Angers, 1st September 2015

Advanced Methods in Mathematical Finance

Monique Jeanblanc, LaMME, Université d'Évry-Val-D'Essonne

Joint work with S. Ankirchner, C. Blanchet-Scalliet and R. Romo.

Enlargement of filtration in discrete time

Financial support from Fédération Bancaire Française

The general problem of enlargement of filtration is the following one : let X be an $\mathbb{F}$-martingale and $\mathbb{G}$ a filtration larger that $\mathbb{F}$. Find conditions such that X is a $\mathbb{G}$-semimartingale and then, give the $\mathbb{G}$-semimartingale decomposition of the process X, i.e. write X as the sum of a $\mathbb{G}$-martingale and a $\mathbb{G}$-predictable bounded variation process.

This is important in finance to exclude arbitrages (for example while studying insider trading) and to study the impact of new information when solving an optimization problem on consumption/ terminal wealth.

Some results are known from the 70's in continuous time, however, in that setting the proofs are not trivial, and one needs to assume specific hypotheses to give a positive answer. Furthermore, many cases are still not solved.

Many references and books are available in continuous time (Jeulin, Jacod, Mansuy and Yor, Yor, Protter)

Recent theses with applications to finance: Amendinger, Ankirchner, Aksamit, Deng, Kreher, Falafala. See also papers by these authors and by Acciaio et al., Coculescu et al., Herdegen and Hermann, Kchia and Protter (all in continuous time).

Some results (presented as particular cases of continuous time) in a discrete time setting can be found in Deng's thesis and the related paper *Tahir Choulli and Jun Deng : Non-arbitrage for Informational Discrete Time Market Models*, available on Arxiv.

In this talk, based on work in progress with Ankirchner, Blanchet-Scalliet and Romo, we study enlargement of filtration in discrete time.

Our goal is to compute more explicitly the semimartingale decomposition, and to show, with elementary computation, that we recover the classical general formula established in the literature in continuous time.
We also study particular case of immersion in a progressive enlargement framework, immersion, pseudo-stopping times and give some results on arbitrages opportunities.

The interest is mainly from a pedagogical point of view.

Basic Facts

Doob's decomposition: Any discrete time process is a semimartingale in any filtration for which it is adapted: $X = M^{\mathbb{F}} + V^{\mathbb{F}}$ where $M^{\mathbb{F}}$ is an $\mathbb{F}$-martingale and $V^{\mathbb{F}}$ is $\mathbb{F}$-predictable, $V_0^{\mathbb{F}} = 0$ and

$$\Delta V_n^{\mathbb{F}} := V_n^{\mathbb{F}} - V_{n-1}^{\mathbb{F}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1}) \, .$$

Setting $V^{\mathbb{F}}$ as above, it remains to check that $M^{\mathbb{F}}$ is a martingale. Note that

$$\Delta M_n^{\mathbb{F}} := M_n^{\mathbb{F}} - M_{n-1}^{\mathbb{F}} = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1}) \, .$$

If X is an $\mathbb{F}$-martingale, and $\mathbb{G}$ any filtration such that $\mathbb{F} \subset \mathbb{G}$, it is a $\mathbb{G}$-semimartingale with decomposition $X = M^{\mathbb{G}} + V^{\mathbb{G}}$ where $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $V^{\mathbb{G}}$ is $\mathbb{G}$-predictable with

$$\Delta V_n^{\mathbb{G}} := V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1}) = \mathbb{E}(\Delta X_n|\mathcal{G}_{n-1})$$

Basic Facts

Doob's decomposition: Any discrete time process is a semimartingale in any filtration for which it is adapted: $X = M^{\mathbb{F}} + V^{\mathbb{F}}$ where $M^{\mathbb{F}}$ is an $\mathbb{F}$-martingale and $V^{\mathbb{F}}$ is $\mathbb{F}$-predictable, $V_0^{\mathbb{F}} = 0$ and

$$\Delta V_n^{\mathbb{F}} := V_n^{\mathbb{F}} - V_{n-1}^{\mathbb{F}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1}) \,.$$

Setting $V^{\mathbb{F}}$ as above, it remains to check that $M^{\mathbb{F}}$ is a martingale. Note that

$$\Delta M_n^{\mathbb{F}} := M_n^{\mathbb{F}} - M_{n-1}^{\mathbb{F}} = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1}) \,.$$

If X is an $\mathbb{F}$-martingale, and $\mathbb{G}$ any filtration such that $\mathbb{F} \subset \mathbb{G}$, it is a $\mathbb{G}$-semimartingale with decomposition $X = M^{\mathbb{G}} + V^{\mathbb{G}}$ where $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $V^{\mathbb{G}}$ is $\mathbb{G}$-predictable with

$$\Delta V_n^{\mathbb{G}} := V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1}) = \mathbb{E}(\Delta X_n|\mathcal{G}_{n-1})$$

Predictable bracket of two martingales

If X and Y are two $\mathbb{F}$-martingales, there exists an $\mathbb{F}$-predictable process K such that $XY - K$ is a martingale and

$$\Delta K_n = \mathbb{E}(Y_n \Delta X_n | \mathcal{F}_{n-1}) = \mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{F}_{n-1})$$

Indeed, from Doob's decomposition the predictable part of the semimartingale XY is $\Delta K_n = \mathbb{E}(X_n Y_n - X_{n-1} Y_{n-1} | \mathcal{F}_{n-1}$.

Predictable bracket of two martingales

If X and Y are two $\mathbb{F}$ martingales, there exists an $\mathbb{F}$-predictable process K such that $XY - K$ is a martingale and

$$\Delta K_n = \mathbb{E}(Y_n \Delta X_n | \mathcal{F}_{n-1}) = \mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{F}_{n-1})$$

Indeed, from Doob's decomposition , XY is an $\mathbb{F}$-semimartingale with predictable part $\Delta K_n = \mathbb{E}(X_n Y_n - X_{n-1} Y_{n-1} | \mathcal{F}_{n-1})$.

The covariation process of two processes $[X, Y]$ is defined by $\Delta [X, Y]_n = \Delta Y_n \Delta X_n$.

Predictable bracket of two semi martingales The predictable bracket of two semimartingales X, Y is defined as the dual predictable projection of the covariation process $\langle X, Y\rangle = [X, Y]^p$.

For discrete time semimartingales, $[X, Y]_n = \sum_{k=1}^n \Delta X_k \, \Delta Y_k$, and $[X, Y]^p$ is the only predictable (bounded variation) process such that $[X, Y] - [X, Y]^p$ is a martingale, i.e. $[X, Y]^p$ is the predictable part of $[X, Y]$.

From Doob's decomposition

$$(\Delta[X, Y]^p)_n = \mathbb{E}([X, Y]_n - [X, Y]_{n-1} | \mathcal{F}_{n-1}) = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{F}_{n-1})$$

Then,

$$\Delta\langle X, Y\rangle^{\mathbb{F}}_n = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{F}_{n-1}) \,.$$

There are, in continuous time, mainly two kinds of enlargement

Initial enlargement, where L is a random variable and

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(L) .$$

Progressive enlargement, where τ is a positive random variable and

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(t \wedge \tau) .$$

Initial Enlargement

Initial enlargement: an example (Bridge)

Let $X_n = \sum_{k=1}^n Y_k$, where $(Y_k, k \geq 1)$ are i.i.d. centered, be a martingale (an $\mathbb{F}^X$-martingale) and let $\mathcal{G}_n := \mathcal{F}_n^X \vee \sigma(X_N)$, for $n \leq N$.

We need to compute $\Delta A_n = \mathbb{E}(\Delta X_n | \mathcal{F}_{n-1} \vee \sigma(X_N))$. Using the fact that $(Y_k, k \leq N)$ are i.i.d, we have that for any $j \geq n$

$$(Y_j, X_1, \cdots, X_{n-1}, X_N) \stackrel{\text{loi}}{=} (Y_n, X_1, \cdots, X_{n-1}, X_N)$$

hence

$$\begin{aligned}
\mathbb{E}(Y_n | \mathcal{F}_{n-1} \vee \sigma(X_N)) &= \mathbb{E}(Y_j | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)} \mathbb{E}(Y_n + \cdots + Y_j + \cdots + Y_N | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)} \mathbb{E}(X_N - X_{n-1} | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{X_N - X_{n-1}}{N-(n-1)}
\end{aligned}$$

Hence,

$$X_n - \sum_{k=1}^{n} \frac{X_N - X_{k-1}}{N - (k-1)}$$

is a $\mathbb{G}$-martingale.

In continuous time: Brownian bridge. For $\mathbb{G} = \mathbb{F} \vee \sigma(B_1)$,

$$B_t = B_t^{\mathbb{G}} + \int_0^t \frac{B_1 - B_s}{1-s} ds, \; t \leq 1$$

Hence,

$$X_n - \sum_{k=1}^{n} \frac{X_N - X_{k-1}}{N - (k-1)}$$

is a $\mathbb{G}$-martingale.

In continuous time: Brownian bridge. For $\mathbb{G} = \mathbb{F} \vee \sigma(B_1)$,

$$B_t = B_t^{\mathbb{G}} + \int_0^t \frac{B_1 - B_s}{1-s} ds, \; t \leq 1$$

Initial enlargement: another example

Let X be a martingale, L be a r.v. taking values in $\mathbb{Z}$ and $p_n(j) = \mathbb{P}(L = j | \mathcal{F}_n^X)$. Define $\mathcal{G}_n = \mathcal{F}_n^X \vee \sigma(L)$.

Then $\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n | \mathcal{F}_n \vee \sigma(L))$ and

$$\begin{aligned}
\Delta V_n^{\mathbb{G}} \mathbb{1}_{\{L=j\}} &= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta X_n \mathbb{1}_{\{L=j\}} | \mathcal{F}_{n-1})}{\mathbb{P}(L = j | \mathcal{F}_{n-1})} = \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta X_n \mathbb{E}(\mathbb{1}_{\{L=j\}} | \mathcal{F}_n) | \mathcal{F}_{n-1})}{p_{n-1}(j)} \\
&= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta \langle X, p(j) \rangle_n | \mathcal{F}_{n-1})}{p_{n-1}(j)} .
\end{aligned}$$

It can be proved that, on the set $\{L = j\}$, one has $p_{n-1}(j) \neq 0, \forall n$.

In continuous time, under Jacod's hypothesis $\mathbb{P}(\tau \in du | \mathcal{F}_t) = p_t(u) \mathbb{P}(\tau \in du)$

$$X_t = X_t^{\mathbb{G}} + \int \frac{d\langle X, p(u) \rangle_s}{p_{s-}(u)} |_{L=u}$$

Initial enlargement: another example

Let X be an $\mathbb{F}$-martingale, L be a r.v. taking values in $\mathbb{Z}$ and $p_n(j) = \mathbb{P}(L = j|\mathcal{F}_n^X)$ and let $\mathcal{G}_n = \mathcal{F}_n^X \vee \sigma(L)$.

Then $X = X^{\mathbb{G}} + V^{\mathbb{G}}$ where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n|\mathcal{F}_n \vee \sigma(L))$ and

$$\begin{aligned}\Delta V_n^{\mathbb{G}} \mathbb{1}_{\{L=j\}} &= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta X_n \mathbb{1}_{\{L=j\}}|\mathcal{F}_{n-1})}{\mathbb{P}(L = j|\mathcal{F}_{n-1})} \\ &= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(p_n(j)\Delta X_n|\mathcal{F}_n)}{p_{n-1}(j)} = \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta\langle X, p(j)\rangle_n|\mathcal{F}_{n-1})}{p_{n-1}(j)} .\end{aligned}$$

It can be proved that, on the set $\{L = j\}$, one has $p_{n-1}(j) \neq 0, \forall n$.

In continuous time, under Jacod's hypothesis $\mathbb{P}(L \in du|\mathcal{F}_t) = p_t(u)\mathbb{P}(L \in du)$

$$X_t = X_t^{\mathbb{G}} + \int_0^t \frac{d\langle X, p(u)\rangle_s|_{L=u}}{p_{s-}(L)}$$

Progressive Enlargement

Progressive enlargement

We assume here that τ is a random variable valued in $\mathbb{N} \cup \{+\infty\}$, and introduce

$$\mathcal{G}_n := \mathcal{F}_n \vee \sigma(\tau \wedge n) \, .$$

If $Y_n \in \mathcal{G}_n$, then there exists $y_n \in \mathcal{F}_n$ such that $Y_n 1\!\!1_{\{n<\tau\}} = y_n 1\!\!1_{\{n<\tau\}}$. Any $\mathbb{G}$-predictable process can be written as

$$V_n = V_n^b 1\!\!1_{\{n \le \tau\}} + V_n^a(\tau) 1\!\!1_{\{\tau < n\}}$$

where, V^b is $\mathbb{F}$-predictable and for any u, $V^a(u)$ is $\mathbb{F}$-predictable.

We introduce two supermartingales

$$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \ge n | \mathcal{F}_n)$$

and the Doob-Meyer decomposition of $\widetilde{Z} = \widetilde{M} - \widetilde{A}$ where M is an $\mathbb{F}$-martingale and $\widetilde{A}$ an $\mathbb{F}$-predictable increasing process.

We shall use the trivial equalities $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_n), Z_n = \mathbb{P}(\tau \geq n+1|\mathcal{F}_n)$.

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive.

The proof follows from simple arguments

$$\begin{aligned} \mathbb{E}(1\!\!1_{\{n\leq\tau\}} 1\!\!1_{\{Z_{n-1}=0\}}) &= \mathbb{E}(\mathbb{P}(n \leq \tau|\mathcal{F}_{n-1}) 1\!\!1_{\{Z_{n-1}=0\}}) \\ &= \mathbb{E}(\mathbb{P}(n-1 < \tau|\mathcal{F}_{n-1}) 1\!\!1_{\{Z_{n-1}=0\}}) = \mathbb{E}(Z_{n-1} 1\!\!1_{\{Z_{n-1}=0\}}) = 0 \end{aligned}$$

On the set $\{n > \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1.

We shall use the trivial equalities $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_n), Z_n = \mathbb{P}(\tau \geq n+1|\mathcal{F}_n)$.

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive.

The proof follows from simple arguments

$$\begin{aligned}\mathbb{E}(1\!\!1_{\{n\leq\tau\}}1\!\!1_{\{Z_{n-1}=0\}}) &= \mathbb{E}(\mathbb{P}(n \leq \tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}}) \\ &= \mathbb{E}(\mathbb{P}(n-1 < \tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}}) = \mathbb{E}(Z_{n-1}1\!\!1_{\{Z_{n-1}=0\}}) = 0\end{aligned}$$

On the set $\{n > \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1.

We shall use the trivial equalities $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_n), Z_n = \mathbb{P}(\tau \geq n+1|\mathcal{F}_n)$.

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive.

The proof follows from simple arguments

$$\begin{aligned} \mathbb{E}(1\!\!1_{\{n\leq\tau\}}1\!\!1_{\{Z_{n-1}=0\}}) &= \mathbb{E}(\mathbb{P}(n\leq\tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}}) \\ &= \mathbb{E}(\mathbb{P}(n-1<\tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}})\mathbb{E}(Z_{n-1}1\!\!1_{\{Z_{n-1}=0\}}) = 0 \end{aligned}$$

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive. On the set $\{n > \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1.

For any random time τ, if Y is integrable

$$\mathbb{E}(Y|\mathcal{G}_n)\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n)}{Z_n}$$

Indeed, on $\{n < \tau\}$, any $\mathcal{G}_n$ measurable random variable Y_n is equal to an $\mathcal{F}_n$ measurable random variable so that, for $Y_n\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}y_n$ which leads to

$$\mathbb{E}(Y_n\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n))y_n$$

One has, for any random time τ, if Y is integrable

$$\mathbb{E}(Y|\mathcal{G}_n)\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n)}{Z_n}$$

Indeed, on $\{n < \tau\}$, any $\mathcal{G}_n$ measurable random variable Y_n is equal to an $\mathcal{F}_n$ measurable random variable so that, for $Y_n\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}y_n$ which leads to

$$\mathbb{E}(Y_n\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n)y_n$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

For $Y_n \in \mathcal{F}_n$

$$\begin{aligned} \mathbb{E}(Y_n \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \widetilde{Z}_n | \mathcal{F}_{n-1}) \\ \mathbb{E}(\frac{Y_n}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \mathbb{1}_{\{\widetilde{Z}_n > 0\}} | \mathcal{F}_{n-1}) \, . \end{aligned}$$

Only the second equality requires a proof

$$\begin{aligned} \mathbb{E}(\frac{Y_n}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \frac{1}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{F}_{n-1}) \\ &= \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \frac{1}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} \mathbb{1} \widetilde{Z}_n > 0 | \mathcal{F}_{n-1}) = \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \mathbb{1}_{\{\widetilde{Z}_n \end{aligned}$$

$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n)$

For $Y_n \in \mathcal{F}_n$

$$\begin{aligned} \mathbb{E}(Y_n|\mathcal{G}_{n-1})1\!\!1_{\{\tau\geq n\}} &= 1\!\!1_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\widetilde{Z}_n|\mathcal{F}_{n-1}) \\ \mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}1\!\!1_{\{\tau\geq n\}}|\mathcal{G}_{n-1}) &= 1\!\!1_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n 1\!\!1_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,. \end{aligned}$$

Only the second equality requires a proof

$$\begin{aligned} \mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}1\!\!1_{\{\tau\geq n\}}|\mathcal{G}_{n-1}) &= 1\!\!1_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}1\!\!1_{\{\tau\geq n\}}|\mathcal{F}_{n-1}) \\ &= \frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}1\!\!1_{\{\tau\geq n\}}1\!\!1_{\widetilde{Z}_n>0}|\mathcal{F}_{n-1}) \\ &= 1\!\!1_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n 1\!\!1_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,. \end{aligned}$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Arbitrages

If $Z > 0$, there are no arbitrages before τ.

We prove that if S is an $\mathbb{F}$-martingale, then, there exists a positive $\mathbb{G}$-martingale L such that $S^\tau L$ is a local martingale.

The process

$$L_n = \prod_{k=1}^{n} (1 + \Delta N_k) = L_{n-1}(1 + \Delta N_n)$$

where $\Delta N_k = 1\!\!1_{\tau \geq k}\left(\left(Z_{k-1}/\widetilde{Z}_k\right) - 1\right)$ is a positive $\mathbb{G}$-martingale.

Indeed,

$$\begin{aligned}
\mathbb{E}(1 + \Delta N_n | \mathcal{G}_{n-1}) &= 1 + \mathbb{E}(1\!\!1_{\tau \geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n} - 1)|\mathcal{G}_{n-1}) \\
&= 1 + 1\!\!1_{\tau > n-1} \frac{1}{Z_{n-1}} \mathbb{E}(1\!\!1_{\tau \geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n} - 1)|\mathcal{F}_{n-1}) \\
&= 1 + 1\!\!1_{\tau > n-1} \frac{1}{Z_{n-1}} \left(\mathbb{E}(\widetilde{Z}_n \frac{Z_{n-1}}{\widetilde{Z}_n} |\mathcal{F}_{n-1}) - Z_{n-1}\right) = 1
\end{aligned}$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$, $\quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

The process $S^\tau L$ is a $(\mathbb{G},\mathbb{P})$ martingale.

Indeed

$$\begin{aligned}
&\mathbb{E}(S_{(n+1)\wedge\tau}(1+1\!\!1_{\tau\geq n+1}(\frac{Z_n}{\widetilde{Z}_{n+1}}-1))|\mathcal{G}_n) \\
&= \mathbb{E}(S_{n+1}1\!\!1_{\tau\geq n+1}(1+\frac{Z_n}{\widetilde{Z}_{n+1}}-1)|\mathcal{G}_n) + \mathbb{E}(S_\tau 1\!\!1_{\tau<n+1}|\mathcal{G}_n) \\
&= \mathbb{E}(S_{n+1}1\!\!1_{\tau\geq n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{G}_n) + S_\tau 1\!\!1_{\tau<n+1} \\
&= 1\!\!1_{\tau>n}\frac{1}{Z_n}\mathbb{E}(S_{n+1}\widetilde{Z}_{n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{F}_n) + S_\tau 1\!\!1_{\tau\leq n} = S_{n\wedge\tau}
\end{aligned}$$

A necessary and sufficient condition can be found in Choulli-Deng for any S satisfying NA($\mathbb{F}$), S satisfies NA($\mathbb{G}$) is equivalent to $\{\tilde{Z}_n=0\}=\{Z_{n-1}=0\}$

$Z_n=\mathbb{P}(\tau>n|\mathcal{F}_n), \quad \widetilde{Z}_n=\mathbb{P}(\tau\geq n|\mathcal{F}_n)$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n\wedge\tau}$ is a $\mathbb{G}$-martingale, where $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}}$.

Assume now that $\tau = \inf\{n : \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$. If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$.

If Γ is not predictable,

$$\Delta\Lambda_n = 1 - \mathbb{E}(e^{-\Delta\Gamma_n}|\mathcal{F}_{n-1})$$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n\wedge\tau}$ is a $\mathbb{G}$ martingale, where $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}}$.

Assume now that $\tau = \inf\{n : \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$.

If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$

If Γ is not predictable,

$$\Delta\Lambda_n = e^{-\Gamma_{n-1}} \frac{1 - \mathbb{E}(e^{-\Delta\Gamma_n}|\mathcal{F}_{n-1})}{Z_{n-1}}$$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n \wedge \tau}$ is a $\mathbb{G}$ martingale, where $\Delta\Lambda_n = -\frac{\Delta Z_n}{Z_{n-1}}$.

Assume now that $\tau = \inf\{n : \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$.

If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$

If Γ is not predictable,

$$\Delta\Lambda_n = e^{-\Gamma_{n-1}} \frac{1 - \mathbb{E}(e^{-\Delta\Gamma_n} | \mathcal{F}_{n-1})}{Z_{n-1}}$$

$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n)$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n\wedge\tau}$ is a $\mathbb{G}$ martingale, where $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}}$.

Case of Cox Model: $\tau = \inf\{n : \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$.

If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$

If Γ is not predictable,

$$\Delta\Lambda_n = 1 - \mathbb{E}(e^{-\Delta\Gamma_n}|\mathcal{F}_{n-1})$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$, $\quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Immersion in progressive enlargement

$\mathbb{F}$ is immersed in $\mathbb{G}$ iff any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale, or equivalently if $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_\infty) = \mathbb{P}(\tau > n|\mathcal{F}_k)$ for any $k \geq n$.

This is the case in the Cox Model.

$\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Then,

$$\begin{aligned}\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n) \quad &=: \quad \mathbb{P}(\tau > n-1|\mathcal{F}_n) = \mathbb{P}(\tau > n-1|\mathcal{F}_{n-1}) = \mathbb{P}(\tau > n-1|\mathcal{F}_\infty) \\ &= \quad \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)\end{aligned}$$

where the third and the next to last equality follow from immersion assumption. The third equality establishes the predictability of $\widetilde{Z}$. Note that one has $\widetilde{Z}_n = Z_{n-1}$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Assume now that $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_\infty)$. Then, $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_{n-1})$ and

$$\begin{aligned} \mathbb{P}(\tau > n | \mathcal{F}_n) &= \mathbb{P}(\tau \geq n+1 | \mathcal{F}_n) = \widetilde{Z}_{n+1} \\ &= \mathbb{P}(\tau > n | \mathcal{F}_\infty) \end{aligned}$$

Suppose $\mathbb{F} \hookrightarrow \mathbb{G}$. Then the following assertions are equivalent

(i) Z is $\mathbb{F}$-predictable.

(ii) $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = 0$ for all $n \geq 0$.

(iii) $\mathbb{E}\Big(\sum_{k=1}^n U_k \Delta N_k \Big| \mathcal{F}_n\Big) = 0$; for any $\mathbb{G}$-predictable process U.

(iv) Any $\mathbb{F}$-martingale X is orthogonal to N.

Pseudo-stopping times We assume that $\mathcal{F}_0$ is trivial. A finite random time τ is an $\mathbb{F}$-pseudo stopping time if $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$ for any bounded $\mathbb{F}$-martingale X (see [**?**]). We denote by H^o the $\mathbb{F}$ dual optional projection of H

$$H^o_n := \sum_{k=1}^{n} \mathbb{E}(\Delta H_k | \mathcal{F}_k) = \sum_{k=1}^{n} \mathbb{P}(\tau = k | \mathcal{F}_k)\ , \quad \forall n \geq 1,\ H^0_0 = 0,$$

and introduce the martingale $\mu_n := \mathbb{E}(H^o_\infty | \mathcal{F}_n) = Z_n + H^o_n$. Note that $\mu_0 = Z_0$.

The following statements are equivalent:

(i) τ is an $\mathbb{F}$ pseudo-stopping time.

(ii) $\mu = Z_0$.

(iii) $\widetilde{Z}$ is predictable.

(iv) $H^0_\infty = Z_0$.

(v) Every $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.

$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n)$

Semi martingale decomposition, Before τ

Any $\mathbb{F}$-martingale X stopped at τ is a $\mathbb{G}$-semimartingale with decomposition

$$X^\tau = X^{\mathbb{G}} + \sum_{k=0}^{\cdot\wedge\tau} \frac{1}{Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k$$

where $\widetilde{Z} = \widetilde{M} - \widetilde{A}$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

The $\mathbb{G}$ predictable part of the $\mathbb{G}$ semimartingale X is $V^{\mathbb{G}}$ with $\Delta V_n^{\mathbb{G}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1})$. We apply previous results on the set $n-1 < \tau$,

$$
\begin{aligned}
\mathbb{1}_{\{\tau>n-1\}}(V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}}) &= \mathbb{1}_{\{\tau>n-1\}}\mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\{\tau>n-1\}}(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>n-1\}}|\mathcal{F}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1})\,.
\end{aligned}
$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Using now the Doob-Meyer decomposition of $\widetilde{Z}$, and the martingale property of X, we obtain

$$\begin{aligned}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) &= \mathbb{E}((\widetilde{M}_n - \widetilde{A}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(\widetilde{M}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) = \Delta\langle \widetilde{M}, X\rangle_n\end{aligned}$$

and finally

$$\mathbb{1}_{\{\tau>n-1\}}(V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}}) = \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\Delta\langle \widetilde{M}, X\rangle_n\,.$$

In continuous time

$$X^\tau = X^{\mathbb{G}} + \int_0^{\cdot\wedge\tau} \frac{1}{Z_{s-}} d\langle X, \widetilde{M}\rangle$$

where $\widetilde{M} = \widetilde{Z} - A^0_-$, and A^0 is the dual optional projection of $\mathbb{1}_{\tau\leq t}$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Using now the Doob-Meyer decomposition of $\widetilde{Z}$, and the martingale property of X, we obtain

$$\begin{aligned}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) &= \mathbb{E}((\widetilde{M}_n - \widetilde{A}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(\widetilde{M}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) = \Delta\langle \widetilde{M}, X\rangle_n\end{aligned}$$

and finally

$$\mathbb{1}_{\{\tau>n-1\}}(A_n^{\mathbb{G}} - A_{n-1}^{\mathbb{G}}) = \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\Delta\langle \widetilde{M}, X\rangle_n\,.$$

In continuous time

$$X^\tau = X^{\mathbb{G}} + \int_0^{\cdot\wedge\tau}\frac{1}{Z_{s-}}d\langle X, \tilde{M}\rangle_s$$

where $\widetilde{M} = \widetilde{Z} - A^0_-$, *with* A^0 *being the* $\mathbb{F}$-*dual optional projection of* $\mathbb{1}_{\tau\leq t}$ *and* $\widetilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

After τ, Honest times

We now consider the case where τ is honest (and valued in $\mathbb{N}$). We recall the definition and some of the main properties.

A random time is honest, if, for any $n \in \mathbb{N}$, there exists an $\mathcal{F}_n$-measurable random variable $\tau(n)$ such that

$$\mathbb{1}_{\{\tau \leq n\}} \tau = \mathbb{1}_{\{\tau \leq n\}} \tau(n)$$

or equivalently if there exists $\widehat{\tau}(n)$ such that

$$\mathbb{1}_{\{\tau < n\}} \tau = \mathbb{1}_{\{\tau < n\}} \widehat{\tau}(n)$$

It follows that any $\mathbb{G}$-predictable process V can be written as
$V_n = V_n^b \mathbb{1}_{\{n \leq \tau\}} + V_n^a \mathbb{1}_{\{\tau < n\}}$ where V^a, V^b are $\mathbb{F}$-predictable processes.

If τ is honest, $Z_n = \tilde{Z}_n$ on the set $n > \tau$. Furthermore, τ is honest if and only if $\widetilde{Z}_\tau = 1$

For any n,

$$\begin{aligned}
\mathbb{P}(\tau = n|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} &= \mathbb{P}(\tau = n|\mathcal{F}_n)1\!\!1_{\{n>\tau;n>\tau(n)\}} = \mathbb{E}(1\!\!1_{\{\tau=n\}}1\!\!1_{\{n>\tau(n)\}}|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} \\
&= \mathbb{E}(1\!\!1_{\{\tau=n\}}1\!\!1_{\{n>\tau(n)\}}1\!\!1_{\{n>\tau\}}|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} = 0
\end{aligned}$$

It follows that $Z_n 1\!\!1_{\{\tau<n\}} = \widetilde{Z}_n 1\!\!1_{\{\tau<n\}}$.

Furthermore,

$$\begin{aligned}
\widetilde{Z}_n 1\!\!1_{\{\tau=n\}} &= 1\!\!1_{\{\tau=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) = 1\!\!1_{\{\tau=n\}}1\!\!1_{\{\tau(n)=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) \\
&= 1\!\!1_{\{\tau=n\}}\mathbb{E}(1\!\!1_{\{\tau(n)=n\}}1\!\!1_{\{\tau\geq n\}}|\mathcal{F}_n) = 1\!\!1_{\{\tau=n\}}
\end{aligned}$$

which implies $\widetilde{Z}_\tau = 1$.

Let $\ell(n) = \sup\{k \leq n : \tilde{Z}_k = 1\}$. Then, on $\tau \leq n$ one has $\tau = \ell(n)$.

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Decomposition in the enlarged filtration.

Let X be an $\mathbb{F}$-martingale. Then,

$$X = \widehat{X} + \sum_{k=0}^{\cdot\wedge\tau} \frac{1}{Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k - \sum_{k=\tau}^{\cdot} \frac{1}{1-Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k$$

where $\widehat{X}$ is a $\mathbb{G}$-martingale.

One has

$$\mathbb{1}_{\tau\leq n}(V^{\mathbb{G}}_{n+1} - V^{\mathbb{G}}_{n}) = \mathbb{E}(\mathbb{1}_{\tau\leq n}(X_{n+1} - X_n)|\mathcal{G}_n)\,.$$

We now take the conditional expectation w.r.t. $\mathcal{F}_n$. From the property of honest times, there exists $V^{\mathbb{F}}$, an $\mathbb{F}$-predictable process, such that $V^{\mathbb{G}}_n \mathbb{1}_{\tau\leq n} = V^{\mathbb{F}}_n \mathbb{1}_{\tau\leq n}$. Taking into account that $V^{\mathbb{F}}$ is predictable, one has

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\tau\leq n}|\mathcal{F}_n)(V^{\mathbb{G}}_{n+1} - V^{\mathbb{G}}_{n}) &= \mathbb{E}(\mathbb{1}_{\tau\leq n}(X_{n+1} - X_n)|\mathcal{F}_n) \\
&= \mathbb{E}(\mathbb{E}(\mathbb{1}_{\tau\leq n}|\mathcal{F}_{n+1})(X_{n+1} - X_n)|\mathcal{F}_n)
\end{aligned}$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

$$\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_n)(V^{\mathbb{F}}_{n+1}-V^{\mathbb{F}}_n) \quad = \quad \mathbb{E}(\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_{n+1})(X_{n+1}-X_n)|\mathcal{F}_n)$$

Now, using the fact that

$$\begin{aligned}
\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_n) &= 1-\mathbb{E}(1\!\!1_{\tau>n}|\mathcal{F}_n)=1-Z_n \\
\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_{n+1}) &= 1-\mathbb{E}(1\!\!1_{\tau>n}|\mathcal{F}_{n+1})=1-\mathbb{E}(1\!\!1_{\tau\geq n+1}|\mathcal{F}_{n+1})=1-\widetilde{Z}_{n+1}
\end{aligned}$$

and that X is an $\mathbb{F}$-martingale, we obtain

$$(1-Z_n)(V^{\mathbb{G}}_{n+1}-V^{\mathbb{G}}_n)=-\mathbb{E}(\tilde{Z}_{n+1}(X_{n+1}-X_n)|\mathcal{F}_n)=-\Delta\langle\widetilde{M},X\rangle_n\,.$$

It seems important to note that the Doob-Meyer decomposition of Z is not needed.

$Z_n=\mathbb{P}(\tau>n|\mathcal{F}_n),\quad \widetilde{Z}_n=\mathbb{P}(\tau\geq n|\mathcal{F}_n)$

Brackets in $\mathbb{F}$ and $\mathbb{G}$.

Let X and Y be $\mathbb{F}$ adapted processes (hence, semi martingales)

$$\Delta\langle X,Y\rangle^{\mathbb{G}}_n \mathbb{1}_{n\leq\tau} = \mathbb{1}_{n\leq\tau}\frac{1}{Z_{n-1}}\mathbb{E}(\widetilde{Z}_n\Delta X_n\Delta Y_n|\mathcal{F}_{n-1})$$

Let τ be an honest time. Then

$$\Delta\langle X,Y\rangle^{\mathbb{G}}_n \mathbb{1}_{\tau<n} = \mathbb{1}_{\tau<n}\frac{1}{1-Z_{n-1}}\mathbb{E}((1-\widetilde{Z}_n)\Delta X_n\Delta Y_n|\mathcal{F}_{n-1})$$

$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n)$

Enlargement with a process

For $n \geq 0$ let $U_n(dy)$ be a regular conditional distribution of the random vector $\widehat{Y}_{n-1} = (Y_0, \ldots, Y_{n-1})$ with respect to $\mathcal{F}_n$. Moreover, for $n \geq 1$ let $V_n(dy)$ be a regular conditional distribution of $\widehat{Y}_{n-1}$ with respect to $\mathcal{F}_{n-1}$.

Assume that $U_n(dy)$ is absolutely continuous w.r.t. $V_n(dy)$ for all $n \geq 1$ and $d_n(y) := \frac{U_n(dy)}{V_n(dy)}$. Then, the information drift of X w.r.t. to $(\mathcal{G}_n)$ is given by

$$A_n = \sum_{k=1}^{n} \Delta\langle X, d(z)\rangle\rangle_k \Big|_{z=(Y_0,\ldots,Y_{k-1})}.$$

THANK YOU FOR YOUR ATTENTION