Continuity Problems for Boundary Crossing Probabilities

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One of motivations: pricing general barrier options.

Aim: to analyse the accuracy of approximations of boundary crossing probabilities where the originally given boundary is replaced with some other boundaries (hopefully of a form making the computation of the boundary crossing probability feasible).

[A related problem: discrete monitoring vs the continuous one.]

We:

- discuss approximation rates in stability theorems for 1-dim Brownian motion (BM) boundary crossing problems (that were also extended to general time-homogeneous diffusions);
- go beyond approximation rates in that case: Gâteaux differentiability of the crossing probability as a functional of the boundary;
- present a general result on approximation rate in the multivariate case.

- In the general Black–Scholes framework, the problem of "fair pricing" of (two-sided) barrier options can be reduced to calculating

$$P = P(g_-, g_+; B) := \mathbf{P}\big(g_-(t) < W_t < g_+(t),\ t \in [0,T];\ W_T \in B\big),$$

where $\{W_t\}$ is the standard BM.

- Well-known: $P = u(0,0)$, where $u = u(t,x)$ solves

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < t < T, \quad g_-(t) < x < g_+(t),$$

subject to the boundary conditions

$$u(t,x) = \begin{cases} 0 & \text{if } (t,x) \in L_- \cup L_+ \cup L(B^c), \\ 1 & \text{if } (t,x) \in L(B), \end{cases}$$

where $L_\pm = \{(t,x) : 0 < t < T,\ x = g_\pm(t)\}$, $L(A) = \{(T,x) :\ x \in A\}$.

- Similarly in the multivariate case.

- Practitioners say: solving this, in real time (domain changes, barriers move etc.), and to the required precision, is no fun.

- Moreover, Monte Carlo is also quite time-consuming (as this is a path-dependent option, of course), one needs something faster.

- What about trying approximations? For some boundaries, there are closed-form expressions for P. For instance, can find P very quickly for piece-wise linear $g_\pm$ (esp. in the one-sided case): if, say, $g_- = -\infty$, g_+ consists of k pieces of straight lines, the problem of calculating P in the case of a call option reduces (by conditioning on the process' values at the boundary "junction points") to calculating the values of k-dimensional normal CDFs (Novikov et al.).

- If g is regular enough, can approximate it well with piece-wise linear boundaries. But how well will P be approximated by the resp. probabilities for the new boundaies?

- First: Potzelberger & Wang (2001). Then Borovkov & Novikov (2005): *If* $g_\pm \in \mathrm{Lip}_K$ *and, for some* $f_\pm : [0,T] \to \mathbb{R}$*, one has* $\|g_\pm - f_\pm\|_\infty \le \varepsilon$ *for an* $\varepsilon > 0$*, then*

$$|P(-\infty, g_+; B) - P(-\infty, f_+; B)| \le (2.5K + 2T^{-1/2})\varepsilon \tag{1}$$

and

$$|P(g_-, g_+; B) - P(f_-, f_+; B)| \le (5K + 4T^{-1/2})\varepsilon. \tag{2}$$

- Sharp: the coefficient of ε on the RHS on (1) cannot be less than $2K + \sqrt{2\pi^{-1}}T^{-1/2}$ (check the straight line boundary $g_+(t) = \varepsilon + Kt$).

- So, if $g_\pm \in C^1[0,T]$, $K = \max\{\|g'_-\|, \|g'_+\|\}$, and $g'_\pm$ is absolutely continuous with $|g''_\pm| \le \gamma < \infty$ a.e., then for a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ of rank $\delta = \max_{0<i\le n} |t_i - t_{i-1}|$ and $f_\pm$ piece-wise linear functions with nodes at $(t_i, g_\pm(t_i))$, one has

$$|P(g_-, g_+; B) - P(f_-, f_+; B)| \le (0.313K + 0.25T^{-1/2})\gamma\delta^2.$$

How to show that? A simple observation: say, for $P(g_-, g_+) := P(g_-, g_+; \mathbf{R})$,

$$P(g_- + \varepsilon, g_+ - \varepsilon) \le P(f_-, f_+) \le P(g_- - \varepsilon, g_+ + \varepsilon)$$

and then, setting $\tau := \inf\{t > 0 : W_t > g_+(t)\}$ and assuming $T = 1$,

$$\begin{aligned}
P(g_-, g_+ + \varepsilon) - P(g_-, g_+) \\
&= \mathbf{P}\Big(0 \le \sup_{0\le t\le 1} (W_t - g_+(t)) < \varepsilon,\ \inf_{0\le t\le 1} (W_t - g_-(t)) > 0\Big) \\
&\le \mathbf{P}\Big(0 \le \sup_{0\le t\le 1} (W_t - g_+(t)) < \varepsilon\Big) \\
&= \int_0^1 \mathbf{P}(\tau \in dt)\, \mathbf{P}\Big(\sup_{t\le s\le 1} (W_s - g_+(s)) < \varepsilon \Big| W_t = g_+(t)\Big) \\
&\le \int_0^1 \mathbf{P}(\tau \in dt)\, \mathbf{P}\Big(\sup_{0\le s\le 1-t} (W_s - Ks) < \varepsilon\Big).
\end{aligned}$$

The last probability is known & doesn't exceed $\sqrt{2/(\pi(1-t))}\varepsilon + 2K\varepsilon$, so just need an upper bound of the density of τ.

- This bound can be extended to time-homogeneous diffusions

$$dX_t = \mu(X_t)dt + dW_t. \tag{3}$$

- We basically showed that $P(g) := \mathbf{P}_x \left(\sup_{0\le s\le T}(X_s - g(s)) < 0\right)$ is a "locally Lipschitz" functional in $\|\cdot\|_\infty$:

$$\left|P(g+h) - P(g)\right| \le C(g) \sup_{0\le t\le 1} |h(t)|.$$

- The next theorem proves that $P(g)$ is actually Gâteaux differentiable and, moreover, gives a representation for the derivative in terms of the Brownian meander process $\{W_s^{\oplus}\}_{0\le s\le 1}$.
- BTW: both results are new for the boundary problem theory for PDEs.

Recall that the Brownian meander can be defined as follows: letting $\zeta = \sup\{t \le 1 : W_t = 0\}$ be the last zero of the BM in $[0, 1]$, we set

$$W_s^{\oplus} := (1-\zeta)^{-1/2} \big| W_{\zeta+(1-\zeta)s} \big|, \qquad 0 \le s \le 1.$$

Alternatively, its distribution can be thought of as the weak limit, as $\varepsilon \downarrow 0$, of the distribution of $\{W_s\}_{0\le s\le 1}$ conditioned to stay above $-\varepsilon$ on $[0, 1]$:

$$\{W_s^{\oplus}\}_{0\le s\le 1} \Leftarrow \Big(\{W_s\}_{0\le s\le 1} \Big| \inf_{0\le s\le 1} W_s > -\varepsilon \Big)$$

(see e.g. Durrett et al. (1977)).

This is a continuous non-homogeneous Markov process that appears as a limit in a number of conditional functional central limit theorems.

Let (3) *have a unique strong non-explosive solution with diffusion interval* $\mathbb{R}$, *and let* $\mu \in C^1$ *satisfy* $\mu'(y) + \mu^2(y) \geq -Q(y)$, $y \in \mathbb{R}$, *where* $\limsup_{y \to -\infty} y^{-2} Q(y) < 1$. *Assume that* g *and* h *are both* C^2. *Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \Big[P(g + \varepsilon h) - P(g) \Big] = \int_0^1 h(v) \psi(v) p_\tau(v) dv, \tag{4}$$

where

$$\psi(v) = \left(\frac{2}{\pi(1-v)} \right)^{1/2} \mathbf{E} \exp \Big\{ G\left(g(1) - \sqrt{1-v} W_1^{\oplus}\right) - G(g(v)) + \sqrt{1-v} W_1^{\oplus} g'(1) + \overline{N}_{1-v}(1-v) \Big\},$$

where $G'(y) = \mu(y)$ *and we put* $g_{0,u}(s) := g(1-u+s)$, $0 \leq s + u \leq 1$,

$$\overline{N}_u(t) := -\frac{1}{2} \int_0^t \Big[\mu'(-\sqrt{t} W_{s/t}^{\oplus} + g_{0,u}(s)) + \mu^2(-\sqrt{t} W_{s/t}^{\oplus} + g_{0,u}(s)) \Big] ds - \sqrt{t} \int_0^t g''_{0,u}(s) W_{s/t}^{\oplus} ds - \frac{1}{2} \int_0^t (g'_{0,u}(s))^2 ds.$$

The proof of this theorem consists of four steps; for the first three steps, we assume that $h(t) \geq 0$, $0 \leq t \leq 1$.

(i) Observe that the difference $P(g + \varepsilon h) - P(g)$ can be written as an integral by conditioning on the first crossing time τ of g.

(ii) Transform the integrand (using Girsanov's theorem, transforming the trajectory space and then using Girsanov's theorem once again) so it is written as the product of an expectation of a functional of the BM conditioned to stay below a (very) low constant level and a well-known boundary non-crossing probability for the BM.

(iii) Calculate the limit of the ratio of the thus obtained expression to ε as $\varepsilon \to 0$. This requires Theorem 2.1 from Durrett et al. (1977) and involves careful treatment near the right end point of the integration interval.

(iv) Finally, we show how to extend the result to general h which are twice continuously differentiable.

EXAMPLE. For $X_t \equiv W_t$, consider the case of linear functions $g(t) = a_1 + b_1 t$ and $h(t) = a_2 + b_2 t$, where $b_1, a_2, b_2 \in \mathbb{R}$ and $a_1 > 0$. Then (4) becomes

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \Big[P(g + \varepsilon h) - P(g) \Big]$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{a_2 + b_2(1-t)}{\sqrt{t}} \mathbf{P}_x(1 - \tau \in dt) \mathbf{E}\, e^{\sqrt{t} b_1 W_1^{\oplus} - \frac{1}{2} b_1^2 t}. \qquad (5)$$

Here we know that, for a boundary $a + bt$ with $a, b > 0$,

(a)

$$P(a + bt) = \overline{\Phi}(-a - b) - e^{-2ba} \Phi(-a + b),$$

(b)

$$p_\tau(t) = \frac{a}{\sqrt{2\pi} t^{3/2}} \exp\left\{ -\frac{(a + bt)^2}{2t} \right\},$$

Furthermore,

(c)

$$\mathbf{P}(W_1^{\oplus} \in dy) = y e^{-y^2/2} dy, \qquad y > 0.$$

In the special case $b_1 = b_2 = 0$, one can evaluate both sides of (5) and find that they each give $(2/\pi)^{1/2} a_2 e^{-a_1^2/2}$.

In the general case, we evaluate the required Laplace transform:

$$\mathbf{E}\, e^{\lambda W_1^{\oplus}} = 1 + \sqrt{2\pi}\lambda e^{\lambda^2/2}\overline{\Phi}(-\lambda), \qquad \lambda \in \mathbb{R}.$$

to find that (5) is equivalent to the following curious identity:

$$\begin{aligned} a_2\sqrt{\frac{2}{\pi}}e^{-(a_1+b_1)^2/2} &+ 2(a_2 b_1 + a_1 b_2)e^{-2a_1 b_1}\Phi(b_1 - a_1) \\ &= \frac{a_1}{\pi}\int_0^1 \frac{a_2 + b_2(1-t)}{\sqrt{t}(1-t)^{3/2}} \exp\left\{-\frac{(a_1 + b_1(1-t))^2}{2(1-t)} - \frac{b_1^2 t}{2}\right\} \\ &\qquad\qquad \times \left(1 + \sqrt{2\pi t}\, b_1 e^{t b_1^2/2}\overline{\Phi}(-\sqrt{t} b_1)\right) dt. \end{aligned}$$

For given values of a_1, a_2, b_1 and b_2, one can verify the identity by numerically evaluating the integral on its RHS. It holds.

Now turn to the multivariate case and introduce notation:

- $\{\boldsymbol{W}_t = (W_t^{(1)}, \ldots, W_t^{(m)})\}_{t \geq 0}$ is the standard m-dim BM;
- for $A \subset [0,T] \times \mathbb{R}^m$, set

$$A_t := \{\boldsymbol{x} \in \mathbb{R}^m : (t, \boldsymbol{x}) \in A\}, \qquad t \in [0,T];$$

- for $B \subset \mathbb{R}^m$ and $\varepsilon > 0$, set $\rho(\boldsymbol{x}, B) := \inf_{\boldsymbol{y} \in B} \|\boldsymbol{x} - \boldsymbol{y}\|$ and

$$B^{(+\varepsilon)} := \{\boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, B) < \varepsilon\}, \quad B^{(-\varepsilon)} := \{\boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, B^c) \leq \varepsilon\}^c;$$

- the Hausdorff distance b/w sets $B, \widetilde{B} \subset \mathbb{R}^m$ is

$$\rho_H(B, \widetilde{B}) := \inf\{\varepsilon > 0 : B \subset \widetilde{B}^{(+\varepsilon)}, \ \widetilde{B} \subset B^{(+\varepsilon)}\};$$

- we also set $\overline{\rho}_H(B, \widetilde{B}) := \rho_H(B, \widetilde{B}) \vee \rho_H(B^c, \widetilde{B}^c)$;
- for a(n open) set $G \subset [0,T] \times \mathbb{R}^m$, let

$$\boxed{P(G) := \mathbf{P}\big(\boldsymbol{W}_t \in G_t, \ t \in [0,T]\big).}$$

Denote by $\mathcal{G}$ the class of all open sets $G \subset [0, T] \times \mathbb{R}^m$ with $(0, \mathbf{0}) \in G$. For positive numbers K, β, γ, introduce the class $\mathcal{G}_{K,\beta,\gamma} \subset \mathcal{G}$ of sets G satisfying the following conditions **[G1]**–**[G3]** on their cross-sections.

[G1] There exists a $K < \infty$ s.t. $\overline{\rho}_H(G_s, G_t) \leq K(t - s)$, $0 \leq s < t \leq T$.

It is easy to show that **[G1]** implies that ∂G_t is K-Lipschitz in ρ_H.

[G2] For any $t \in (0, T)$ and $\boldsymbol{x} \in \partial G_t$, there exists a ball $B_\beta(\boldsymbol{y})$ (of radius β, centered at $\boldsymbol{y}$) s.t. $B_\beta(\boldsymbol{y}) \subset G_t^c$ and $\boldsymbol{x} \in \partial B_\beta(\boldsymbol{y})$.

The above condition prevents the boundary of G_t from “folding outwards”.

[G3] For any $t \in (0, T)$, there exists a $v_0 < \infty$ s.t.

$$\mathbf{E}\left[1 + \|\boldsymbol{W}_t\|\,;\, 0 < \rho(\boldsymbol{W}_t, G_t^c) < v\right] \leq \gamma v, \quad 0 < v < v_0.$$

This condition means that, as one "peels" the set G_t, the "layers" of a given thickness are uniformly "light" (when weighed using the distr'n of $\boldsymbol{W}_t$, with or without the additional weight $\|\boldsymbol{W}_t\|$). Roughly speaking, that means that the boundary ∂G_t "cannot have too many folds".

It is not hard to show that **[G2]** & **[G3]** are satisfied when G_t are convex.

Theorem *If* $G \in \mathcal{G}_{K,\beta,\gamma}$ *then there exists a constant* $c = c(T, K, \beta, \gamma) < \infty$ *such that*

$$|P(G^{(\varepsilon)}) - P(G)| \leq c\varepsilon, \quad \varepsilon > 0.$$

Corollary *Assume that* $G \in \mathcal{G}$ *satisfies* **[G1]** *and* G_t *is convex for any* $t \in (0, T)$. *Then* G *also satisfies* **[G2]** *with any* $\beta > 0$ *and* **[G3]** *for some* $\gamma < \infty$, *and so the bound from the above theorem holds true.*

Corollary *Suppose that* $G \in \mathcal{G}_{K,\beta,\gamma}$. *For any* $\varepsilon > 0$, *if sets* $G', G'' \in \mathcal{G}$ *are such that* $G \subset G' \subset G^{(\varepsilon)}$, $G \subset G'' \subset G^{(\varepsilon)}$, *then*

$$|P(G') - P(G'')| < c\varepsilon, \quad \varepsilon > 0,$$

for some constant $c = c(T, K, \beta, \gamma) < \infty$.

How to prove that? Assume $T = 1$ (w.l.o.g.).

Then there are two major steps:

(i) Need bounds for the density $p(t)$ of $\tau := \inf\{t > 0 : \boldsymbol{W}_t \in \partial G_t\}$ to use in

$$P\big(G^{(\varepsilon)}\big) - P(G) = \int_{(0,1)} \mathbf{P}\big(\tau^{(\varepsilon)} = 1 | \tau = t\big) \mathbf{P}(\tau \in dt). \tag{6}$$

And here they are: $p(t) \leq$

$$8m^2\gamma \begin{cases} \sqrt{\frac{1}{\pi t}} + \frac{m-1}{2\beta - Kt} + 2K + \frac{2}{t}, & t \in (0, (\beta/K) \wedge 1], \\ \sqrt{\frac{K}{\pi\beta}} + \frac{\beta+2}{2t-\beta/K} + \frac{m-1}{\beta} + K, & t \in [(\beta/K) \wedge 1, 1]. \end{cases}$$

The bound is of independent interest.

To prove the bound, note that, for any $t \in (0,1)$, setting $\tau_t := \inf\{s > t : (s, \boldsymbol{W}_s) \in \partial G\}$, one has, for $0 < h < 1 - t$,

$$\begin{aligned}\mathbf{P}(\tau \in (t, t+h)) &= \int_{G_t} \mathbf{P}(\tau \in (t, t+h) | \boldsymbol{W}_t = \boldsymbol{z}) \mathbf{P}(\boldsymbol{W}_t \in d\boldsymbol{z}) \\ &= \int_{G_t} \mathbf{P}(\tau > t | \boldsymbol{W}_t = \boldsymbol{z}) \mathbf{P}(\tau_t < t + h | \boldsymbol{W}_t = \boldsymbol{z}) \mathbf{P}(\boldsymbol{W}_t \in d\boldsymbol{z}).\end{aligned}$$

Next one (carefully) bounds the two factors in the integrand.

(ii) Combine that with the bound for the integrand in (6): setting

$$C(v,u) := \{(s,\boldsymbol{x}) \in [0,v] \times \mathbb{R}^m : \|\boldsymbol{x}\| \le u - Ks\},$$

one has, for $\boldsymbol{x} \in \mathbb{R}^m$ with $x := \|\boldsymbol{x}\| > \beta$,

$$\frac{\mathbf{P}_{\boldsymbol{x}}\big(\tau(C(t,\beta)) > t | \boldsymbol{W}_t = \boldsymbol{y}\big)}{\|x\| - \beta} \le$$

$$\le \begin{cases} \sqrt{\frac{2}{\pi u}} + 2\Big(\frac{2(\|\boldsymbol{y}| - \beta)}{t} + \frac{m-1}{2\beta - Kt} + 2K\Big)^+, & u \le t/2,\ t < \beta/K, \\ \sqrt{\frac{2}{\pi u}} + 2\Big(\frac{\|\boldsymbol{y}\| - \beta/2}{t - \beta/(2K)} + \frac{m-1}{\beta} + K\Big)^+, & u \le \beta/(2K),\ t \ge \beta/K, \end{cases}$$

where $x^+ := \max\{0, x\}$.

Note that the above upper bounds agree at $t = \beta/K$.