Martingale Optimal Transport and Robust Hedging

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Outline

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

The problem of "Déblais et Remblais" (Monge 1781)

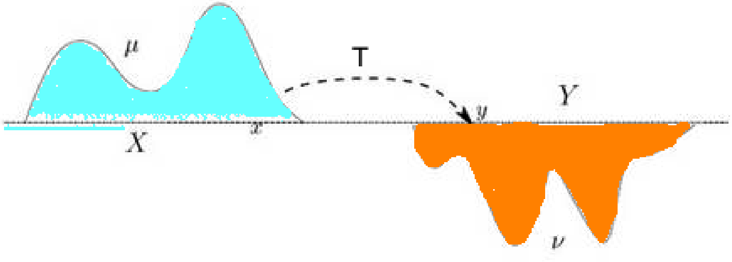


Figure: Mass transport. Ref : C. Villani

Analytic formulation (Monge 1781)

- Initial distribution : probability measure μ on $\mathcal{X}$
- Final distribution : probability measure ν on $\mathcal{Y}$

Problem : find an optimal transference plan T^*

$$P^M := \sup_{T \in \mathcal{T}(\mu,\nu)} \int c(x, T(x))\mu(dx)$$

where $\mathcal{T}(\mu, \nu)$ of all maps $T : x \longmapsto y = T(x)$ such that

$$\nu \quad = \quad \mu \circ T^{-1}$$

Randomization of mass transfert (Kantorovich 1942)

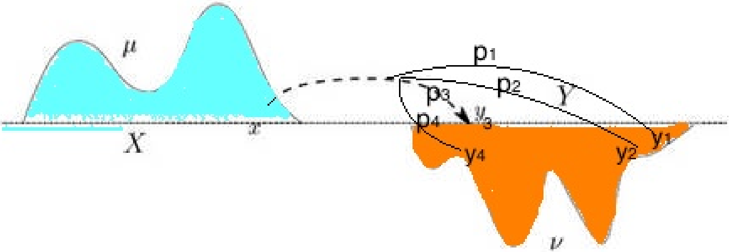


Figure: Randomized mass transport

Probabilistic formulation (Kantorovich 1942)

Randomization of transference plans :

$$\overline{P}^K := \sup_{\mathbb{P}\in\mathcal{P}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big]$$

where $\Omega = \mathcal{X}\times\mathcal{Y}$, $X(x,y)=x$, $Y(x,y)=y$, and

$$\mathcal{P}(\mu,\nu) := \big\{\mathbb{P}\in \mathrm{Prob}_\Omega : X \sim_{\mathbb{P}} \mu \text{ and } Y \sim_{\mathbb{P}} \nu\big\}$$

Kantorovich duality

Duality in linear programming, Legendre-Fenchel duality...

$$D^0 := \inf_{(\varphi,\psi)\in\mathcal{D}^0} \{\mu(\varphi) + \nu(\psi)\}$$
$$\mathcal{D}^0 := \{(\varphi,\psi)\in\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu) : \varphi\oplus\psi \geq c\}$$

where $\mu(\varphi) := \int \varphi d\mu$, $\nu(\psi) := \int \psi d\nu$, and

$$\varphi\oplus\psi(x,y) := \varphi(x)+\psi(y), \quad x\in\mathcal{X}, \; y\in\mathcal{Y}$$

No reference probability measure on the product space

The duality

Theorem

Let $c \geq 0$ be measurable. Then

- *duality : $D^0 = P^K$*
- *existence holds for D^0*
- *If in addition $c \in USC$, existence also holds for P^K*

In this generality, the result is due to Kellerer '84

Financial interpretation

- $X \sim \mu$ and $Y \sim \nu$ prices of two assets at time 1

- μ and ν identified from market prices of call options :

$$C_\mu(K) = \int (x-K)^+ \mu(dx), \quad C_\nu(K) = \int (y-K)^+ \nu(dy), \quad K \geq 0$$

Breeden-Litzenberger 1978 : $\mu = C''_\mu$ and $\nu = C''_\nu$

-

$\varphi(X)$	Vanilla position in	X	with market price	$\mu(\varphi)$
$\psi(Y)$	$\cdots$	Y	$\cdots$	$\nu(\psi)$

Financial interpretation : no reference probability

- $c(X, Y)$ payoff of derivative security
- Robust static hedging strategies for the derivative $c(X, Y)$:

$$\mathcal{D}^0 := \{(\varphi, \psi) \in \mathbb{L}^1(\mu) \times \mathbb{L}^1(\nu) : \ \varphi \oplus \psi \geq c\}$$

Robust superhedging cost of $c(X, Y)$ is the Kantorovitch dual :

$$D^0 = \inf_{(\varphi,\psi)\in\mathcal{D}^0} \{\mu(\varphi) + \nu(\psi)\}$$

The primal Monge-Kantorovitch problem is :

$$P^K = \sup_{\mathbb{P}\in\mathcal{P}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X, Y)]$$

Outline

MARTINGALE OPTIMAL TRANSPORT

initiated by Pierre Henry-Labordère, Preprints Soc. Gén.

- Beiglböck, Henry-Labordère & Penkner
- Galichon, Henry-Labordère & NT

One asset observed at two future dates

Our interest now is on the case where

$$X = X_0 \quad \text{and} \quad Y = X_1$$

are the prices of the same asset at two future dates 0 and 1

Interest rate is reduced to zero

This setting introduces a new feature :

possibility of dynamic trading the asset between times 0 and 1

Superhedging problem $\equiv$ Kantorovitch dual

Robust super hedging problem naturally formulated as :

$$D(\mu,\nu) \quad := \quad \inf_{(\varphi,\psi,h)\in\mathcal{D}} \{\mu(\varphi)+\nu(\psi)\}$$

where

$$\mathcal{D} := \{(\varphi,\psi,h)\in\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu)\times\mathbb{L}^0 : \ \varphi\oplus\psi+h^{\otimes}\geq c\}$$

where

$$\varphi\oplus\psi(x,y) := \varphi(x)+\psi(y) \quad \text{and} \quad h^{\otimes}(x,y) := h(x)(y-x)$$

The Martingale Optimal Transport Problem

Theorem (Beiglböck, Henry-Labordère, Penkner)

Assume $c \in$ USC and bounded from above. Then $P = D$, and existence holds for $P(\mu, \nu)$.

where the dual problem is :

$$P(\mu, \nu) \quad := \quad \sup_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\left[c(X, Y)\right]$$

with

$$\mathcal{M}(\mu, \nu) := \left\{\mathbb{P} \in \mathcal{P}(\mu, \nu) : \mathbb{E}^{\mathbb{P}}[Y|X] = X\right\}$$

Strassen '65 : $\mathcal{M}(\mu, \nu) \neq \emptyset$ iff $\mu \preceq \nu$, i.e. $\mu(g) \leq \nu(g)$ $\forall$ g convex

Existence of optimal hedge does not hold in general

• There are easy examples where existence for the dual fails, even for bounded c, bounded support... (Beiglböck, Henry-Labordère & Penkner, Beiglböck, Nutz & NT)

• Let $\mu = \nu$, then $\mathcal{M}(\mu, \mu) = \{\mathbb{P}^*\}$ where $Y = X$, $\mathbb{P}^*-$a.s.

- $P(\mu, \mu) = \mathbb{E}^\mu\big[c(X, X)\big]$
- The derivative is in fact $c(X, X)$, primal problem carries no information about $c(x, y)$ outside the diagonal $y = x$
- One is only interested in hedging along the diagonal

$$\varphi(X) + \psi(X) + h(X)(X - X) \;\; \geq \;\; c(X, X), \;\; \mu - \text{a.s.}$$

Duality under more general payoff functions

- The condition $c \in \text{USC}$ is not innocent...
- Consider the following example of LSC payoff

$$c(x,y) := \mathbb{1}_{\{x \neq y\}}, \quad x, y \in [0,1] \times [0,1]$$

Let $\mu = \nu =$ Lebesgue measure on $[0,1]$. Then

- $\mathcal{M}(\mu,\mu) = \{\mathbb{P}^*\}$ uniform distribution on the diagonal of the square $[0,1]^2$
- Then $P(\mu,\mu) = 0$
- However, we may prove that $D(\mu,\mu) = 1$!

Quasi-sure robust superhedging

Definition

$\mathcal{M}(\mu,\nu)-$q.s. (quasi surely) means $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{M}(\mu,\nu)$

- The quasi-sure robust superhedging cost

$$D^{qs} := \inf_{(\varphi,\psi,h)\in\mathcal{D}^{qs}} \{\mu(\varphi)+\nu(\psi)\}$$
$$\mathcal{D}^{qs} := \{(\varphi,\psi,h) : \varphi\oplus\psi+h^{\otimes}\geq c,\ \mathcal{M}(\mu,\nu)-\text{q.s.}\}$$

is more natural...

- Then, $D(\mu,\nu)\geq D^{qs}(\mu,\nu)\ \geq\ P(\mu,\nu)$

so if the duality $P=D$ holds, it follows that $D=D^{qs}$

Structure of polar sets in (standard) optimal transport

Theorem (Kellerer)

For $N \subset \mathbb{R} \times \mathbb{R}$, *TFAE :*

• $\mathbb{P}[N] = 0$ *for all* $\mathbb{P} \in \mathcal{P}(\mu, \nu)$

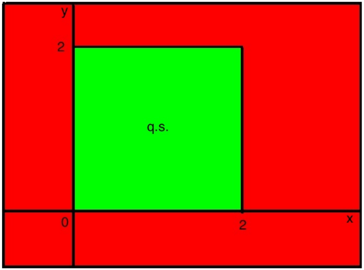
• $N \subset (N_1 \times \mathbb{R}) \cup (\mathbb{R} \times N_2)$ *for some* $N_1, N_2 \subset \mathbb{R}$,
$\mu(N_1) = \nu(N_2) = 0$

$\Longrightarrow$ no difference between the pointwise and the quasi-sure formulations in standard optimal transport

Pointwise versus Quasi-sure superhedging I

Suppose $\mathrm{Supp}(\mu) = [0,2] = \mathrm{Supp}(\nu) = [0,2]$, then

- $\mathcal{M}(\mu,\nu)-$q.s. only involves the values $(x,y) \in [0,2]^2$
- Pointwise superhedging involves all values $(x,y) \in \mathbb{R}^2$

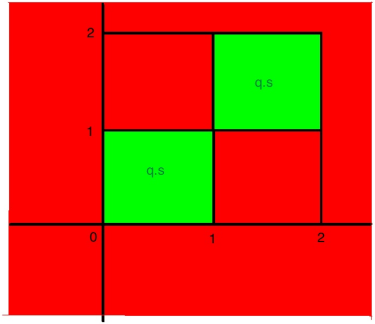


Pointwise versus Quasi-sure superhedging II

Suppose $\mathrm{Supp}(\mu) = \mathrm{Supp}(\nu) = [0,2]$, and $C_\mu(1) = C_\nu(1)$

$$\mathbb{E}\big[(X-1)^+\big] \;=\; \mathbb{E}\big[(Y-1)^+\big] \;\leq\; \mathbb{E}\big[(X-1)^+\big]$$

by Jensen's inequality, and then $\{X \geq 1\} = \{Y \geq 1\}$



Structure of polar sets in martingale optimal transport

Consider the partition :

$$\{C_\mu < C_\nu\} = \cup_{k\geq 0} I_k, \quad I_k = (a_k, b_k), \quad J_k := I_k \cup \{\nu - \text{atoms}\}$$

Theorem (Beiglböck, Nutz & NT '15)

For $N \subset \mathbb{R} \times \mathbb{R}$, *TFAE :*

- $\mathbb{P}[N] = 0$ *for all* $\mathbb{P} \in \mathcal{M}(\mu, \nu)$
- $N \subset (N_1 \times \mathbb{R}) \cup (\mathbb{R} \times N_2) \cup \{\Delta \cup (I_k \times J_k)\}^c$ *for some* $N_1, N_2 \subset \mathbb{R}$, $\mu(N_1) = \nu(N_2) = 0$

Duality and existence under quasi-sure robust superhedging

Theorem (Beiglböck, Nutz & NT '15)

Let $\mu \preceq \nu$ and $c \geq 0$ measurable. Then

$$P(\mu, \nu) \quad = \quad D^{qs}(\mu, \nu)$$

and existence holds for D^{qs}

Outline

Geometry of optimal transport plans : cyclic monotonicity

In standard optimal transport, optimality of a transport plan is a property of its support...

Theorem

For an optimal transport plan $\mathbb{P}^*$, there exists $\Gamma \subset \mathcal{X} \times \mathcal{Y}$:

- $\mathbb{P}^*[\Gamma] = 1$,
- *and for all finite subset $(x_i, y_i)_{i \leq n} \subset \Gamma$:*

$$\sum_{i=1}^{n} c(x_i, y_i) \geq \sum_{i=1}^{n} c(x_i, y_{i+1}) \quad \textit{with} \quad y_{n+1} = y_1$$

(necessary and sufficient, under slight conditions)

Back to the original Monge formulation

- $P^K \geq P^M$: Kantorovitch formulation $\equiv$ relaxation of Monge one

Theorem (Rachev & Rüschendorf)

μ without atoms, $c \in C^1$ with $c_{xy} > 0$ (Spence-Mirrlees/Twist condition). Then there is a unique optimal transference plan :

$$\mathbb{P}^*(dx, dy) = \mu(dx)\delta_{\{T^*(x)\}}(dy) \quad \textit{with} \quad T^* = F_\nu^{-1} \circ F_\mu$$

Consequently $P^M = P^K$, and T^* solves both problems.

- T^* : monotone rearrangement, Fréchet-Hoeffding coupling

- Extension to $\mathbb{R}^d$ (Brenier) : $\mathbb{P}^*$ concentrated on the graph of the gradient of some $c-$convex function

On the Spence-Mirrlees condition

The solution of the Kantorovich optimal transportation problem

$$P^K := \sup_{\mathbb{P}\in\mathcal{P}(\mu,\nu)} \int c(x,y)\mathbb{P}(dx,dy)$$

is not modified by the change of performance criterion :

$$c(x,y) \longrightarrow \hat{c}(x,y) := c(x,y) + a(x) + b(y)$$

Spence-Mirrlees condition $c_{xy} > 0$ stable by this transformation

Martingale monotonicity condition

Optimality of martingale transport is also a property of its support...

Theorem (Beiglböck & Juillet '12, Zaev '14)

Let $\mathbb{P}^*$ *be a solution of* $P(\mu, \nu)$. *Then*

- *there exists* $\Gamma \subset \Omega$, $\mathbb{P}^*[\Gamma] = 1$ *such that*
- *for all* $\mathbb{P}_0^*$ *with* $Supp(\mathbb{P}_0^*) = \{\omega_1, \ldots, \omega_N\} \subset \Gamma$, *we have :*

$$\mathbb{E}^{\mathbb{P}_0^*}\left[c(X, Y)\right] \geq \mathbb{E}^{\mathbb{P}}\left[c(X, Y)\right]$$

whenever

$$\mathbb{P} \circ X^{-1} = \mathbb{P}_0^* \circ X^{-1}, \quad \mathbb{P} \circ Y^{-1} = \mathbb{P}_0^* \circ Y^{-1}$$
$$\text{and} \quad \mathbb{E}^{\mathbb{P}}[Y|X] = \mathbb{E}^{\mathbb{P}_0^*}[Y|X]$$

Necessary & sufficient Martingale monotonicity

Theorem (Beiglböck, Nutz & NT '15)

Let $(\hat{\varphi}, \hat{\psi}, \hat{h})$ *be solution of* $D^{qs}(\mu, \nu)$, *and set*

$$\Gamma := \{\hat{\varphi} \oplus \hat{\psi} + \hat{h}^{\otimes} = c\}$$

Then $\mathbb{P}^*$ *solution of* $P(\mu, \nu)$ *iff* $\mathbb{P}^*[\Gamma] = 1$

Worst Case Financial Market – Brenier Theorem

• Solution $\mathbb{P}^* \in \mathcal{M}(\mu, \nu)$ exists for $c \in$ USC. Is there an optimal transfert map, i.e. optimal transport of μ to ν through a map T^* ?

NO, unless $\mu = \nu$!

Is there a transference plan along a minimal randomization ?

$$X \quad \begin{cases} Y = T_u(X) & \text{with probability } \; q(X) \\ Y = T_d(X) & \text{with probability } \; 1 - q(X) \end{cases}$$

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NO, unless $\mu = \nu$!

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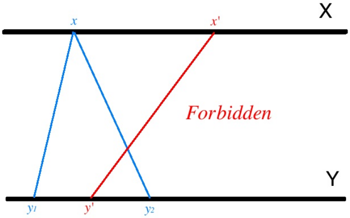
X →
$Y = T_u(X)$ with probability $q(X)$
$Y = T_d(X)$ with probability $1 - q(X)$

Left-monotone martingale transport

Definition (Beiglbock & Juillet 2012)

$\mathbb{P} \in \mathcal{M}(\mu, \nu)$ is **left-monotone** if $\mathbb{P}[(X, Y) \in \Gamma] = 1$, for some $\Gamma \subset \mathbb{R} \times \mathbb{R}$, and

$$\text{for all } (x, y_1), (x, y_2), (x', y') \in \Gamma : \quad x < x' \implies y' \notin (y_1, y_2)$$



Existence and uniqueness of left-monotone martingale transport

Theorem (Beiglböck, Henry-Labordère & NT '15)

Let $\mathbb{P}^* \in \mathcal{M}(\mu,\nu)$ be solution of $P(\mu,\nu)$. If c_x strictly convex in y, $\mu-$a.e. x, then $\mathbb{P}^*$ is left-monotone

Theorem (Beiglböck & Juillet '12, Beiglböck, Henry-Labordère & NT '15)

Assume μ has no atoms. Then, there is a unique left-monotone $\mathbb{P} \in \mathcal{M}(\mu,\nu)$ with distribution concentrated on two graphs

Henry-Labordère & NT '13 provide an explicit description of

- the left-monotone transport plan
- and corresponding semi-static robust superhedging strategy

The martingale version of the Spence-Mirrlees condition

... is $c_{xyy} > 0$

• Notice that the solution of the Martingale Transport problem is not altered by the change of payoff :

$$c(x,y) \quad \longrightarrow \quad \hat{c}(x,y) := c(x,y) + a(x) + b(y) + h(x)(y-x)$$

• $\hat{c}_{xyy} = c_{xyy}$

Concluding remarks

• Extension to $\mathbb{R}^d$, i.e. μ and ν prob. meas. on $\mathbb{R}^d$:

- Duality for $c \in$ USC and existence for $P(\mu, \nu)$
- Duality for c meas. and existence for $D(\mu, \nu)$? ?
- Martingale version of the Brenier theorem, see Ghoussoub, Kim & Lim 2015

• Extension to **finite discrete-time**, possibly **finitely-many marginals** constraints

Concluding remarks II

• **Continuous-Time Martingale Transport** : substitute for the return from a dynamic hedge $h^{\otimes}$ is the stochastic integral

$$\int_0^T H_s dX_s$$

But without reference probability ? ? $\Longrightarrow$ Two viewpoints

- Pointwise definition : restrict H to have finite variation, then :

$$\int_0^T H_s dX_s \; := \; \int_0^T X_s dH_s + H_T X_T - H_0 X_0$$

  Dolinsky & Soner

- Quasi-sure definition : under any $\mathbb{P} \in \mathcal{M}(\mu, \nu)$, $\int_0^T H_s dX_s$ is defined by the standard stochastic analysis...

Guo, Tan & NT 2015

THANK YOU FOR YOUR ATTENTION