(\tau)$$

and

$$\Phi(t(\tau))\frac{t'(\tau)}{t(\tau)} = 4f(2\tau) - f(\tau)$$

where

$$f(\tau) = \frac{\eta(5\tau)^{10}}{\eta(\tau)\eta(25\tau)} + 5\,\eta(\tau)^2\eta(5\tau)^4\eta(25\tau)^2\,.$$

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2. In the cases $N = 8, 9, 16, 25, 27, 36, 72$ the following lifts produce order 3, degree 3 recursions:

N	AESZ	Lift
8	3	$\binom{2k}{k}^4 \rightsquigarrow \frac{(2k+n)!(2k)^3}{(k+n)!k!^7}$
9	11	$\binom{3k}{k}\frac{(4k)!}{k!^4} \rightsquigarrow \frac{(3k+n)!(4k)!}{(k+n)!k!^4(2k)!}$
16	6	$\binom{2k}{k}\frac{(4k)!}{k!^4} \rightsquigarrow \frac{(2k+n)!(4k)!}{(k+n)!k!^5}$
25	1	$\frac{(5k)!}{k!^5} \rightsquigarrow \frac{(5k+n)!}{k!^4(k+n)!}$
27	4	$\frac{(3k)!^2}{k!^6} \rightsquigarrow \frac{(3k+n)!(3k)!}{(k+n)!k!^5}$
36	5	$\binom{2k}{k}^2\frac{(3k)!}{k!^3} \rightsquigarrow \frac{(2k+n)!(2k)!(3k)!}{k!^6(k+n)!}$
72	14	$\binom{2k}{k}\frac{(6k)!}{k!^3(3k)!} \rightsquigarrow \frac{(2k+n)!(6k)!}{(k+n)!k!^4(3k)!}$

3. In the cases $N = 8, 9, 16, 27, 32, 36, 108, 144, 216$ there exist modular hypergeometric lifts.