Variational View to Optimal Stopping with Application to Real Options

Vadim Arkin, Alexander Slastnikov

Central Economics and Mathematics Institute,
Russian Academy of Sciences, Moscow

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Outline

- Optimal stopping problem for diffusion processes
- Variational approach to solving OS problem
- One-dimensional diffusion. Two classes of threshold stopping times. Necessary and sufficient conditions for optimality
- Application to investment timing problem and optimal abandonment problem
- Relations between solutions to free-boundary problem and optimal stopping problem in "threshold case"

Optimal stopping problem

Let $X_t,\ t \geq 0$ be a diffusion process with values in $D \subseteq \mathbb{R}^n$, defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$.

Let us consider an optimal stopping problem for this process:

$$U(x) = \sup_{\tau \in \mathfrak{M}} \mathbf{E}^x g(X_\tau) e^{-\rho\tau} \mathbf{1}_{\{\tau<\infty\}}, \qquad (1)$$

where:

$g:\ D \to \mathbb{R}^1$ is payoff function,
$\rho > 0$ is discount rate,
$\mathbf{1}_A$ is indicator function of the set A,
$\mathbf{E}^x$ means the expectation for the process X_t starting from the state x,
$\mathfrak{M}$ is some class of stopping times τ (which can take infinite values with positive probability).

Usually $\mathfrak{M}$ is the class of *all* stopping times.

Two main approaches to solving an optimal stopping problem for diffusion processes.

1. Markovian (or "mass") approach embeds underlying optimal stopping problem into the family of problems (1) with all initial states $X_0 = x$ of the process X_t. In this case to solve problem (1) means to find the value function $U(x)$ and optimal stopping time $\tau^*(x)$ for all x.
(Excessive characterization, iterative methods, free-boundary problem,...)
2. Martingale approach solves an optimal stopping problem (1) for fixed initial state $X_0 = x$.
(Snell's envelope, Beibel–Lerche method, ...)

Variational approach. I

We propose to find a solution to optimal stopping problem (1) over the class $\mathfrak{M}$ of first exit times of the process X_t from the sets belonging to the given family, and to make optimization over this family of sets.

Arguments in favour of such a reduction of the class of stopping times:

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Arguments in favour of such a reduction of the class of stopping times:

• Under enough general assumptions an optimal stopping time can be find as the first exit time of the process X_t out of the open "continuation" set $\{U(x) > g(x)\}$. Hence, in one-dimensional case optimal stopping problem can be reduced to finding optimal first exit time from intervals (a, b), which contain starting point x of the process X_t.

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• For a lot of optimal stopping problems (especially in multi-dimensional case) the exact solution is very hard for calculations and interpretations. Thus, if finding an optimal stopping decision is not the final goal of study (for example, in investment models), then it makes sense to restrict considerations to simple class of stopping times in order to obtain any "reasonable" solution which will be tractable and suitable for analysis.

Variational approach. II

Let $\mathcal{G} = \{G\}$ be a given family (class) of regions in $\mathbb{R}^n$,
$\tau_G = \tau_G(x) = \inf\{t \geq 0 : \ X_t \notin G\}$ be a first exit time of process X_t from the region G (obviously, $\tau_G = 0$ whenever $x \notin G$), and
$\mathfrak{M}(\mathcal{G}) = \{\tau_G, \ G \in \mathcal{G}\}$ be a set of first exit times for all regions from the class $\mathcal{G}$.

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Under fixed initial value x define the following function of sets $G \in \mathcal{G}$

$$V_G(x) = \mathbf{E}^x g(X_{\tau_G}) e^{-\rho\tau_G} \mathbf{1}_{\{\tau_G < \infty\}}.$$

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Thus, an optimal stopping problem (1) over a class of stopping times $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$ can be converted to the following variational problem:

$$V_G(x) \to \sup_{G \in \mathcal{G}}. \qquad (2)$$

If G^* is an optimal region in (2), then τ_{G^*} will be the optimal stopping time for the problem (1) over the class $\mathfrak{M} = \mathfrak{M}(\mathcal{G})$.

One-parametric family of regions

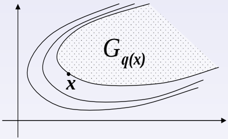
Let $\mathcal{G} = \{G_p,\ p \in P \subset \mathbb{R}^1\}$ be one-parametric family of regions in $\mathbb{R}^n$,
$\tau_p = \inf\{t \geq 0 : X_t \notin G_p\}$,

$$V(p;x) = V_{G_p}(x) = \mathbf{E}^x g(X_{\tau_p}) e^{-\rho\tau_p} \mathbf{1}_{\{\tau_p<\infty\}}. \tag{3}$$

The function $V(p;x)$ is defined on $P \times D$; $V(p;x) = g(x)$ for $x \notin G_p$.

Let a family of regions $\mathcal{G} = \{G_p\}$ satisfy the conditions:

1. Monotonicity. $G_{p_1} \subset G_{p_2}$ whenever $p_1 < p_2$.
2. Thickness. Every point $x \in D$ belongs to the boundary of the unique set $G_{q(x)} \in \mathcal{G}$.



The variational problem (2) can be written as one-dimensional optimization:

$$V(p;x) \to \sup_{p \in P}. \tag{4}$$

Necessary and sufficient conditions for maximization of $V(p;x)$ in p:

Theorem 1

i) If $p^* = p^*(x)$ is the solution to the problem (4) then the following conditions hold:

$$V(p;x) \le V(p^*;x) \quad \text{whenever } p < p^*,\ x \in G_p \cup \partial G_p, \tag{5}$$

$$V(p;x) \le g(x) \quad \text{whenever } p > p^*,\ x \in G_p \setminus G_{p^*}. \tag{6}$$

ii) If for some $p^* = p^*(x)$

$$V(p_1;x) \ge V(p_2;x) \quad \text{whenever } p^* \le p_1 < p_2,\ x \in G_{p_2} \tag{7}$$

and condition (5) hold, then p^* is the solution to the problem (4).

One-dimensional diffusion processes. I

Let $X_t, \ t \geq 0$ be a homogeneous diffusion process with values in the segment $D \subseteq \mathbb{R}^1$ with boundary points l and r, where $-\infty \leq l < r \leq +\infty$, open or closed (it may be $(l, r), \ [l, r), \ (l, r], $ or $[l, r]$), which describes by the stochastic differential equation

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{8}$$

where $a(x)$ and $\sigma(x)$ are the drift and diffusion functions and W_t is the standard Wiener process.

The process X_t is assumed to be regular; i.e., starting from an arbitrary point $x \in \operatorname{int} D = (l, r)$, the process reaches any point $y \in (l, r)$ in finite time with positive probability. The regularity of a process is guaranteed, if at any $x \in (l, r)$:

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |a(y)|}{\sigma^2(y)} dy < \infty \quad \text{for some } \varepsilon > 0.$$

One-dimensional diffusion processes. II

The process X_t is associated with the infinitesimal operator

$$\mathbb{L}f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

It is known that under regularity conditions there exist (unique up to constant positive multipliers) increasing and decreasing functions $\psi(x)$ and $\varphi(x)$ (resp.) with absolutely continuous derivatives, $\psi(x)$ and $\varphi(x)$ are the fundamental solutions to the ODE

$$\mathbb{L}u(x) = \rho u(x) \qquad (9)$$

almost sure (in Lebesque measure) on the interval (l, r).

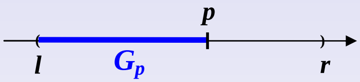
Moreover, $0 < \psi(x),\ \varphi(x) < \infty$ for $x \in (l, r)$.

If functions $a(x),\ \sigma(x)$ are continuous, then $\psi,\ \varphi \in C^2(l, r)$.

Two one-parametric families of sets for one-dimensional diffusions.

1. l-intervals

As the first family of sets we take intervals of the type
$G_p = \{x \in D : x < p\}, \ \ p \in (l, r)$, which we call l-intervals.



l-interval G_p is $[l, p)$ or (l, p) in dependence on $l \in D$ or not.

Obviously, the class of l-intervals satisfies conditions **(A1)**–**(A2)**.

The class of 'threshold' stopping times $\mathfrak{M}_l = \{\tau_p, \ p \in (l, r)\}$,
where $\tau_p = \inf\{t \geq 0 : X_t \geq p\}$ is the first exit time from the l-interval G_p.

Threshold stopping times. Optimality conditions

An optimal stopping problem over the class $\mathfrak{M}_l$ (induced by l-intervals) can be written as a problem of one-dimensional optimization:

$$V(p;x) := \mathbf{E}^x g(X_{\tau_p}) e^{-\rho\tau_p} \mathbf{1}_{\{\tau_p<\infty\}} \to \sup_{l<p<r} . \qquad (10)$$

Theorem 1 above gives a necessary and sufficient conditions for the optimality over class of threshold stopping times.

Theorem 2

Threshold stopping time τ_{p^*} is optimal in the problem (1) for all $x \in (l, r)$ over the class $\mathfrak{M}_l$ of threshold stopping times if and only if the following conditions hold:

$$\frac{g(p)}{\psi(p)} \le \frac{g(p^*)}{\psi(p^*)} \quad \text{whenever } p < p^*; \qquad (11)$$

$$\frac{g(p)}{\psi(p)} \quad \text{does not increase for } p > p^*. \qquad (12)$$

Threshold stopping times. Smooth pasting

So, the optimal threshold p^* is a point of maximum for the function $h(p) = g(p)/\psi(p)$. This implies a simple proof (under minor assumptions) of the well-known smooth pasting principle.

Define $V(x) = \sup\limits_{l<p<r} \mathbf{E}^x g(X_{\tau_p}) e^{-\rho\tau_p} \mathbf{1}_{\{\tau_p<\infty\}}$.

Corollary. *Let τ_{p^*}, where $p^* \in (l, r)$, be the optimal stopping time over the class $\mathfrak{M}_l$ and the payoff $g(x)$ be differentiable at the point p^*. Then the function $V(x)$ is differentiable at the point p^*, and $V'(p^*)=g'(p^*)$.*

Optimality of threshold stopping time over all stopping times

The extended conditions of Theorem 2 remain necessary and sufficient for a threshold structure of a solution to stopping problem (1) over *all* stopping times.

Theorem 3

Let for some $p^* \in (l, r)$ and set of isolated points $\{a_1, ..., a_n, ...\}$, where $p^* < a_1 < a_2 < ... < r$, the derivative of payoff function $g'(p)$ be absolutely continuous on the intervals (p^*, a_1), (a_i, a_{i+1}), $i \geq 1$ and there exist one-sided derivatives $g'(p^*+0)$, $g'(a_i \pm 0)$, $i \geq 1$, such that $\sum_{i \geq 1} \sigma^2(a_i)|g'(a_i+0) - g'(a_i-0)| < \infty$. Then threshold stopping time τ_{p^*} is the optimal stopping time in problem (1) for all $x \in (l, r)$ over class of all stopping times, if and only if the following conditions hold:

$$\frac{g(p)}{\psi(p)} \leq \frac{g(p^*)}{\psi(p^*)} \text{ for } p < p^*; \tag{13}$$

$$\psi'(p^*)g(p^*) \geq \psi(p^*)g'(p^*+0); \tag{14}$$

$$\mathbb{L}g(p) \leq \rho g(p) \quad \text{a.s. (in Lebesque measure)} \quad \text{for } p > p^*; \tag{15}$$

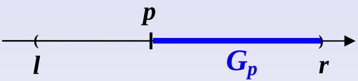
$$g'(a_i+0) - g'(a_i-0) \leq 0, \quad i \geq 1. \tag{16}$$

These conditions can be viewed as a solution to 'inverse optimal stopping problem' when it is required to find such a diffusion process and payoff function that optimal decision will have a threshold structure.

Another class of threshold stopping times

2. r-intervals.

$G_p = \{x \in D : \; x > p\}, \;\; p \in (l, r)$, which we call r-intervals.



Define another class $\mathfrak{M}_r$ of threshold stopping times
$\bar{\tau}_p = \inf\{t \geq 0 : X_t \notin G_p\} = \inf\{t \geq 0 : X_t \leq p\}$ — the class of first times when process X_t falls below threshold p, $\;\; p \in (l, r)$.

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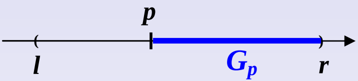

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Threshold stopping times $\bar{\tau}_p$ appear, in particular, in optimal stopping problems with both integral and terminal payoffs:

$$\mathbf{E}^x \left(\int_0^\tau g_1(X_t) e^{-\rho t}\, dt + g_0(X_\tau) e^{-\rho\tau} \right) \to \sup_\tau, \qquad (17)$$

where $g_0(x)$, $g_1(x)$ are given functions.

Necessary and sufficient conditions of optimality of threshold stopping time in the problem with both integral and terminal payoff functions:

Theorem 4

Let for some $p^* \in (l, r)$ functions a, σ, g_1 be continuous on segment $(l, p^*]$, g_0 be twice differentiable on $(l, p^*]$ and $g_0(x) \geq R(x)$ on $(l, p^*]$. Then threshold stopping time $\bar{\tau}_{p^*} = \inf\{t \geq 0 : X_t \leq p^*\}$ is the optimal stopping time in problem (17) over the class of all stopping times if and only if the following conditions hold:

$$[g_0(p) - R(p)]\varphi(p^*) \leq [g_0(p^*) - R(p^*)]\varphi(p) \quad \text{for } p > p^*;$$

$$J(p^*)S'(p^*) = g_0'(p^*)\varphi(p^*) - g_0(p^*)\varphi'(p^*);$$

$$\mathbb{L}g_0(p) - \rho g_0(p) \leq \mathbb{L}R(p) - \rho R(p) \quad \text{for } p < p^*,$$

Here, $R(x) = \mathbf{E}^x \int_0^\infty g_1(X_t)e^{-\rho t}\,dt$, $\quad J(x) = \int_x^r \varphi(y)g_1(y)H(y)\,dy$,

$S'(x) = \exp\left\{-\int 2a(x)/\sigma^2(x)dx\right\}$, $\quad H(y) = 2/[\sigma^2(y)S'(y)]$,

and $\varphi(p)$ — decreasing solution to 'characteristic ODE': $\mathbb{L}u(x) = \rho u(x)$.

Application to Real Options Theory

1. Investment timing problem

An infinitely-lived investor faces a problem of choosing a time when to finance some investment project. Investment is considered to be instantaneous and irreversible, and the project begins to produce profits just after the investment is made.

Let diffusion process X_t describe the Present Value of the implemented project started at time t,

I be a cost of investment required for beginning the project.

The investor solves the following *investment timing problem* : to find such a stopping time τ (investment rule), that maximizes the net present value (NPV) from the project:

$$\mathbf{E}(X_\tau - I)e^{-\rho\tau}\mathbf{1}_{\{\tau<\infty\}} \to \max_\tau,$$

where the maximum is taken over all possible stopping times τ.

The majority of results on this problem has a threshold structure: to invest when PV from the project exceeds the certain threshold. This is so for geometric and arithmetic Brownian motions, mean-reverting process and some other.

The general question: For what processes of PV from the project an optimal decision of investment timing problem will have a threshold structure?

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We give the necessary and sufficient conditions for optimality of threshold strategy in investment timing problem:

Theorem 5

Threshold stopping time τ_{p^*}, $p^* \in (l, r)$, is optimal in the investment timing problem if and only if the following conditions hold:

$$(p - I)\psi(p^*) \le (p^* - I)\psi(p) \quad \text{for } p < p^*;$$
$$\psi(p^*) = (p^* - I)\psi'(p^*);$$
$$a(p) \le \rho(p - I) \quad \text{for } p > p^*,$$

where $a(p)$ is the drift function of the process X_t, and $\psi(p)$ is an increasing solution to 'characteristic ODE': $\mathbb{L}u(x) = \rho u(x)$.

Classic case

Let X_t be geometric Brownian motion with rate of drift α and volatility σ:

$$dX_t = X_t(\alpha dt + \sigma dw_t).$$

In this case, $\psi(x) = x^{\beta}$, where β is the positive root of the equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta = \rho$. If $\rho > \alpha$ then $\beta > 1$.

Theorem 4 implies that optimal threshold is $p^* = \dfrac{\beta}{\beta-1}I$.

Note, if $\sigma \to \infty$ then $\beta \to 1$ and, therefore, $p^* \to \infty$. Hence, threshold for investing increases when volatility grows.

2. Abandonment problem

Let X_t be the price (at time t) of the good produced by the firm, and function $\pi(x)$ describes a dependence of firm's current profit on current price x.

The optimal abandonment problem is, using the available information about current production prices X_t, to find a moment τ for terminating of firm's activity such that the net present value of the firm be maximal:

$$\mathbf{E}^x \left(\int_0^\tau \pi(X_t) e^{-\rho t} \, dt - A e^{-\rho \tau} \right) \to \max_\tau,$$

where $A \geq 0$ is the abandonment cost.

The above results allow to give the following necessary and sufficient conditions for optimality of threshold stopping time $\bar{\tau}_{x^*} = \inf\{t \geq 0 : X_t \leq x^*\}$ in an abandonment problem:

$$[A + R(x)]\varphi(x^*) \geq [A + R(x^*)]\varphi(x) \quad \text{for } x > x^*; \tag{18}$$

$$J(x^*)S'(x^*) = A\varphi'(x^*); \tag{19}$$

$$\mathbb{L}R(x) \geq \rho R(x) + \rho A \quad \text{for } x < x^*, \tag{20}$$

where $R(x) = \mathbf{E}^x \int_0^\infty \pi(X_t)e^{-\rho t}\, dt, \quad J(x) = \int_x^r \varphi(y)\pi(y)H(y)\, dy,$

$S'(x) = \exp\left\{-\int 2a(x)/\sigma^2(x)dx\right\}, \quad H(y) = 2/[\sigma^2(y)S'(y)],$

$\varphi(p)$ — decreasing solution to 'characteristic ODE'.

If profit $\pi(x)$ increases when current price rises, then from (19) we have $\pi(x^*) < 0$. It means that a firm should terminate an activity only when its revenue falls below some negative level.

Classic case

Let prices X_t is modelled by geometric Brownian motion with rate of drift $\alpha < 0$ and volatility σ, and $\pi(x) = x - c$. In this case, $R(x) = x/(\rho - \alpha) - c/\rho, \quad \varphi(x) = x^{\beta_1}$, where β_1 is the negative root of the equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta = \rho$.

The optimal threshold is $x^* = \dfrac{\beta_1}{\beta_1 - 1} \cdot \dfrac{\rho - \alpha}{\rho}(c - \rho A)$.

It is easy to see that $x^* < c - \rho A$ and conditions (18)–(20) hold.

Note, that if volatility σ tends to $+\infty$, then $\beta_1 \to 0$ and, therefore, $x^* \to 0$. It means that large volatility implies low level of price for shut down and more long period before abandonment of firm's activity, even though current profits $X_t - c$ will be negative.

Relationship between solutions to optimal stopping problem and free-boundary problem

A free-boundary problem for threshold case:
to find threshold p^*, $l < p^* < r$ and twice differentiable function $H(x)$, $l < x < p^*$, such that

$$\mathbb{L}H(x) = \rho H(x), \quad l < x < p^*; \tag{21}$$

$$H(p^* - 0) = g(p^*), \tag{22}$$

$$H'(p^* - 0) = g'(p^*). \tag{23}$$

Conditions (21)–(22) hold for the function

$$H(x) = h(p^*)\psi(x), \quad l < x < p^* \tag{24}$$

where $\psi(x)$ is an increasing solution to ODE (9) and $h(p) = g(p)/\psi(p)$.
Smooth pasting condition (23) at the point p^* is equivalent to $h'(p^*) = 0$.

On the other hand, the optimal threshold must be a point of maximum of the function $h(p)$.

Example. A solution to FB problem can not give a solution to OS problem.

Geometric Brownian motion $X_t = x \exp\{w_t\}$, payoff function $g(x)=(x-1)^3 + x^\delta$ for $x \geq 0$, and discount rate $\rho = \delta^2/2 \ \ (\delta > 1)$.

For this case the free-boundary problem is the following one:

$$\begin{cases} \frac{1}{2}x^2 H''(x) + \frac{1}{2}xH'(x) = \rho H(x), \qquad 0 < x < p^* \\ H(p^*) = g(p^*), \qquad H'(p^*) = g'(p^*) \end{cases} \tag{25}$$

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For $\delta \leq 3$ the free-boundary problem (25) has the unique solution:
$H(x) = x^\delta, \quad p^* = 1$, however, the stopping time τ_1 is not optimal.

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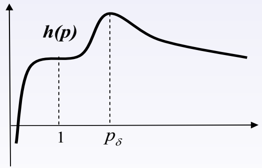
For $\delta > 3$ the free-boundary problem (25) has two solutions:

(a) $H(x){=}x^\delta,\ p^* = 1$

(b) $H(x){=}h(p_\delta)x^\delta,\ p^*{=}p_\delta{=}\delta/(\delta{-}3)$.



Note that $h(p_\delta) > 1$. Thus, the solution (a) does not give a solution to the optimal stopping problem, but (b) gives.

Relationship between solutions

Second-order conditions:

If τ_{p^*} is optimal stopping time over the class $\mathfrak{M}_l$ (induced by l-intervals), and function $g(x)$ is twice differentiable at the point p^*, then there exists a solution to free-boundary problem $(H(x), p^*)$ and $H''(p^* - 0) \geq g''(p^*)$

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and vice versa

If a pair $(H(x), p^*)$ is the unique solution to free-boundary problem, such that $H''(p^*-0)>g''(p^*)$, then τ_{p^*} is optimal in stopping problem over the class $\mathfrak{M}_l$.

But we can not say about the optimality of τ_{p^*} over class of all stopping times without the additional conditions on payoff function $g(x)$.

In this sense an optimality over threshold stopping times can be viewed as a minimal (or weakest) optimality which follows from the uniqueness of the solution to free-boundary problem.

References

- Peskir G., Shiryaev A. *Optimal stopping and free-boundary problems.* Birkhauser, 2006.
- Dixit A.K., Pindyck R.S. *Investment under Uncertainty.* Princeton: Princeton University Press, 1994.
- Alvarez L.H.R. Reward functionals, salvage values, and optimal stopping. — Mathematical Methods of Operations Research, 2001, v. 54, p. 315-337.
- Arkin V.I., Slastnikov A.D. *A variational approach to an optimal stopping problems for diffusion processes.* — Probability Theory and Applications, 2009, v. 53, No. 3.
- Arkin V.I., Slastnikov A.D. *Threshold Stopping Rules for Diffusion Processes and Stefan's Problem.* — Doklady Mathematics, 2012, v. 86, No. 2, pp. 1–4.
- F. Crocce, E. Mordecki. Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions. — Stochastics, 86 (3), 491–509, 2014.
- Villeneuve S. *On the threshold strategies and smooth-fit principle for optimal stopping problems.* — Journal of Applied Probability, 2007, v. 44, No. 1, pp. 181–198.

Thank you for your attention !