On the dual problem of utility maximization

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The market model

- Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions, where T is a finite horizon.
- The market consists of a bond and a stock, where the bond is of zero interest rate and the stock-price process S is a strictly positive semimartingale.
- "No arbitrage condition": $\mathcal{M}^e(S) \neq \emptyset$.
- For an initial value x and a predictable S-integrable trading strategy H, the value process $X = (X_t)_{0\leq t\leq T}$ is given by

$$X_t = x + (H \cdot S)_t, \quad 0 \leq t \leq T.$$

- We call H admissible if for $0 \leq t \leq T$, the associated terminal value $X_t \geq -M$, for some positive M.
- The agent receives an exogenous random endowment $e_T \in \mathcal{F}_T$ at time T, satisfying $\rho := \|e_T\|_\infty < \infty$.

Utility maximization on the positive half line

A utility function $U : (0, \infty) \to \mathbb{R}$ represents the agent's preferences over the terminal wealth.

The function U is assumed to be strictly concave, strictly increasing and continuously differentiable satisfying the Inada conditions:

$$U'(0) := \lim_{x \to 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0.$$

and the condition of reasonable asymptotic elasticity (RAE):

$$AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

The aim of the agent is to maximize the expected utility from the terminal wealth:

$$u(x) := \sup_{H \text{ adm}} \mathbb{E}[U(x + (H \cdot S)_T + e_T)], \quad x > 0,$$

- The Itô framework: [Karatzas-Lehoczky-Shreve-Xu, 1991], [Cvitanić-Karatzas, 1992], etc...;
- The general semimartingale framework:

	$U:(0,\infty)\to\mathbb{R}$
$e_T=0$	[Kramkov-Schachermayer, 1999]
bounded e_T	[Cvitanić-Schachermayer-Wang, 2001]
unbounded e_T	[Hugonnier-Kramkov, 2004]

- $U:\mathbb{R}\to\mathbb{R}$:
  - locally bounded semimartingale models: [Schachermayer, 2001], [Owen, 2002], [Owen-Žitković, 2009];
  - general semimartingale models: [Biagini-Frittelli, 2008], [Biagini-Frittelli-Grasselli, 2011].
- Optimal consumption, with constraints, etc...

Duality method: $e_T = 0$

Define

$$\mathcal{X}(x) := \{X : X = x + (H \cdot S),\ H \text{ adm},\ X_T \geq 0\}$$

and

$$\mathcal{C}(x) := \left\{g \in \mathbf{L}^0_+(\mathcal{F}_T) :\ 0 \leq g \leq X_T,\ \text{for some } X \in \mathcal{X}(x)\right\},$$

where the latter is the set of positive terminal values, which can be dominated by some admissible strategies initialed from $x > 0$.

Then, the maximization problem (primal problem) can be rewritten into

$$u(x) := \sup_{H \text{ adm}} \mathbb{E}[U(x + (H \cdot S)_T)] = \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)].$$

Definition (Supermartingale deflators)

We call a positive semimartingale Y a supermartingale deflator, if for each $X \in \mathcal{X}(1)$, XY is a supermartingale. Moreover, we denote by $\mathcal{Y}(y)$ the collection of all such processes starting from y, namely,

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y,\ XY \text{ is a supermartingale}, \forall\ X \in \mathcal{X}(1)\}.$$

Note

$$\mathcal{D}(y) := \{h \in \mathbf{L}^0_+(\mathcal{F}_T) : 0 \leq h \leq Y_T,\ \text{for some } Y \in \mathcal{Y}(y)\}.$$

Then, the dual problem is formulated as

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\big[V(Y_T)\big] = \inf_{h \in \mathcal{D}(y)} \mathbb{E}\big[V(h)\big],$$

where

$$V(y) := \sup_{x>0}\{U(x) - xy\},\ y > 0.$$

Theorem ([Kramkov-Schachermayer, 1999])

Assume

- *"No arbitrage condition": $\mathcal{M}^e(S) \neq \emptyset$,*
- *U satisfies the Inada conditions and RAE,*
- *$u(x) < \infty$, for some $x > 0$.*

Then

- *The value functions u, v have the same properties as U and V.*
- *For any $y > 0$, there exists a unique dual optimizer $\widehat{h} \in \mathcal{D}(y)$.*
- *Let $\widehat{y} := u'(x)$, then there exists a unique primal solution $\widehat{g} \in \mathcal{C}(x)$, which is defined by $\widehat{g} := (U')^{-1}(\widehat{h}_{\widehat{y}})$.*
- *$\mathbb{E}[\widehat{g}\widehat{h}] = x\widehat{y}$.*

In general, the optimal element $\widehat{Y}$ from $\mathcal{Y}(\widehat{y})$ associated with the dual optimizer $\widehat{h}_{\widehat{y}}$ is not a martingale. An example can be found in [Kramkov-Schachermayer, 1999] even with an one-period model. However, under certain condition, $\widehat{Y}$ is proved a local martingale.

Theorem ([Larson-Žitković, 2007])

In addition to the conditions for the above theorem, suppose that S is continuous, then the dual optimizer $\widehat{h}_{\widehat{y}}$ is attained by a local martingale from the set of supermartingale deflators.

Outline of the proof: That S is continuous and $\mathcal{M}^e(S) \neq \emptyset$ implies the following representation ([Delbean-Schachermayer, 1995]):

$$S_t = 1 + M_t + \int_0^t \lambda_u d\langle M \rangle_u, \ 0 \leq t \leq T,$$

where M is a local martingale and λ is a predicable M-integrable process.

Proposition

For any $Y \in \mathcal{Y}(y)$, we have the following multiplicative decomposition

$$Y = y\mathcal{E}(-\lambda \cdot M)\mathcal{E}(L)D,$$

where L is a càdlàg local martingale satisfying $\langle M, L \rangle \equiv 0$, and D is a predictable, non-increasing, strictly positive, càdlàg process with $D_0 = 1$.

It can be verified by Itô's formula that $y\mathcal{E}(-\lambda \cdot M)\mathcal{E}(L) \in \mathcal{Y}(y)$. Then, from the fact that V is strictly decreasing, one can deduce that

$$\widehat{Y} = \widehat{y}\mathcal{E}(-\lambda \cdot M)\mathcal{E}(\widehat{L}),$$

which is a local martingale, namely, $\widehat{D} \equiv 1$. □

We would like to provide an alternative method to prove the same theorem [Larson-Žitković, 2007]. Based on the same idea, we could generalize this theorem to the case of bounded random endowment in the next section.

The idea is as follows: we first stop the process $\widehat{X}$ by a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$, such that before each τ_k, $\widehat{X}$ is bounded away from 0. Precisely, define a localizing sequence

$$\tau_k := \inf\{t : \widehat{X}_t < 1/k\} \wedge T.$$

Since $\widehat{X}$ is continuous, $\widehat{X}_{\tau_k} \geq 1/k$ and $\mathbf{P}(\lim_k \tau_k = T) = 1$.

Then, we shall construct a process $\widehat{Y}$, such that $\widehat{Y}_T = \widehat{h}_{\widehat{y}}$ and prove that the stopped process $\widehat{Y}_{\cdot\wedge\tau_k}$ is a martingale by means of that $\widehat{X}\widehat{Y}$ is a uniformly integrable martingale. Finally, we prove that $\widehat{Y} \in \mathcal{D}(\widehat{y})$.

From the result in [Kramkov-Schachermayer, 1999], one can find a sequence $\{\mathbf{Q}^n\}_{n=1}^{\infty}$ from $\mathcal{M}^e(S)$ such that

$$\widehat{y}\frac{d\mathbf{Q}^n}{d\mathbf{P}} \longrightarrow \widehat{h}_{\widehat{y}},\ a.s..$$

We denote by Y^n the associated density process, which is a martingale, i.e.,

$$Y_t^n := \widehat{y}\frac{d\mathbf{Q}^n}{d\mathbf{P}}\bigg|_{\mathcal{F}_t}.$$

Then, we construct a process $\widehat{Y}$ in terms of $\{Y^n\}_{n=1}^{\infty}$, such that $\widehat{Y}_T = \widehat{h}_{\widehat{y}}$. To this end, we need the following lemma.

Lemma ([Czichowsky-Schachermayer, 2014])

Let $\{Y^n\}_{n=1}^{\infty}$ *be a sequence of non-negative optional strong supermartingales* $Y^n = \{Y^n_t\}_{0\leq t\leq T}$ *starting at* $Y^n_0 = y$. *Then there is a sequence* $\{\widetilde{Y}^n\}_{n=1}^{\infty}$ *of convex combinations*

$$\widetilde{Y}^n \in conv(Y^n, Y^{n+1}, \cdots)$$

and a non-negative optional strong supermartingale $\widehat{Y} = \{\widehat{Y}_t\}_{0\leq t\leq 1}$ *such that for every* $[0,T]$*-valued stoppting time* τ, *we have convergence in probability, i.e.,*

$$\widetilde{Y}^n_\tau \longrightarrow \widehat{Y}_\tau.$$

WLOG, we may choose a subsequence such that for each k, $\widetilde{Y}^n_{\tau_k} \longrightarrow \widehat{Y}_{\tau_k}$, a.s..

In our case, $\widetilde{Y}^n$ are all true martingales associated with the equivalent local martingale measure $\widetilde{\mathbf{Q}}^n$ defined by $\widehat{y}\frac{d\widetilde{\mathbf{Q}}^n}{d\mathbf{P}} = \widetilde{Y}_T$. Obviously, $\widetilde{Y}^n_T \longrightarrow \widehat{Y}_T = \widehat{h}_{\widehat{y}}$.

Fixing k, by super-replication theorem, we have

$$\mathbb{E}[\widehat{X}_{\tau_k}\widehat{Y}^n_{\tau_k}] \leq x\widehat{y}.$$

On the other hand, by applying the above lemma again, one can see that $\widehat{X}\widehat{Y}$ is an optional strong supermatingale. Furthermore, $\widehat{X}_0\widehat{Y}_0 = \mathbb{E}[\widehat{X}_T\widehat{Y}_T] = \mathbb{E}[\widehat{g}\widehat{h}] = x\widehat{y}$, then $\widehat{X}\widehat{Y}$ is a true martingale.

Therefore,

$$\mathbb{E}[\widehat{X}_{\tau_k}\widehat{Y}_{\tau_k}] = x\widehat{y}.$$

Lemma

Let $X \in \mathbf{L}^0(\Omega, \mathcal{F}, \mathbf{P})$, $X \geq a > 0$, a.s., and $\{Y^n\}_{n=1}^{\infty} \subset \mathbf{L}^1_+(\Omega, \mathcal{F}, \mathbf{P})$, $Y^n \to Y$, a.s.. If

$$\mathbb{E}[XY] \geq \liminf_n \mathbb{E}[XY^n].$$

Then, $\{Y^n\}_{n=1}^{\infty}$ is uniformly integrable.

By the above lemma, we know from

$$\begin{cases} \mathbb{E}[\widehat{X}_{\tau_k} Y^n_{\tau_k}] & \leq x\widehat{y}; \\ \mathbb{E}[\widehat{X}_{\tau_k} \widehat{Y}_{\tau_k}] & = x\widehat{y}. \end{cases}$$

that $\{Y^n_{\tau_k}\}_{n=1}^{\infty}$ is uniformly integrable and thus, the stopped process $\widehat{Y}_{\cdot \wedge \tau_k}$ is a true martingale. Thus, $\widehat{Y}$ is a local martingale and has a càdlàg version. Moreover, by the lemma in [Czichowsky-Schachermayer, 2014] again, we can verify that for each $X \in \mathcal{X}(1)$, $X\widehat{Y}$ is a supermatingale.

Duality method: e_T bounded

Recall the primal problem of the utility maximization

$$u(x) := \sup_{H \text{ adm}} \mathbb{E}[U(x+(H\cdot S)_T+e_T)] = \sup_{g\in\mathcal{C}_0} \mathbb{E}[U(x+g+e_T)], \quad x>0,$$

where $\mathcal{C}_0 := \{g : g = (H\cdot S)_T, H \text{ adm}\}$.

The dual problem is formulated by

$$v(y) := \inf_{Q \in \mathcal{D}} \left\{ \mathbf{E}\left[V\left(y \frac{dQ^r}{d\mathbf{P}} \right) \right] + y\langle Q, e_T \rangle \right\}, \ y > 0,$$

where

$$\mathcal{D} := \Big\{ Q \in (\mathbf{L}^\infty)^*_+ : \ \|Q\|_{(\mathbf{L}^\infty)^*} = 1, \ \langle Q, g \rangle \le x, \text{ for all } g \in \mathcal{C}(x), \text{for all } x > 0 \Big\}.$$

From the result in [Yosida-Hewitt, 1952], for any $Q \in (\mathbf{L}^\infty)^*_+$, Q can be uniquely decomposed into $Q = Q^r + Q^s$, where Q^r is countably additive and Q^s is purely finitely additive.

Theorem ([Cvitanić-Schachermayer-Wang, 2001])

Assume

- *"No arbitrage condition":* $\mathcal{M}^e(S) \neq \emptyset$,
- U *satisfies the Inada conditions and RAE,*
- $u(x) < \infty$, *for some* $x > \rho$.

Then

- *The value functions* u, v *have the same properties as* U *and* V.
- *The dual solution* $\widehat{Q}_y \in \mathcal{D}$ *exists for all* $y > 0$ *and* $\widehat{Q}_y^r$ *is unique.*
- *For all* $x > x_0 := \sup_{Q \in \mathcal{D}} \langle Q, -e_T \rangle$, $\widehat{g} := I\left(\widehat{y}\frac{d\widehat{Q}^r_{\widehat{y}}}{d\mathbf{P}}\right) - x - e_T \in \mathcal{C}_0$ *is the solution to the primal problem, where* $\widehat{y} = u'(x)$.
- *Denote by* $\widehat{H}$ *the corresponding optimal strategy, then*

$$\langle \widehat{Q}^r_{\widehat{y}}, x + (\widehat{H}\cdot S)_T + e_T \rangle = \langle \widehat{Q}_{\widehat{y}}, x + (\widehat{H}\cdot S)_T + e_T \rangle = x + \langle \widehat{Q}_{\widehat{y}}, e_T \rangle.$$

Theorem (Main result)

In addition to the conditions for the above theorem, we assume that the filtration is Brownian, then the regular part of the dual optimizer $\widehat{Q}^r_{\widehat{y}}$ can be attained by some local martingale $\widehat{Y} \in \mathcal{Y}(1)$.

Outline of the proof: for simplicity of notation, we drop the subscript $\widehat{y}$ in $\widehat{Q}_{\widehat{y}}$.

- We prove that the dual optimizer $\widehat{Q}$ can be "approximated" by a sequence $\{\mathbf{Q}^n\}_{n\in\mathbb{N}}$ from $\mathcal{M}^e(S)$ such that

$$\frac{d\mathbf{Q}^n}{d\mathbf{P}} \longrightarrow \frac{d\widehat{Q}^r}{d\mathbf{P}},\ a.s.,\ \text{and } \langle Q^n, e_T\rangle \longrightarrow \langle \widehat{Q}, e_T\rangle.$$

- For each n, denote by Y^n the density process associated with $\mathbf{Q}^n$. We choose a sequence $\{\widetilde{Y}^n\}_{n=1}^{\infty}$ of convex combinations

$$\widetilde{Y}^n \in conv(Y^n, Y^{n+1}, \cdots),$$

and a non-negative optional strong supermartingale $\widehat{Y}$, such that $\widetilde{Y}^n_\tau \longrightarrow \widehat{Y}_\tau$ in probability, for any finite stopping time τ.

- For each n, denote by $\widetilde{\mathbf{Q}}^n$ the equivalent martingale measure determined by $\widetilde{Y}^n$. We define a fictional wealth process by

$$\widetilde{W}^n_t := x + (\widehat{H} \cdot S)_t + \mathbf{E}^{\widetilde{\mathbf{Q}}^n}[e_T|\mathcal{F}_t] = x + (\widehat{H} \cdot S)_t + \widetilde{e}^n_\tau.$$

Then, the fictional optimal wealth process can be construct in a similar way as the step above:

$$\widehat{W}_t := x + (\widehat{H} \cdot S)_t + \widehat{e}_t,$$

where for any finite stopping time τ, $\widetilde{e}^n_\tau \longrightarrow \widehat{e}_\tau$ in probability.

- The process $\widehat{W}\widehat{Y}$ can be proved a martingale. Thanks to the assumption on the filtration, one can find a sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ such that on $[\![0,\tau_k]\!]$, $\widehat{W}$ stays above $1/k$.
- Consider a cluster point Q^* of $\{\widetilde{\mathbf{Q}}^n\}_{n\in\mathbb{N}}$, which is still a dual optimizer and $Q^{*r} \equiv \widehat{Q}^r$ by the uniqueness. We can prove that
$$\widehat{Y}_\cdot = \frac{d(Q^*|_{\mathcal{F}_\cdot})^r}{d\mathbf{P}}.$$
- Fixing k,
$$\langle (Q^*|_{\mathcal{F}_{\tau_k}})^r, x + (\widehat{H}\cdot S)_{\tau_k} + \widehat{e}_{\tau_k}\rangle = x + \langle Q^*, e_T\rangle.$$

- It also can be shown that $(\widehat{H}\cdot S)$ and $\widehat{e}$ is a "martingale" under the finitely additive measure Q^*, namely,

$$\langle Q^*|_{\mathcal{F}_{\tau_k}}, x+(\widehat{H}\cdot S)_{\tau_k}+\widehat{e}_{\tau_k}\rangle = x+\langle Q^*, e_T\rangle.$$

- Because $x+(\widehat{H}\cdot S)_{\tau_k}+\widehat{e}_{\tau_k}\geq 1/k$, we can compare the above equality with

$$\langle (Q^*|_{\mathcal{F}_{\tau_k}})^r, x+(\widehat{H}\cdot S)_{\tau_k}+\widehat{e}_{\tau_k}\rangle = x+\langle Q^*, e_T\rangle,$$

and deduce that $(Q^*|_{\mathcal{F}_{\tau_k}})^s\equiv 0$, which implies that $\mathbb{E}[\widehat{Y}_{\tau_k}]=1$.

- By Scheffé's lemma, we conclude that $\{\widetilde{Y}^n_{\tau_k}\}_{n\in\mathbb{N}}$ is uniformly integrable and thus, $\widehat{Y}$ is a local martingale from $\mathcal{Y}(1)$. $\square$

- In the case that $e_T \neq 0$, we need a condition on the filtration instead of only assuming that S is continuous. That is because we do not have enough information on the fictional process $\widehat{e}$ so that it is difficult to stop the fictional optimal wealth process $\widehat{W}$ and let it stay away from 0.
- if we could do better? namely, could we find a martingale associated with the dual optimizer?
  - If $e_T = 0$, [Kramkov-Weston, 2015] have a positive answer under some (A_p) condition over the dual domain.
  - If e_T is uniformly bounded, [Larsen-Soner-Žitković, 2015] have a counterexample with a geometric brownian motion stock price process.

- In the case that U supports the whole real line and S is locally bounded, Bellini, Frittelli, Owen, Schachermayer, Žitković, observe that the dual optimizer does not lose any mass. However, it may not be equivalent to $\mathbf{P}$ (only absolutely continuous).
- If we consider the numeraire based model in a market with proportional transaction cost, we can deduce a similar result when $e_T = 0$, i.e., if S is continuous and satisfying (NUPBR), the dual optimizer is attained by some local martingale from the set of supermatingale deflators. The case that e_T is uniformly bounded is under consideration.

Thank you for your attention!