Multidimensional quadratic BSDEs with separated generators

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joint work with Asgar Jamneshan and Michael Kupper

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Backward Stochastic Differential Equations

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$
- W d-dimensional Brownian motion

- Pardoux and Peng (1990): g Lipschitz continuous, $\xi \in L^2$.

Quadratic BSDEs

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T \theta_1 |Z_s^1|^2 + \vartheta_1 |Z_s^2|^2 ds - \int_t^T Z_s^1 dW_s, \\ Y_t^2 = \xi^2 + \int_t^T \vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2 ds - \int_t^T Z_s^2 dW_s. \end{cases}$$

- θ_1 and θ_2: growth coefficients.
- ϑ_1 and ϑ_2: strength of coupling.
- Interplay of ξ^i, θ_i and ϑ_i.

Existing results

- Counterexample of Frei and Dos Reis (2011): $\xi^2 = 0$, $\theta_1 = \vartheta_1 = 0$, $\theta_2 = -1$ and $\vartheta_2 = -1/2$, ξ^1.

- $\vartheta_1 = \vartheta_2 = 0$, Kobylanski (2000) and Briand and Hu (2006,2008).

- $\vartheta_1 \neq 0, \ \vartheta_2 \neq 0$: Tevzadze (2008), $\sqrt{\|\xi^1\|_\infty^2 + \|\xi^2\|_\infty^2} < \frac{1}{64\lambda}$ where $\lambda = \max\{\theta_1, \vartheta_1, \theta_2, \vartheta_2\}$.

Existing results (Cont'd)

- Cheridito and Nam (2015): Markovian and projectable quadratic BSDEs
- Hu and Tang (2014): Diagonally quadratic BSDEs
- Frei (2014): Split solution
- Kardaras, Xing and Žitković (2015): Close to Pareto equilibrium

Jamneshan, Kupper and L.: First set of conditions

If $\theta_1 = \theta_2 = 0$ and

(i) $8|\vartheta_2|\|\xi^1\|_\infty^2 \leq \|\xi^2\|_\infty$ and $8|\vartheta_1|\|\xi^2\|_\infty^2 \leq \|\xi^1\|_\infty$,

(ii) $16|\vartheta_1|\|\xi^2\|_\infty \leq 1$ and $16|\vartheta_2|\|\xi^1\|_\infty \leq 1$,

then

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T \vartheta_1 |Z_s^2|^2 ds - \int_t^T Z_s^1 dW_s, \\ Y_t^2 = \xi^2 + \int_t^T \vartheta_2 |Z_s^1|^2 ds - \int_t^T Z_s^2 dW_s, \end{cases}$$

has a unique solution s.t. Y is bounded and $\|Z^i \cdot W\|_{BMO} \leq 2\|\xi^i\|_\infty$.

Sketch of the proof

- For $z \cdot W \in BMO$, BSDE

$$Y_t = \xi + \int_t^T \vartheta |z_s|^2 ds - \int_t^T Z_s dW_s$$

  has a unique solution in $\mathcal{S}^2 \times \mathcal{H}^2$.

- Indeed, taking conditional expectation w.r.t. $\mathcal{F}_t$,

$$|Y_t| = \left| E\left[\xi + \int_t^T \vartheta |z_s|^2 ds \middle| \mathcal{F}_t \right] \right|$$
$$\leq \|\xi\|_\infty + |\vartheta| \|z \cdot W\|_{BMO}^2.$$

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$$\begin{aligned} |Y_t| &= \left| E\left[\xi + \int_t^T \vartheta |z_s|^2 ds \middle| \mathcal{F}_t \right] \right| \\ &\leq \|\xi\|_\infty + |\vartheta| \|z \cdot W\|_{BMO}^2. \end{aligned}$$

Sketch of the proof

- Itô's formula implies that

$$E\left[\int_t^T Z_s^2 ds \middle| \mathcal{F}_t\right] \leq E\left[\xi^2 + 2\vartheta \int_t^T Y_s z_s^2 ds \middle| \mathcal{F}_t\right]$$
$$\leq \|\xi\|_\infty^2 + 2|\vartheta|\|\xi\|_\infty \|z \cdot W\|_{BMO}^2 + 2|\vartheta|^2 \|z \cdot W\|_\infty^4.$$

- Condition (i) guarantees the existence of the set of candidate solutions

$$M = \{(z^1, z^2) : \|z^1 \cdot W\|_{BMO} \leq 2\|\xi^1\|_\infty, \\ \|z^2 \cdot W\|_{BMO} \leq 2\|\xi^2\|_\infty\}.$$

s.t. $I : M \to M, \quad I(z^1, z^2) = (Z^1, Z^2)$ is well-defined.

- By condition (ii), I is a contraction.

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- By condition (ii), I is a contraction.

For $i = 1, 2$, let $\theta_i > 0$ and assume

(i) $\frac{4\theta_1|\vartheta_1|e^{2\theta_2\|\xi^2\|_\infty}}{\theta_2^2} \leq 1$ and $\frac{4\theta_2|\vartheta_2|e^{2\theta_1\|\xi^1\|_\infty}}{\theta_1^2} \leq 1$,

(ii) $\frac{4L_4^4c_2^2|\vartheta_1|^2e^{2\theta_2\|\xi^2\|_\infty^2}}{c_1\theta_2^2} \leq 1$ and $\frac{4L_4^4\bar{c}_2^2|\vartheta_2|^2e^{2\theta_1\|\xi^1\|_\infty^2}}{\bar{c}_1\theta_1^2} \leq 1$.

Then

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T \theta_1|Z_s^1|^2 + \vartheta_1|Z_s^2|^2 ds - \int_t^T Z_s^1 dW_s, \\ Y_t^2 = \xi^2 + \int_t^T \vartheta_2|Z_s^1|^2 + \theta_2|Z_s^2|^2 ds - \int_t^T Z_s^2 dW_s, \end{cases}$$

has a unique solution s.t. Y is bounded and $\|Z^i \cdot W\|_{BMO} \leq \frac{e^{\theta_i\|\xi^i\|_\infty}}{\theta_i}$.

Assume

(i) $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ predictable,

(ii) $f^i(t, y, z) = f^i(t, y, z^i)$,

(iii) there exist $C \geq 0, \theta > 0, \beta > 0$ s.t.

$$|f(t, 0, 0)| \leq C,$$
$$|f(t, y, z) - f(t, y', z')| \leq \beta|y - y'| + \theta(1 + |z| + |z'|)|z - z'|,$$

(iv) $\xi \in L^\infty$.

Then

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

has a unique solution s.t. Y is bounded and $Z \cdot W \in BMO$.

Sketch of the proof

- For $y \in \mathcal{S}^\infty$, for $i = 1, \ldots n$, BSDE

$$Y_t^i = \xi + \int_t^T f^i(s, y_s, Z_s^i)ds - \int_t^T Z_s^i dW_s$$

has a unique solution solution in $\mathcal{S}^\infty \times BMO$.
- Define $I : \mathcal{S}^\infty \to \mathcal{S}^\infty$ mapping $y \longmapsto I(y) = Y$.
- Consider the difference

$$Y_t^i - \bar{Y}_t^i = \int_t^T f^i(s, y_s, Z_s^i) - f^i(s, \bar{y}_s, \bar{Z}_s^i)ds - \int_t^T Z_s^i - \bar{Z}_s^i dW_s$$
$$= \int_t^T f^i(s, y_s, Z_s^i) - f^i(s, \bar{y}_s, Z_s^i) + f^i(s, \bar{y}_s, Z_s^i) - f^i(s, \bar{y}_s, \bar{Z}_s^i)ds$$
$$- \int_t^T Z_s^i - \bar{Z}_s^i dW_s$$

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$$- \int_t^T Z_s^i - \bar{Z}_s^i dW_s$$

Sketch of the proof

- there exists a predictable process b_s s.t.

$$f^i(s, \bar{y}_s, Z^i_s) - f^i(s, \bar{y}_s, \bar{Z}^i_s) = b_s(Z^i_s - \bar{Z}^i_s)$$
$$\text{and} \quad |b_s| \leq \theta(1 + |Z^i_s| + |\bar{Z}^i_s|).$$

- $dQ = \mathcal{E}(b \cdot W)_T dP$ defines an equivalent measure and

$$Y^i_t - \bar{Y}^i_t = E^Q \left[\int_t^T f^i(s, y_s, Z^i_s) - f^i(s, \bar{y}_s, Z^i_s) ds \middle| \mathcal{F}_t \right].$$

- Contraction: $\exists$ a fixed constant $\lambda > 0$ s.t. on $[T - \lambda, T]$ one obtains a contraction.
- By recurrence and pasting, there exists a unique solution in $\mathcal{S}^\infty \times BMO$.

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- By recurrence and pasting, there exists a unique solution in $\mathcal{S}^\infty \times BMO$.

The general case

$$Y_t^i = \xi^i + \int_t^T g(s, Y^1, Y^2, Z^1, Z^2) ds - \int_t^T Z_s^i dW_s, \quad i = 1, 2,$$

where the generator is separated:

$$g^i(s, y^1, y^2, z^1, z^2) = f^i(s, z^i) + h^i(s, y^1, y^2, z^1, z^2), \quad i = 1, 2.$$

(i) ξ^i is bounded,

(ii) $f^i(s, 0)$ bounded and
$|f^i(s, z^i) - f^i(t, \bar{z}^i)| \leq \theta_i(1 + |z^i| + |\bar{z}^i|)|z^i - \bar{z}^i|$,

(iii) $h^i(s, 0, 0)$ bounded
$|h^i(s, y, z) - h^i(s, \bar{y}, \bar{z})| \leq \beta_i|y - \bar{y}| + \vartheta_i(1 + |z| + |\bar{z}|)|z - \bar{z}|$.

Conclusion

- Two sets of assumptions on the ingredients are worked out guaranteeing a unique solution.
- For the general case: Jamneshan, Kupper and L. "Multidimensional quadratic BSDEs with separated generators".
- Unbounded terminal conditions can be considered whenever the generator is independent of the value process.

Thank you for your attention!