Information Asymmetries, Volatility, Liquidity, and the Tobin Tax

Albina Danilova◇ Christian Julliard◇

◇London School of Economics

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Outline

In market data:

- Volatility is time varying and clusters at high/medium frequency $\Rightarrow$ ARCH/GARCH/SV models
- Highest degree of clustering at high frequency.
- Market vol $\neq$ fundamental vol (e.g. Campbell-Kyle (1993)).
- A relation between volatility and number/volume of trade (e.g. Gallant-Rossi-Tauchen (1992), Jones-Kaul-Lipson (1994)) & frequency (e.g. Engle-Sun (2007))

$\Rightarrow$ Gaussian log returns under a number of trades (stochastic) time change (Ané-Geman (2000))

- Links between information asymmetries, volatility and liquidity and return dynamics (e.g. Kelly-Ljungqvist (2013), Easly-Hvidkjaer-O'Hara (2002)).
- Transaction taxes (seem to) $\uparrow$ volatility (e.g. Jones-Seguin (1997), Umlauf (1993))
- Joint volatility spikes and liquidity dry-ups (e.g. recent crises)

Our paper: a (non trivial) theory that can explain all the above facts and, more broadly, the equilibrium determinants of volatility (at different frequencies), and liquidity (tightness, depth, resilience).

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Our paper: a (non trivial) theory that can explain all the above facts and, more broadly, the equilibrium determinants of volatility (at different frequencies), and liquidity (tightness, depth, resilience).

(Other) Related Literature

- Sequential trade models e.g. Glosten-Milgrom (1985), Easley-O'Hara (1987), Glosten (1989), Brunnermeier-Pedersen (2009) etc.

But: *a*) complete order book; *b*) dynamic info; *c*) *weakly* exogenous arrivals; *d*) arrival intensity to infinity → approximate continuous market → make arrival process irrelevant;

- Time Deformation and Volatility e.g. Clark (1973), Ghysels-Gouriroux-Jasiak (1995), Yor-Madan-Geman (2002), Andersen-Bollerslev-Dobrev (2007), Kalogeropoulos-Roberts-Dellaportas (2007), etc.

⇒ a distributional characterisation (via stochastic time change) of equilibria on different time scales (trade, calendar, business).

- Information aggregation in markets e.g. Grossman-Stiglitz (1980), Hellwig (1980), Admati (1985), Kyle (1985), Wang (1993), Easley-O'Hara (2004), Vayanos-Wang (2012) etc.

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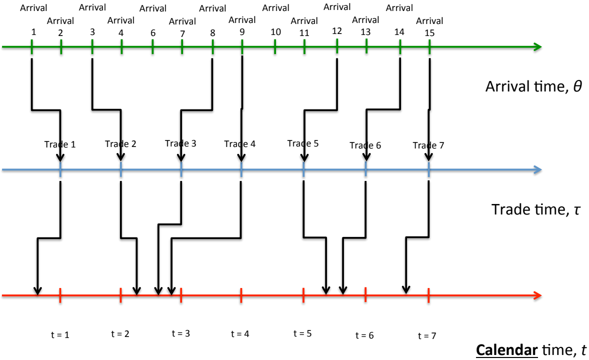
Time Scales and Notation

- 3 time scales, hence needs to define variables accordingly



X_t: on calendar time scale

x_i: on arrival time scale i.e. $x_i = X_{\theta_i}$

$\tilde{x}_i$: on number of trade time scale i.e. $\tilde{x}_i = X_{\mathcal{T}_i}$

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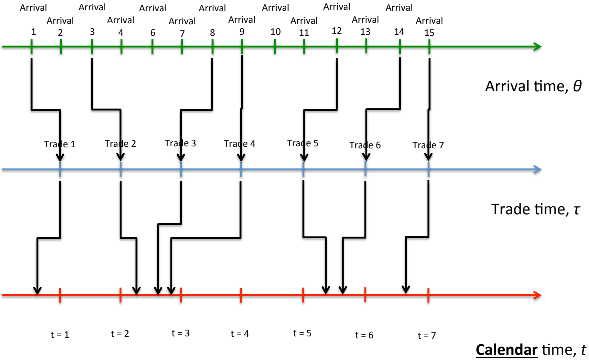

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Market Structure

Assets: a riskless bond ($r = 0$) and a stock with final value e^{D_T}

$$dD_t = \mu dt + \sigma dW_t^d, \quad D_0 = const, \quad W_t^d \text{ is B.M. w.r.t. to } \mathcal{F}_t.$$

Utilities: risk neutral traders and (competitive) market maker (M).

A1: Traders arrive to the market and meet M according to a stochastic counting process, N_t, with stopping time θ_i

- When the trader arrives at time θ_i, she observes bid, $B_{\theta_i}(v^-)$, and ask, $A_{\theta_i}(v^+)$, prices per-share posted by M, and decides if and how much to trade ($v \in \mathbb{R}$).

Friction: proportional transaction cost δ (like Tobin tax), i.e. M receives $v^+ A_{\tau_i}(v^+)(1-\delta)$ (spends $v^- B_{\tau_i}(v^-)(1+\delta)$)

if $v = 0$ M does not observe the arrival.

Notation: L_t= cumulated # of trades, V_t= cumulated volume, P_t = price.

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Information Structure

M: common knowledge: preferences, parameters, and the price and trade history ($\mathcal{G}_t^M := \mathcal{F}_t^P \vee \mathcal{F}_t^V$)

I type: i-th (more) informed trader, in share $1-q$, observes D ($\mathcal{G}_t^{I,i} = \mathcal{G}_t^M \vee \mathcal{F}_t^D \vee \sigma\left\{\theta_i^I \wedge s, s \leq t\right\}$)

U type: i-th uniformed/liquidity/noisy, in share q, with $\delta \in (0,q)$, trader believes/receives a random signal/shock ($\mathcal{G}_t^{U,i} = \mathcal{G}_t^M \vee \sigma\left\{S_{\theta_i^U}\right\} \vee \sigma\left\{\theta_i^U \wedge s, s \leq t\right\}$)

A2: fundamental shocks, arrivals, and noise/liquidity-shocks are conditionally independent.

($\mathcal{F}_T^W$, $\mathcal{F}_T^N$ and S_{θ_i} are conditionally independent given $\mathcal{H}_{i-1} \forall i$, where $\mathcal{H}_i = \mathcal{G}_{\theta_i}$, $\mathcal{G}_t = \mathcal{F}_t^V \vee \mathcal{F}_t^N$) .

A3: The "type" of arrival is independent (I_i is independent of $\mathcal{F}_T^{N,S,D} \vee \sigma\left(U_k\right)_{k \neq i}$)

A4: M cannot distinguish (conditionally) I and U from their orders.

($\mathbb{P}\left(v_i \in C | \mathcal{H}_{i-1}, I_i, \theta_i\right) = \mathbb{P}\left(v_i \in C | \mathcal{H}_{i-1}, U_i, \theta_i\right)$ for $C \in \mathcal{B}(\mathbb{R})$)

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Traders' optimisation problem

- The expected utility from holding v^+ shares until time T for an agent of type $k \in \{I, U\}$ that arrived at θ_i^k is

$$\mathbb{E}\left[v^+ e^{D_T} \middle| \mathcal{H}_i^k \right] =: v^+ z_i^k .$$

- The expected utility from investing in the risk free asset the amount needed to buy v^+ shares at time θ_i^k is $v^+ A_{\theta_i^k}(v^+)$.

⇒ the **expected utility** can be expressed as:

$$\max_{v^+, v^-} \quad v^+ \left[z_i^k - A_{\theta_i^k}(v^+) \right] + v^- \left[B_{\theta_i^k}(v^-) - z_i^k \right] . \qquad (1)$$

Notation: z_i (shadow price) is the expected value of holding one share of the asset for the agent that arrives at time θ_i

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Market Maker's optimisation problem

- M sets time t bid and ask prices as a functions of the order size v:

$$A_t\left(v^{+}\right)(1-\delta)=\underbrace{\sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M,N_{\tau_i}=L_{\tau_i}\right]\Big|_{\tilde{v}_i=v^+,\tau_i=t}}_{M\text{'s "future" valuation}},\qquad(2)$$

$$B_t\left(v^{-}\right)(1+\delta)=\sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M,N_{\tau_i}=L_{\tau_i}\right]\Big|_{\tilde{v}_i=-v^-,\tau_i=t}.\qquad(3)$$

- Need a (non-falsifiable) belief of M about N_t. We assume that $N_t=L_t\Rightarrow M$ doesn't update her beliefs if no trade occurs.

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$$A_t\left(v^{+}\right)(1-\delta)=\underbrace{\sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\Big|_{\tilde{v}_i=v^{+},\tau_i=t}}_{M\text{'s "future" valuation}},\tag{2}$$

$$B_t\left(v^{-}\right)(1+\delta)=\sum_{i=1}^{\infty}\mathbf{1}_{\{i=1+L_{t-}\}}\,\mathbb{E}\left[e^{D_T}|\tilde{\mathcal{H}}_i^M, N_{\tau_i}=L_{\tau_i}\right]\Big|_{\tilde{v}_i=-v^{-},\tau_i=t}.\tag{3}$$

- Need a (non-falsifiable) belief of M about N_t. We assume that $N_t=L_t \Rightarrow M$ doesn't update her beliefs if no trade occurs.

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Equilibrium: Definition

Definition (Equilibrium)

A market equilibrium is a set of policy fuctions $A_t(v^+)$, $B_t(v^-)$ satisfying regularity conditions and $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ such that:

1. $A_t(v^+)$ and $B_t(v^-)$ solve the market maker optimisation problem $\forall v, t$;
2. $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ solves the trader's problem.

Equilibrium: Bid and Ask functions

Proposition (Optimal ask and bid functions)

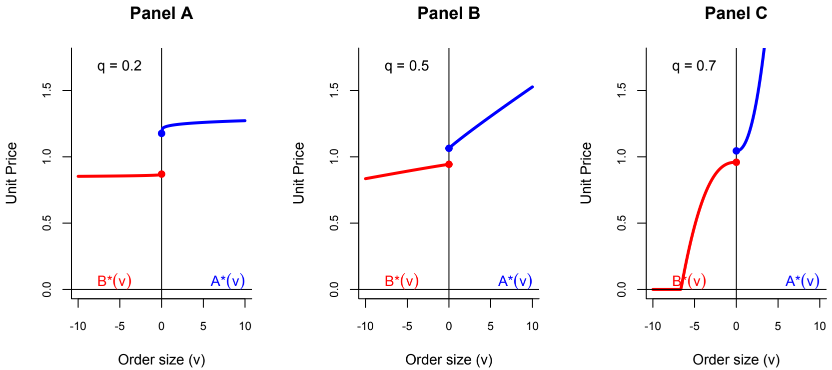
Suppose assumptions A1-A4 are satisfied. Then there exist optimal ask, $A_t(v^+)$, and bid, $B_t(v^-)$, prices that satisfy conditions C1-C5, and the market maker's optimality conditions. Moreover, optimal $A_t(v)$ and $B_t(v)$ have the following forms:

$$A_t^*(v) = \frac{q}{q-\delta}\left(1+\alpha v^{\frac{q-\delta}{1-q}}\right)\underbrace{\sum_{i=0}^{\infty}\mathbf{1}_{\{i=L_{t-}+1\}}Z^M_{\tau_{i-1}}}_{\text{M's "current" valuation}} \qquad (4)$$

$$B_t^*(v) = \begin{cases} \frac{q}{q+\delta}\left(1-\beta v^{\frac{q+\delta}{1-q}}\right)\sum_{i=0}^{\infty}\mathbf{1}_{\{i=L_{t-}+1\}}Z^M_{\tau_{i-1}} & \text{if } \beta v^{\frac{q+\delta}{1-q}} \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad (5)$$

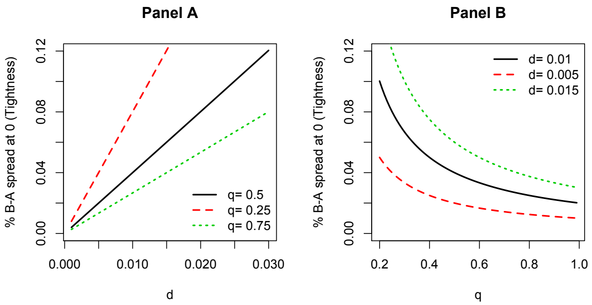
where α and β are strictly positive arbitrary constants, and $Z_t^M := \mathbb{E}\left[e^{D_T}\middle|\mathcal{G}_t^M, N_t = L_t\right]$ denotes the market maker valuation.

Equilibrium: Bid and Ask functions cont'd



- order book interpretation (*M* analogy).
- flexible parametrisation and empirically promising.

Liquidity: Tightness



$$\text{\% Bid-Ask spread at 0} = \frac{2q\delta}{q^2 - \delta^2}$$

B-A ↑ in adverse selection $(1-q)$ and trading cost (δ)

$\Rightarrow$ M uses the spread to recoup both type of costs.

- mutually reinforcing effects

Data: $(1-q)$ & δ explain 86%-100% of the spread (e.g. Stoll (1989), George, Kaul and Nimalendran (1991), Huang and Stoll (1996))

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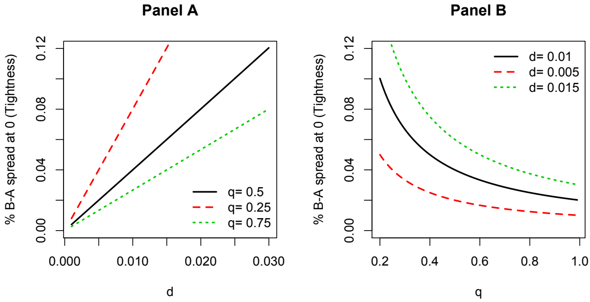

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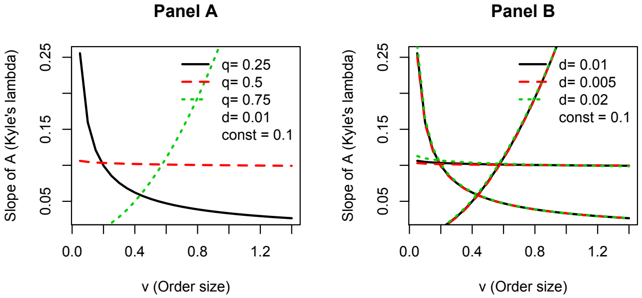
Liquidity: Depth



Slope of the Ask schedule, normalised by M's valuation: $\frac{q}{1-q}\alpha(v^+)^{\frac{2q-\delta-1}{1-q}}$.

High q: price more likely to deviate substantially from fundamental $\Rightarrow$ M's potential losses from trading with I are large $\Rightarrow$ Depth $\downarrow$ in v

Low q: price always close to from fundamental $\Rightarrow$ large orders unlikely to bring large losses $\Rightarrow$ Depth $\uparrow$ in v

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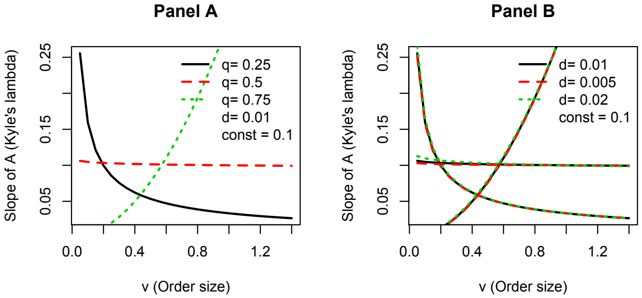

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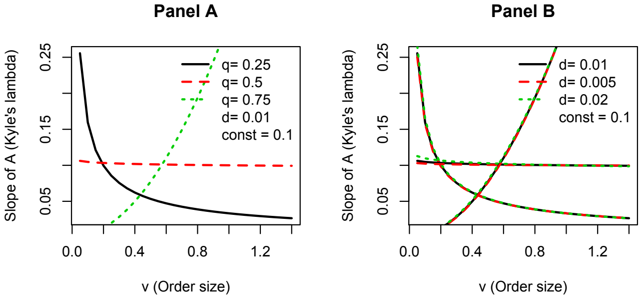

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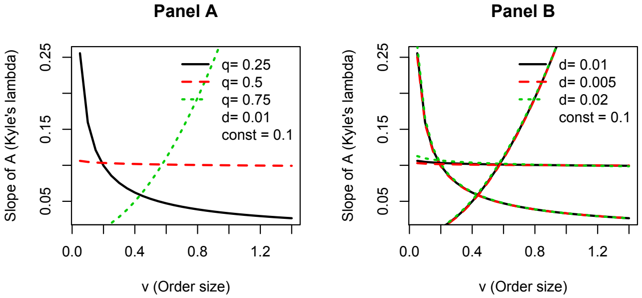

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High Frequency Price Process

- Since trades happen either at ask or bid, we can characterise the price process for any volume process:

$$\log \frac{P_{t+s}}{P_t} = \sum_{i=L_t}^{L_{t+s}} \left\{ \log\left(1 + \xi_i \left| V_{\tau_i} - V_{\tau_{i-1}} \right|^{\gamma_i}\right) + \log c_{1,i} + \log c_{2,i-1} \right\} . \tag{6}$$

where ξ_i, γ_i, $c_{1,i}$ and $c_{2,i}$ are known functions of δ, q, and whether trades are at ask or bid (the latter is a binomial r. v.).

⇒ consistent with non-lin model of Gallant, Rossi, and Tauchen (1992) (Tauchen-Pitts (1983), Epps-Epps (1976), Clark (1973), etc.)

if $|\xi_i| \left| V_{\tau_i} - V_{\tau_{i-1}} \right|^{\gamma_i}$ is

Small ≈ power law relationship (e.g. Farmer and Lillo (2004) and Farmer, Lillo, and Mantegna (2003))

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Equilibrium: Volume

Theorem

Suppose Assumptions A1-A4 are satisfied. For strictly positive constants α and β, there is a <u>*unique*</u> *market equilibrium, $A_t^*(v)$, $B_t^*(v)$, v_i^*, where $A_t^*(v)$ and $B_t^*(v)$ are given, respectively, by equations (4) and (5), and*

$$v_i^* = \begin{cases} \left[\frac{1-q}{\alpha(1-\delta)}\left(\frac{q-\delta}{q}\frac{z_i}{z_i^M} - 1\right)\right]^{\frac{1-q}{q-\delta}} & \text{if } \frac{q}{q-\delta} z_i^M < z_i, \\ -\left[\frac{1-q}{\beta(1+\delta)}\left(1 - \frac{q+\delta}{q}\frac{z_i}{z_i^M}\right)\right]^{\frac{1-q}{q+\delta}} & \text{if } z_i < \frac{q}{q+\delta} z_i^M, \\ 0 & \text{if } \frac{q}{q+\delta} z_i^M \leq z_i \leq \frac{q}{q-\delta} z_i^M \end{cases}$$

where $z_i^M := Z_{\theta_i}^M$

From fundamentals to price process

Lemma (price process as map of fundamentals)

Suppose that Assumptions A1-A5 are satisfied and the market is at the equilibrium. Then the trading times are defined recursively $(\tau_0 = 0)$

$$\tau_i = \inf\left\{\theta_j > \tau_{i-1} : \log z_j - \log \tilde{p}_{i-1} \notin \left(b\left(c_{2,i-1}\right), a\left(c_{2,i-1}\right)\right)\right\},$$

where $a(x) = \log\left(\frac{qx}{q-\delta}\right)$ *and* $b(x) = \log\left(\frac{qx}{q+\delta}\right)$, *and prices are*

$$\tilde{p}_0 = e^{D_0 + \left(\mu + \frac{1}{2}\sigma^2\right)T}, \; \tilde{p}_i = \frac{1}{c_{2,i}}\left[\left(1-q\right)z_i + q\tilde{p}_{i-1}c_{2,i-1}\right], \qquad (7)$$

$$c_{2,i} = \begin{cases} 1-\delta & \text{if } \log \tilde{z}_i - \log \tilde{p}_{i-1} > a\left(c_{2,i-1}\right) \text{ and } i > 0 \\ 1+\delta & \text{if } \log \tilde{z}_i - \log \tilde{p}_{i-1} < b\left(c_{2,i-1}\right) \text{ and } i > 0 \\ 1, & \text{if } i = 0 \end{cases}$$

The process Z (Shadow Price)

- The (de-meaned) log shadow valuation of the i-th agent is:

$$d_i^{tr} = \begin{cases} \log z_i - \left(\mu + \frac{\sigma^2}{2}\right)(T - \theta_i) & \forall i \geq 1 \\ D_0 & i = 0 \end{cases}, \qquad D_t^{tr} = \sum_{i=0}^{\infty} \mathbf{1}_{\{i=N_t\}} d_i^{tr}.$$

- The (conditional) distribution of this process is (Lemma 3):

$$\begin{aligned} \mathbb{P}\left[d_i^{tr} \leq x | \mathcal{H}_{i-1}, \theta_i\right] &= \sum_{j=1}^{i-1} (1-q) q^{i-1-j} \mathbb{P}\left[d_j^{tr} + \varepsilon_{i,j} \leq x | d_j^{tr}, \Delta_{i,j}\right] \\ &+ q^{i-1} \mathbb{P}\left[d_0^{tr} + \varepsilon_{i,0} \leq x | d_0^{tr}, \Delta_{i,0}\right] \end{aligned}$$

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Intuition:
- Last I arrival has all the relevant info

$\Rightarrow$ probability of it being the j-th arrival is $(1-q)q^{i-1-j}$
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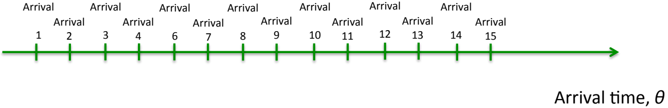
Changing Frequency & Time Changes

Arrival intensity $\rightarrow \infty$ ("business time") $\Rightarrow$ valuation of i-th arrival $(D^{tr}) \xrightarrow{\mathcal{L}}$ to B.M. (Prop. 9) $\Rightarrow$ Trade occurs when B.M. touches B-A bounds $+$ The map from D^{tr} to P is continuous (Lem. 5) $\Rightarrow$ Price $\xrightarrow{\mathcal{L}}$ on trade time (Thm. 7) $\Rightarrow$ Calendar time dist. $=$ time change wrt trades per time (Prop. 9) $\Leftrightarrow$ S.V. driven by $\#$ of trades



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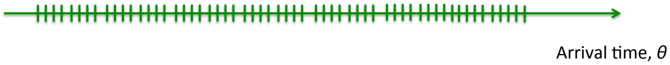

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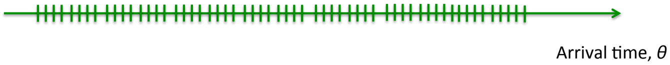

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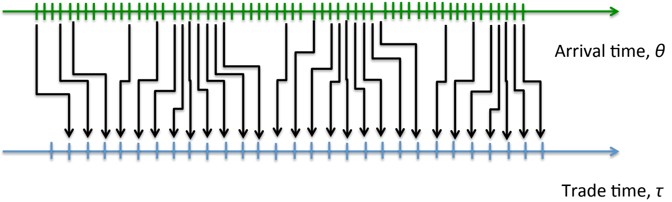

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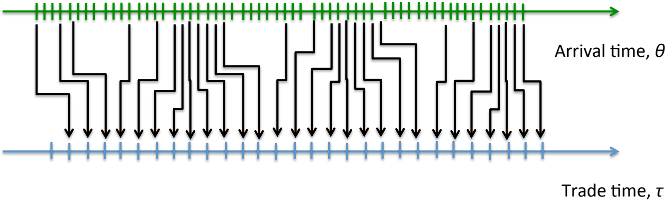

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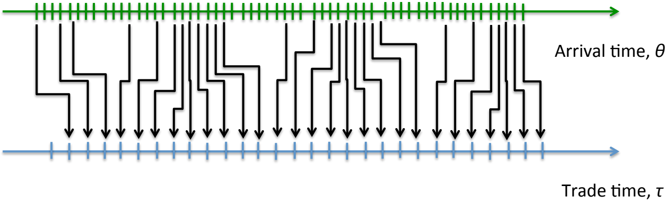

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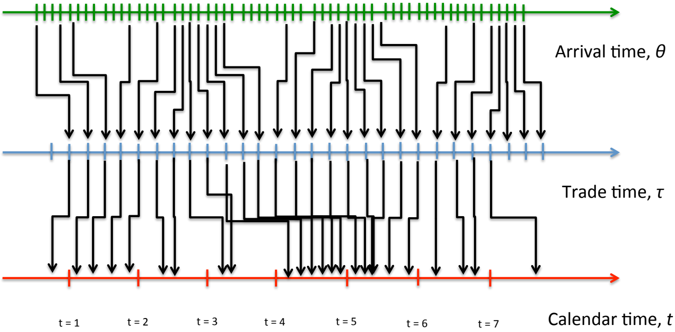
Arrival intensity $\to \infty$ ("business time") $\Rightarrow$ valuation of i-th arrival $(D^{tr}) \xrightarrow{\mathcal{L}}$ to B.M. (Prop. 9) $\Rightarrow$ Trade occurs when B.M. touches B-A bounds + The map from D^{tr} to P is continuous (Lem. 5) $\Rightarrow$ Price $\xrightarrow{\mathcal{L}}$ on trade time (Thm. 7) $\Rightarrow$ Calendar time dist. = time change wrt trades per time (Prop. 9) $\Leftrightarrow$ S.V. driven by # of trades



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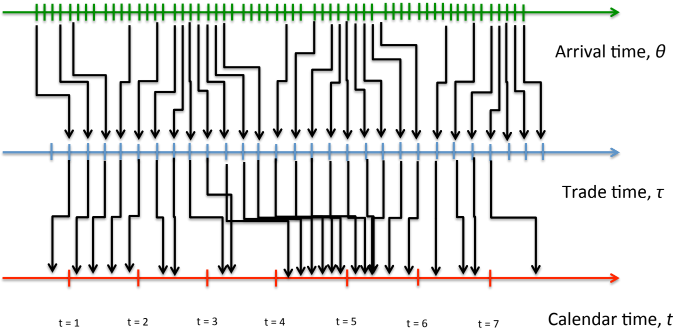

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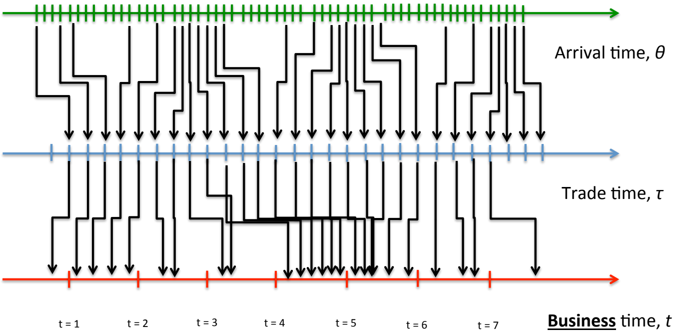
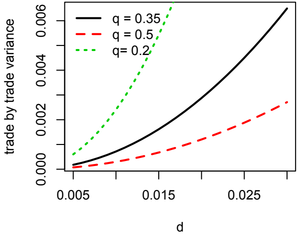


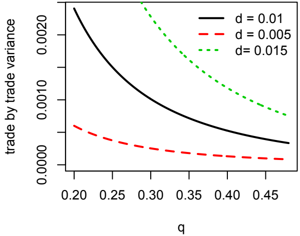
Trade-by-trade Volatility (at Med. Frequency)

Corollary (Volatility of the Limiting Price Process)

The conditional variance on the trade time scale, for $i > 1$, is:

$$Var\left(\frac{\tilde{p}_i}{\tilde{p}_{i-1}}|\mathcal{F}^W_{\tau_{i-1}}\right) = \frac{\delta^2(1-q^2)}{q^2-\delta^2}.$$





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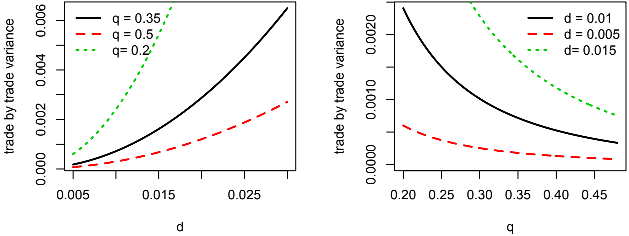

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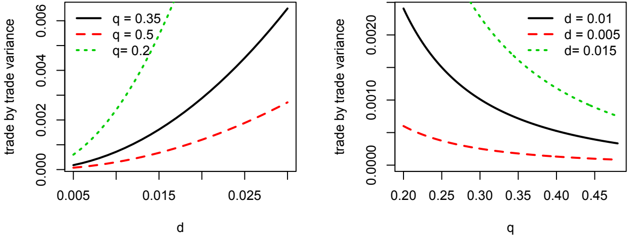

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Proposition (Low & Ultra-Low Frequency Price Distribution)

$$\frac{\log \frac{P_t}{P_s} - \frac{\sigma^2}{2}(s-t)}{\sqrt{L_t^p - L_s^p}} \xrightarrow[t-s\to\infty]{d} \mathcal{N}\left(0, \sigma^2 \mu_\tau\right), \tag{8}$$

where μ_τ is the expected time between trades. And at Ultra-low frequency

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Eq. (8) consistent with Jones, Kaul, and Lipson (1994), Ané and Geman (2000), Engle and Sun (2007) etc.

Note: second result due to $L_t^p/t \xrightarrow[t\to\infty]{a.s.} 1/\mu_\tau$

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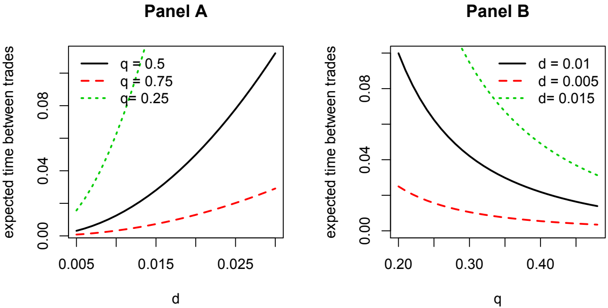

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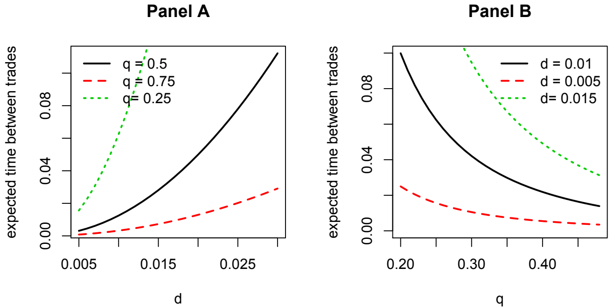

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Liquidity: Resilience

- on the trade time scale, the M's valuation evolves as

$$\tilde{z}_i^M = (1 - q)\,\tilde{z}_i + q\tilde{z}_{i-1}^M. \tag{10}$$

(i.e. an AR(1)) where $\tilde{z}_i$ is the valuation of the i-th trader.

Hence: half-life (reciprocal of resilience) on the calendar time scale:

$$\underbrace{\frac{\ln 1/2}{\ln q}}_{\text{trade-by-trade half-life}} \times \underbrace{\mu_\tau}_{\text{expected time between trades}}$$

w.r.t. δ: same properties as $\mu_\tau \Rightarrow \uparrow$ in δ (consistent with Umlauf (1993)): resilience $\downarrow$ in δ (Tobin Tax)

w.r.t. q: two opposing effects:

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overall: calendar time half-life $\downarrow$ in q: resilience $\downarrow$ in adverse selection

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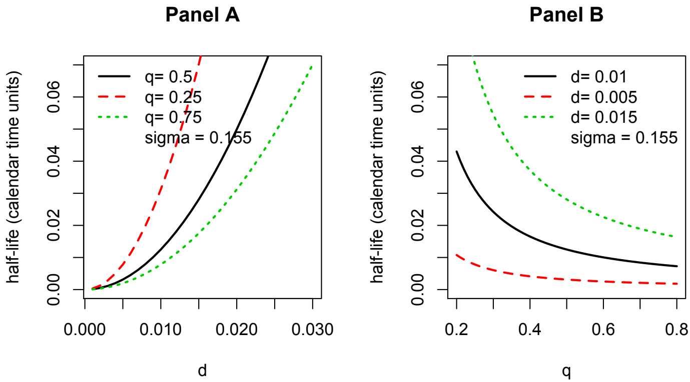
Resilience: calendar time half-life of M's update



negative effects of trade cost (δ) and adverse selection ($1 - q$) on resilience mutually reinforce

Conclusion

A simple and tractable equilibrium framework that:

1. can rationalise a very large set of empirical findings about financial market volatility and returns at different frequencies, including joint volatility spikes and liquidity dry-ups.
2. identifies the equilibrium determinants of the 3 main liquidity dimensions, and can rationalise related empirical findings.
3. delivers policy relevant (and empirically consistent) predictions about the Tobin Tax.
4. can be structurally estimated to pin down asset specific measures of: adverse selection, frictions to trade, liquidity, fundamental vol etc. $\Rightarrow$ empirical follow up.
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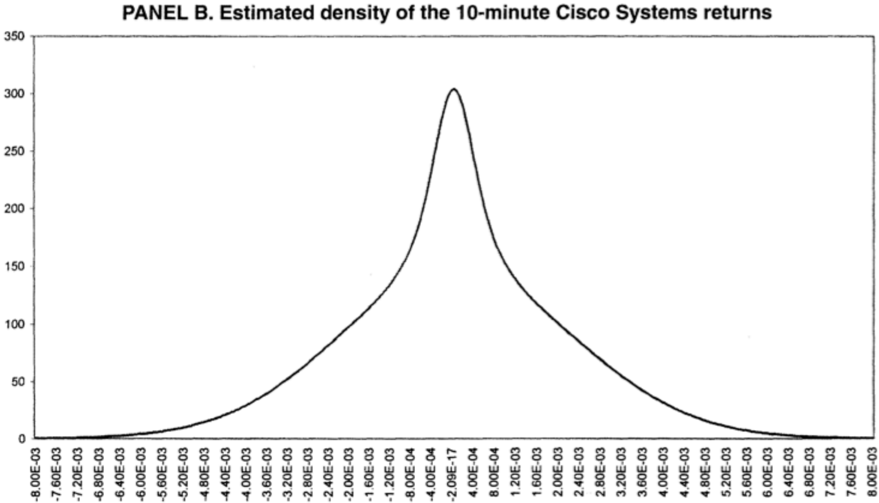
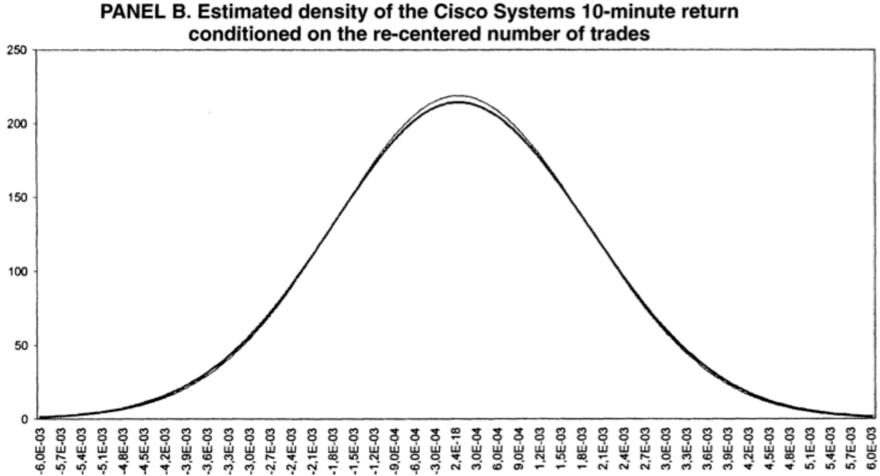
Appendix

4. Additional Figures
   - Ané and Geman (2000)
   - Bid-Ask Curves
5. Notation
6. Extras
   - Time Scales
   - Regularity Conditions
   - Equilibrium Definition



Source: Ané and Geman (2000)



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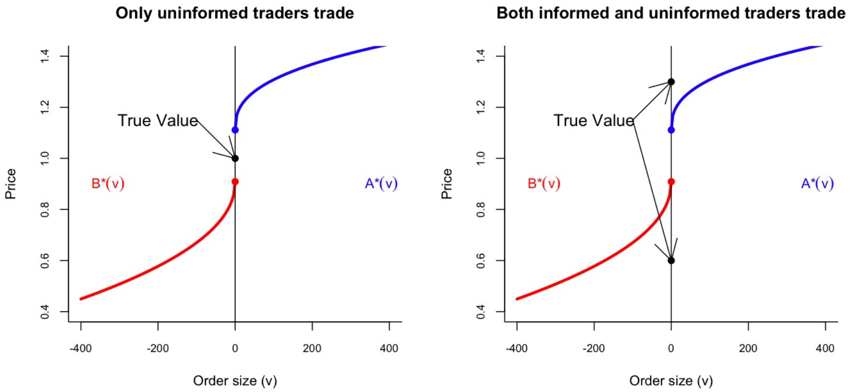
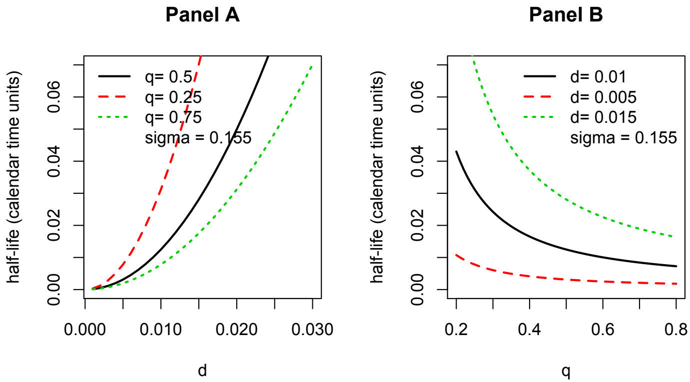

Only uninformed traders trade
Both informed and uninformed traders trade
True Value
True Value
B*(v)
A*(v)
B*(v)
A*(v)
Price
Price
1.4
1.2
1.0
0.8
0.6
0.4
-400
-200
0
200
400
Order size (v)
Order size (v)

Resilience: calendar time half-life of M's update



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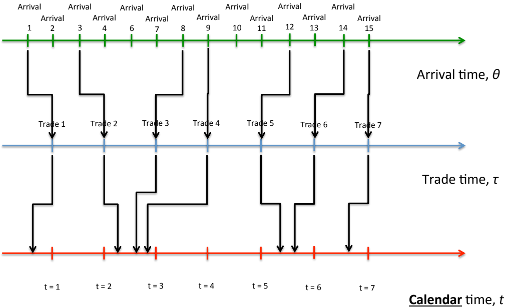
Notation

- 3 time scales, hence needs to define variables accordingly



X_t: on calendar time scale

x_i: on arrival time scale i.e. $x_i = X_{\theta_i}$

$\tilde{x}_i$: on number of trade time scale i.e. $\tilde{x}_i = X_{\tau_i}$

Review of Time Scales

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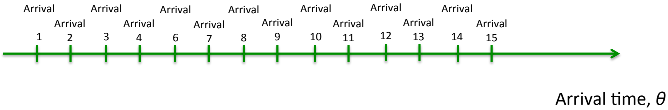

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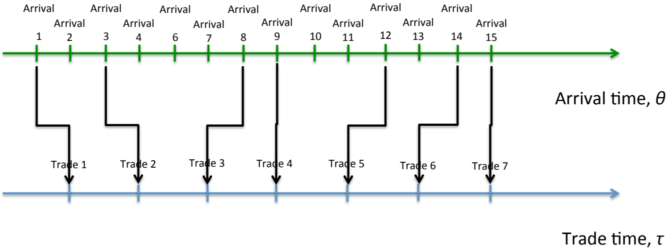

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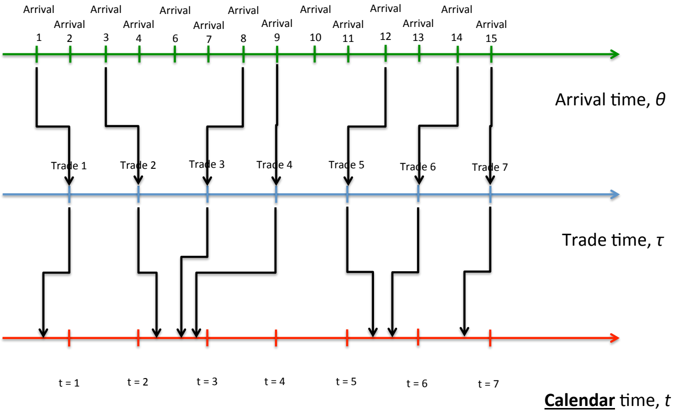

Regularity Conditions

C1 For a fixed v, the processes $B_t\left(v^-\right)$, $A_t\left(v^+\right)$, are *cáglád*

aka: M can change prices at any time but the time of trade.

C2 For a fixed t, $A_t\left(v^+\right) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+ \setminus \{0\}$ is continuous, nondecreasing and $\lim_{v^+ \to \infty} A_t(v^+) = +\infty$.

C3 For a fixed t, $B_t\left(v^-\right) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ is continuous, nonincreasing and $\lim_{v^- \to \infty} B_t(v^-) = 0$.

aka: no: free disposal, infinite trade size, decreasing price per-share.

C4 For a fixed t, $A_t\left(0\right) \geq B_t\left(0\right)$ for all $\omega \in \Omega$.

C5 For any fixed t, $A_t\left(\cdot\right)$ is continously differentiable, and $B_t\left(\cdot\right)$ is continously differentiable on the set $\{v : B_t\left(v\right) > 0\}$

C6 For a fixed t, $vA_t\left(v\right)$ is strictly convex, and $vB_t\left(v\right)$ is striclty concave on the set $\{v : B_t\left(v\right) > 0\}$

aka: C5-C6 ensures strict concavity of traders' problem.

Equilibrium: Definition

Definition (Equilibrium)

A market equilibrium is a set of policy fuctions $A_t(v^+)$, $B_t(v^-)$ satisfying regularity conditions and $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ such that:

1. $A_t(v^+)$ and $B_t(v^-)$ solve the market maker optimisation problem $\forall v, t$;
2. $v_i(A_{\theta_i}(v^+), B_{\theta_i}(v^-))$ solves the trader's problem.