General one-dimensional diffusion: characterization, optimal stopping problem

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Main interest – optimal stopping problem for Markov processes.

The value function is a minimal excessive majorant of the payoff function.

How to find this majorant?

To give characterization of excessive functions.

To guess and use Verification Theorem.

Different approach in Presman (2013).

To construct the value function using sequential modification (approximation from below).
Without guessing and without Verification Theorem.

The optimal stopping problem for Wiener process on the finite interval is very simple.

One can formulate the algorithm in terms of concavity, boundary conditions and the function $E_x g(X_{\tau_{a,b}})$, where $\tau_{a,b}$ is the time of the first exit from the interval $]a, b[$.

For Wiener process the function $E_x g(X_{\tau_{a,b}})$ is linear.

It appears that in the general case the algorithm is the same, only the concavity must be changed to excessivity.

At first we give the known properties of general diffusion.

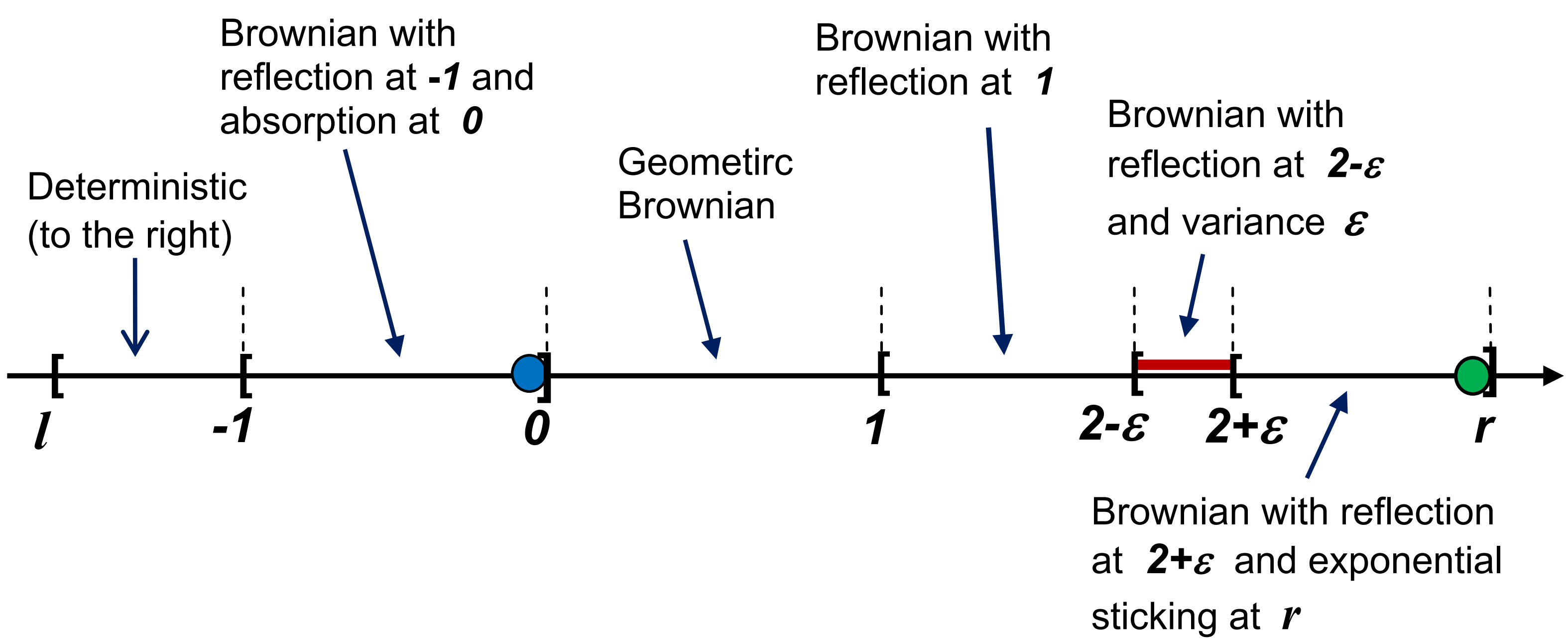
After that we give some results which might be known, but we failed to find them in literature.

After that we give some results about optimal stopping problem.

The general diffusion was investigated by Ito, McKean, Dynkin, Feller...

The general one-dimensional diffusion — is
a homogeneous in time strong Markov process $X_t\ (t \geq 0)$
(Markov family depending on initial state $X_0 = x \in I$)
with continuous trajectories, taking values in $I \bigcup e$
e is absorbing state, I is open (closed, half-open) interval I
(with left end $l \geq -\infty$, and right end $r \leq \infty$).

Example



τ_y — the time of the first hitting the state y,

$\zeta = \tau_e$ — the time of hitting the absorbing state e.

P_x (E_x) – probabilistic measure (expectation) in the set of continuous functions with initial state x
(direct product of that space and half-line corresponding to ζ).

The zero-one law is valid:

$$\forall x \in I \setminus l \;\; \exists \; \lim_{a \uparrow x} E_x \left[e^{-\tau_a} \right] = l_x^-, \text{ and either } l_x^- = 1 \text{ or } l_x^- = 0.$$

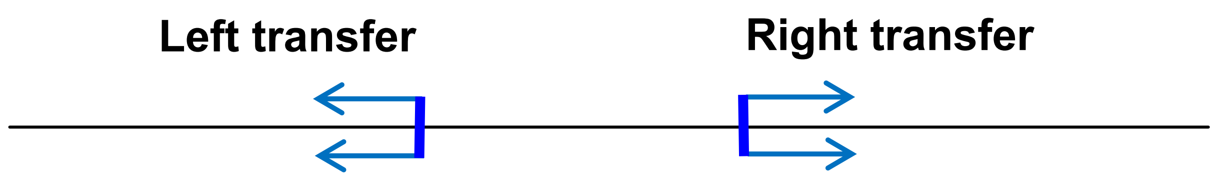
Analogously $\lim_{a \downarrow x} E_x \left[e^{-\tau_a} \right] = l_x^+$.

x is *regular* if $l_x^+ = l_x^- = 1$,

x is *a trap* if $l_x^+ = l_x^- = 0$,

x is *a left transfer* if $l_x^+ = 0, l_x^- = 1$,

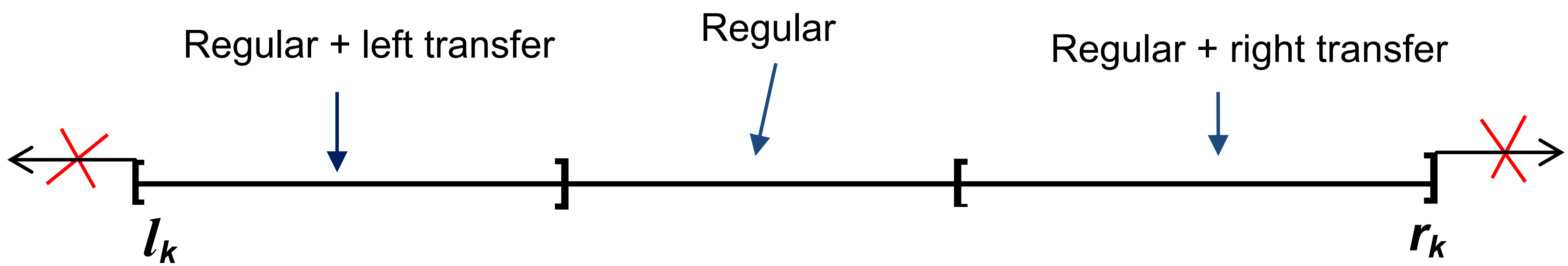
x is *a right transfer* if $l_x^+ = 1, l_x^- = 0$,



The set of regular points is open.

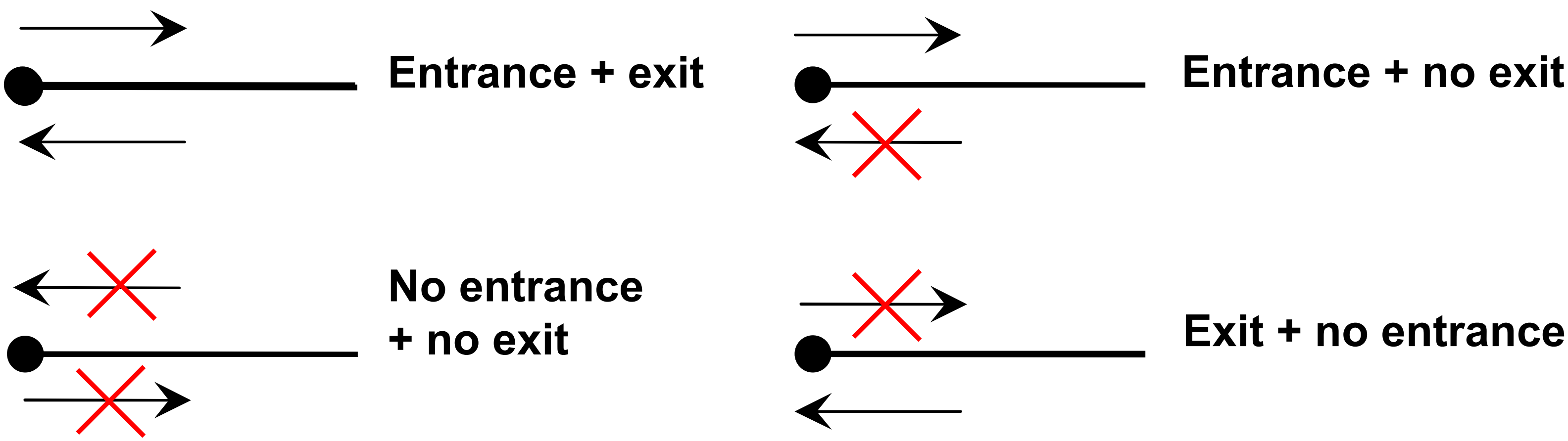
$I = \bigcup_{k=1}^{m} I_k$, where $I_k,\ k \le m \le \infty,$ are open (half-open, closed) intervals with the ends $l_k \le r_k$ such that:

1. for any interval $[a, b] \subset int\, I$: $[a, b] \bigcap I_k \ne \emptyset$ only for finite number of k,
2. for any $k \ne k'$: $I_k \bigcap I_{k'}$ is either empty or consists of one point which is a trap,
3. each I_k has the following structure:



Classifications of the boundaries

Left end l:



- entrance $+$ exit (regular)
- no entrance $+$ no exit (natural)

Analogously for the right end r.

One defines three functions for a general diffusion.

These functions completely define the diffusion.

1) strongly increasing continuous $s(x)$, which generates
a scale measure $S(dx)$, so that $S(]a,b]) = s(b) - s(a),\ l<a<b<r$;

2) strongly increasing continuous from the right $m(x)$, which generates
a speed measure $M(dx)$, so that $M(]a,b]) = m(b) - m(a),\ l<a<b<r$;

3) nondecreasing continuous from the right $k(x)$, which generates
a killing measure $K(dx)$, so that $K(]a,b]) = k(b) - k(a),\ l<a<b<r$.

The first one is defined uniquely up to a linear transformation.
The second and the third ones are defined uniquely up to an additive constant.

In what follows we consider regular diffusions.

The diffusion is conservative if the probability to reach e is equal to zero.

Consider a conservative regular diffusion.
For $l < a < x < b < r$ define

$$p_{b,a}(x) = P_x[\tau_b < \tau_a], \qquad e_{b,a}(x) = E_x[\tau_b \wedge \tau_a].$$

There exists a strongly increasing continuous function $s(x)$ – which does not depend on a and b – such that

$$p_{b,a}(x) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

The function $e_{b,a}(x)$ has strongly decreasing right (left) derivative with respect to $s(x)$, i. e.

$$\frac{d^{\pm} e_{b,a}}{dS}(x) = \lim_{v \downarrow 0} \frac{e_{b,a}(x \pm v) - e_{b,a}(x)}{s(x \pm v) - s(x)},$$

These derivatives does not depend on a and b and generate a positive measure $M(dx)$.

In general case for $l < a < x < b < r$ one consider a conditional diffusion on $[a, b]$ which is stopped at a and b under condition that $\tau_a \wedge \tau_b < \infty$. Such diffusion is conservative.

Corresponding to this conditional diffusion scale and speed measures are denoted by $S^*(dx), M^*(dx)$. Then

$$S(]c, d]) = \int_{]c,d]} \frac{S^*(dx)}{p(x)}, \qquad M(]c, d]) = \int_{]c,d]} \frac{M^*(dx)}{p^2(x)},$$

where $p(x) = P_x[\tau_a \wedge \tau_b < \infty]$.

It was shown that $p(x)$ have an increasing continuous from the right (left) right derivatives with respect to $s(x)$ which is denoted by $\frac{d^{\pm}p}{dS}(x)$.

Denote by $\frac{dp}{dS}(dx)$ corresponding measure. Then

$$K(]c, d]) = \int_{]c,d]} \frac{1}{p(x)} \frac{dp}{dS}(dx).$$

If l (r) is accessible then one needs to define $K(\{l\}), M(\{l\})$ ($K(\{r\}), M(\{r\})$

Let W_t be a Wiener process on $[l, r]$ with some boundary conditions at the ends of interval.

For almost all trajectories there exists a local time $T(t, x) = T(t, x, \omega)$.

For arbitrary measure $\tilde{M}(dx)$ consider

$$T(t) = \int_a^b T(t, v)\tilde{M}(dv).$$

Then the process $Y_t = W_{T^{-1}(t)}$ is a diffusion with the scale $\tilde{s}(x) = x$, and speed measure $\tilde{M}(dx)$.

For arbitrary $s(x)$ and $M(dx)$ one can choose $\tilde{M}(dx)$ such that the process $Z_t = s^{-1}(Y_t)$ with scale $s(x)$ and speed measure $M(dx)$.

For almost every trajectory of the process Z_t one can define a killing in such a way that the resulting process will have a scale $s(x)$, speed measure $M(dx)$ and killing measure $K(dx)$.

Transition operator of diffusion $\mathcal{P}_t f(x) = E_x f(X_t), \; f(e) = 0.$
Transition function $\mathcal{P}_t(x, A) = E_x \chi_A(X_t).$

$$\mathcal{P}_{h+t} = \mathcal{P}_h \mathcal{P}_t, \qquad \frac{\mathcal{P}_{h+t} - \mathcal{P}_t}{h} = \frac{\mathcal{P}_h - E}{h} \mathcal{P}_t.$$

$$\mathcal{G} f(x) = \lim_{h \to 0} \frac{\mathcal{P}_h f(x) - f(x)}{h}.$$

$$\frac{d\mathcal{P}_t}{dt} = \mathcal{G} \mathcal{P}_t, \qquad \mathcal{P}_t = e^{t\mathcal{G}}.$$

For strong markovian process and stopping time τ

$$E_x f(X_\tau) - f(x) = E_x \left[\int_0^\tau \mathcal{G} f(X_t) dt \right].$$

Classical domain of operator $\mathcal{G}$ consists of functions such that

1) Right (and left) derivative of $f(x)$ with respect to $s(x)$

$$\frac{d^+f}{dS}(x) = \lim_{v \downarrow 0} \frac{f(x+v) - f(x)}{s(x+v) - s(x)} \quad \left(\frac{d^-f}{dS}(x) = \lim_{v \downarrow 0} \frac{f(x) - f(x-v)}{s(x) - s(x-v)} \right)$$

exists and is a function of bounded variation on any interval $[a, b[\subset int\, I$.

For functions satisfying 1) consider operator L such that

$$Lf(x) - Lf(a) = \frac{d^+f}{dS}(x) - \frac{d^+f}{dS}(c) - \int_{]c,x]} f(v)K(dv).$$

2) The function $Lf(x)$ is absolutely continuous with respect to $M(dx)$, and Radon–Nikodym derivative $\frac{d}{dM}Lf(x)$ is continuous.

Then

$$\mathcal{G}f(x) = \frac{d}{dM}Lf(x).$$

If l (r) is accessible, then $\mathcal{G}$ must be defined also in l (r).

If $M(dx)$ and $K(dx)$ are absolutely continuous and $s'(x)$ is absolutely continuous, then the infinitesimal generator of X_t has a form

$$\frac{d}{dM}\left[\frac{d^+f}{dS}(x) - \frac{d^+f}{dS}(a) - \int_{]c,x]} f(v)K(dv)\right] = \frac{1}{m'(x)}\frac{d}{dx}\frac{f'(x)}{s'(x)} - \frac{k'(x)}{m'(x)}f(x)$$

$$= \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x) + b(x)\frac{d}{dx}f(x) - \rho(x)f(x),$$

where

$$\frac{\sigma^2(x)}{2} = \frac{1}{s'(x)m'(x)}, \qquad b(x) = -\frac{s''(x)}{m'(x)(s'(x))^2}, \qquad \rho(x) = \frac{k'(x)}{m'(x)},$$

$$\frac{s''(x)}{s'(x)} = -\frac{2b(x)}{\sigma^2(x)}, \qquad m'(x) = \frac{2}{\sigma^2(x)s'(x)}, \qquad k'(x) = \rho(x)m'(x).$$

The regularity means that $\mathrm{P}_x[\tau_y < \zeta] > 0$ for any $x, y \in int\, I$.

If the boundary point l (and/or r) is accessible from some inner point then it is accessible from *any* inner point and for correct definition of diffusion one needs to define the values of measures M and K in respective point.

Since the process X_t $(t \geq 0)$ is strong Markovian and has continuous paths, there exist continuous functions $h_1(x)$ and $h_2(x)$, $x \in int\, I$, such that

$$\mathrm{P}_x[\tau_y < \zeta] = \frac{h_1(x)}{h_1(y)} \text{ for } y > x, \quad \mathrm{P}_x[\tau_y < \zeta] = \frac{h_2(x)}{h_2(y)} \text{ for } y < x. \tag{1}$$

Obviously, $h_1(x) > 0$, and

– either $h_1(x)$ is strictly increasing for all $x \in int\, I$;

– or $h_1(x) = h_1(\bar{x})$ for $l < x < \bar{x}$, $h_1(x)$ is strictly increasing for $\bar{x} < x < r$ (with some $\bar{x} \leq r$).

Analogously, $h_2(x) > 0$ and either it is strictly decreasing for all $x \in int\, I$, or there exists $\hat{x} > l$ such that $h_2(x) = h_2(\hat{x})$ for $\hat{x} < x < r$ and $h_2(x)$ is strictly decreasing for $l < x < \hat{x}$.

If $\bar{x} > \hat{x}$ then $h_1(x) = h_2(x) \equiv 1$ and the optimal stopping time in the problem (3) is the time of the first hitting the set where $g(x)$ achieves its maximum.

In what follows we suppose that $\bar{x} \leq \hat{x}$.

The well-known formula for diffusion X_t:

$$\mathrm{E}_x e^{-\alpha\tau_y} = \begin{cases} \dfrac{\psi_\alpha(x)}{\psi_\alpha(y)} & \text{for } x \le y, \\ \dfrac{\varphi_\alpha(x)}{\varphi_\alpha(y)} & \text{for } y \le x, \end{cases} \qquad \alpha > 0, \tag{2}$$

where $\psi_\alpha(x)$ is continuous and strictly increasing, $\varphi_\alpha(x)$ is continuous and strictly decreasing.

Let us consider the process X_t^α with the same speed and scale measures $M(dx),\ S(dx)$ as for X_t, and killing measure $K^\alpha(dx) = K(dx) + \alpha M(dx)$. Then for the process X_t^α one can take

$$h_1^\alpha(x) = \psi_\alpha(x), \qquad h_2^\alpha(x) = \varphi_\alpha(x).$$

It means that (1) is more general than (2).

Consider the set $\mathcal{D}(\mathcal{G})$ of functions f, such that for all $x \in int\, I$ the limit $\dfrac{d^+f}{dS}(x) = \lim_{v \downarrow 0} \dfrac{f(x+v) - f(x)}{s(x+v) - s(x)}$ exists and is a function of bounded variation on any interval $[a, b[\subset int\, I$, The function $\dfrac{d^+f}{dS}(x)$ is right-continuous.

For functions from $\mathcal{D}(\mathcal{G})$ consider operator L such that

$$Lf(x) - Lf(a) = \frac{d^+f}{dS}(x) - \frac{d^+f}{dS}(c) - \int_{]c,x]} f(v) K(dv).$$

Let us denote:

$$D^+(\mathcal{G}) = \{f \in D(\mathcal{G}) : \text{ the function } Lf \text{ is not decreasing}\};$$
$$D^-(\mathcal{G}) = \{f \in \mathcal{D}(G) : \text{ the function } Lf \text{ is not increasing}\};$$
$$D^0(\mathcal{G}) = D^+(\mathcal{G}) \cap D^-(\mathcal{G}) = \{f \in D(\mathcal{G}) : Lf(x) \equiv 0\}.$$

Any function in $D(\mathcal{G})$ has a unique minimal representation as a difference of two functions from $D^+(\mathcal{G})$.

The following statements hold.

1. Functions $h_1(x)$ and $h_2(x)$ belong to $D^0(\mathcal{G})$, are linearly independent, and any function in $D^0(\mathcal{G})$ is a linear combination of $h_1(x)$ and $h_2(x)$.

2. A function f is excessive (for the process X_t) if and only if $f \in D^-(\mathcal{G})$. If the boundary point l (or/and r) is accessible then one needs to impose additional condition (conditions) at this point (points).

3. The value function $V(x)$ in optimal stopping problem is a minimal (with respect to functions from $D^-(\mathcal{G})$) majorant of $g(x)$.

A problem of optimal stopping consists in finding the value function

$$V(x) = \sup_{\tau} \mathrm{E}_x g(X_\tau), \tag{3}$$

and respective $\tau^*{=}\tau^*(x)$, such that $V(x) = \mathrm{E}_x g(X_{\tau^*})$,
where $\sup$ is taken over all stopping times τ.

The function $g(x)$ is supposed to be lower semi-continuous, $g(e) = 0$.

In the problem (3) the set $D = \{x : V(x)=g(x)\}$ is called *a stopping set*, the set $C = \{x : V(x)>g(x)\}$ is called *a continuation set*.

One says, that the solution of (3) has an threshold structure with right (left) threshold x^*, $l < x^* < r$, if $C = \{x \ : x \in I, x < x^*\}$ ($C = \{x \ : x \in I, x > x^*\}$).

For any y and $x \leq y$ one has an equality

$$\mathrm{E}_x g(X_{\tau_y}) = h_1(x, y) := \frac{h_1(x)}{h_1(y)} g(y).$$

Note that the function $h_1(x, y)$ is defined also for $x > y$.

Theorem 1. *The solution of* (3) *has an threshold structure with right threshold* x^* *iff*:

a) *function* $g(x)$ *is excessive for process* $X_t^* = X_{t \wedge \tau_{x^*}}$, $X_0^* = x \in]x^*, r[$;

b) $g(x) < \mathrm{E}_x g(X_{\tau_{x^*}}) = h_1(x, x^*)$ *for* $l < x < x^*$;

c) $\left.\dfrac{d^+ h_1(x, x^*)}{dS(x)}\right|_{x=x^*} \geq \dfrac{d^+ g}{dS}(x^*)$. (Generalized smooth fitting condition)

Denote

$$g_y(x) = \begin{cases} g(x) & \text{for } x \geq y, \\ \mathrm{E}_x g(X_{\tau_y}) = h_1(x,y) & \text{for } x \leq y. \end{cases}$$

Theorem 2. *Let* $g(x)$ *be excessive for* $x \geq x_1 > l$. *Then either* $g_{x_1}(x)$ *is excessive or there exist unique* $x^* > x_1$, *and* $\tilde{x} \geq x^*$, $\tilde{x} \leq r$, *such that:*

a) *the function* $g_{x^*}(x)$ *is excessive*;

b) $g_y(x) = g_{x^*}(x)$ *for* $x^* \leq y \leq \tilde{x}$, $l < x < r$;

c) $g_y(x) < g_{x^*}(x)$ *for* $x \vee x_1 < y$, $y \notin [x^*, \tilde{x}]$;

d) *for* $x < y$ *the function* $g_y(x)$ *as a function of* y *is not decreasing for* $x_1 < y < x^*$, *and is not increasing for* $y > \tilde{x}$,

$$\left.\frac{d^+ h_1(x,y)}{dS(x)}\right|_{x=y} < \frac{d^+ g}{dS}(y) \text{ for } x_1 < y < x^* \text{ and}$$

$$\left.\frac{d^+ h_1(x,y)}{dS(x)}\right|_{x=y} \geq \frac{d^+ g}{dS}(y) \text{ for } y > \tilde{x}.$$

(Some conditions under which the solution to optimal stopping problem has a threshold structure for less general one-dimensional diffusions were obtained by *Alvarez, Arkin, Arkin&Slastnikov, Crocce&Mordecki, Dayanik&Karatzas, Presman, Salminen, Villeneuve* et al.)

One says, that the solution of (3) has an island structure with ends a, b, $(l<a<b<r)$, if $C =]a, b[$.

For any $x \in [a, b]$ one has an equality

$$\mathrm{E}_x g(X_{\tau_{ab}}) = h(x, a, b)$$
$$:= \frac{h_1(x)[g(a)h_2(b)-g(b)h_2(a)]-h_2(x)[g(a)h_1(b)-g(b)h_1(a)]}{h_1(a)h_2(b) - h_1(b)h_2(a)}.$$

Note that the function $h(x, a, b)$ is defined for all $x,\ l \leq x \leq r$.

Theorem 3. *The solution of* (3) *has an island structure with ends* a, b *iff*:

a) *the function* $g(x)$ *is excessive for the process* $X_t^a = X_{t\wedge\tau_a}$, $X_0^a < a$ *and for the process* $X_t^b = X_{t\wedge\tau_b}$, $X_0^b > b$;

b) $g(x) < \mathrm{E}_x g(X_{\tau_{ab}}) = h(x, a, b)$ *for* $a < x < b$;

c) $\left.\dfrac{d^- h(x, a, b}{dS(x)}\right|_{x=a} \leq \dfrac{d^- g}{dS}(a)$, $\left.\dfrac{d^+ h(x, a, b)}{dS(x)}\right|_{x=b} \geq \dfrac{d^+ g}{dS}(b)$.

Denote

$$g_{[a,b]}(x) = \begin{cases} g(x) & \text{for } x \notin [a,b], \\ \mathrm{E}_x g(X_{\tau_y}) = h(x,a,b) & \text{for } x \in [a,b]. \end{cases}$$

Theorem 4. *Let* $g(x)$ *is excessive for* $x \notin [a_1, b_1]$. *Then either* $g_{[a_1,b_1]}(x)$ *is excessive or there exist unique* $\tilde{a},\ a^*,\ b^*,\ \tilde{b}$ *such that* $l \le \tilde{a} \le a^* \le a_1$, $b_1 \le b^* \le \tilde{b} \le r,\ \ (a_1 - a^*) \vee (b^* - b_1) > 0$, *and :*

a) *the function* $g_{[a^*,b^*]}(x)$ *is excessive*;

b) $g_{[a,b]}(x) = g_{[a^*,b^*]}(x)$ *for* $a \in [\tilde{a}, a^*],\ b \in [b^*, \tilde{b}],\ l < x < r$;

c)

d)

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Thank you for your attention