Pricing step options in an exponential spectrally negative Lévy model

Hélène GUÉRIN

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UNIVERSITÉ DE RENNES 1

Outline

Occupation time options

Let $S = \{S_t, t \geq 0\}$ be the risk-neutral price of an asset.

Introduced by Linetsky in 1999, (barrier) step options are exotic options linked to occupation times of the underlying asset price process.

A (down-and-out call) step option admits the following payoff, for a pre-specified level L,

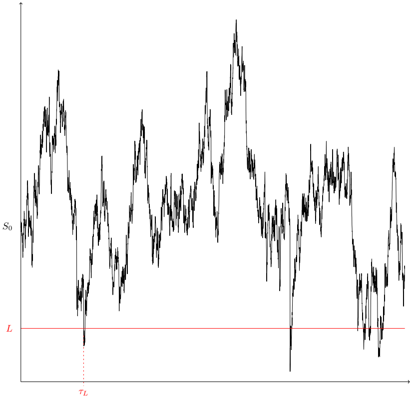
$$\mathrm{e}^{-\rho \int_0^T \mathbf{1}_{\{S_t \leq L\}} \mathrm{d}t} (S_T - K)_+$$

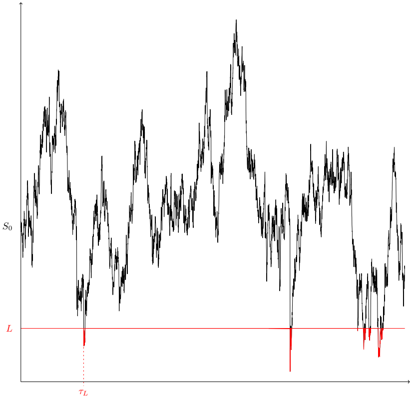
where $\rho > 0$ is called the knock-out rate.

It is interesting to note that we have the following relationship:

$$\mathbf{1}_{\{\tau_L^- > T\}} (S_T - K)_+ \leq \mathrm{e}^{-\rho \int_0^T \mathbf{1}_{\{S_t \leq L\}} \mathrm{d}t} (S_T - K)_+ \leq (S_T - K)_+ ,$$

where $\tau_L^- = \inf\{t \geq 0 \colon S_t < L\}$


S_0
L
τ_L


S_0
L
τ_L

Pricing step options

If $S_t = S_0 \mathrm{e}^{X_t}$, where $X = \{X_t, t \geq 0\}$ is the log-return process, then

$$\int_0^T \mathbf{1}_{\{S_t \leq L\}} \mathrm{d}t = \int_0^T \mathbf{1}_{\{X_t \leq \ln(L/S_0)\}} \mathrm{d}t.$$

For example, in the Black-Scholes-Merton model, X is a Brownian motion with drift.

Mathematically, the payoff of a (down-and-out) proportional step call option can then be written as follows:

$$\mathrm{e}^{-\rho \int_0^T \mathbf{1}_{\{S_t \leq L\}} \mathrm{d}t} \left(S_T - K\right)_+ = \mathrm{e}^{-\rho \int_0^T \mathbf{1}_{\{X_t \leq \ln(L/S_0)\}} \mathrm{d}t} \left(S_0 \mathrm{e}^{X_T} - K\right)_+ .$$

Therefore, its price can be written as

$$\begin{aligned} C(T) &:= \mathrm{e}^{-rT}\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{S_t\le L\}}\mathrm{d}t}\,(S_T-K)_+\right] \\ &= \mathrm{e}^{-rT}\int_{\ln(K/S_0)}^{\infty}(S_0\mathrm{e}^y-K)\,\mathbb{E}\left[\mathrm{e}^{-\rho\int_0^T \mathbf{1}_{\{X_s\le\ln(L/S_0)\}}\mathrm{d}s};X_T\in\mathrm{d}y\right] \end{aligned}$$

with r the risk-free interest rate.

Linetsky (1999) obtained an analytical pricing formula for a down-and-out proportional step call option for the Black-Scholes-Merton model.

Expressions for the price of barrier step options are also available in Kou's jump-diffusion model; see Cai, Chen & Wan (2010).

A general model

We now want to price step options in an exponential spectrally negative Lévy model, that is when the price of the underlying asset is

$$S_t = S_0 \mathrm{e}^{X_t},$$

where the log-return process $X = \{X_t, t \geq 0\}$ is a spectrally negative Lévy process (SNLP).

We now give a brief introduction to this family of processes.

Spectrally negative Lévy processes

Let $X = (X_t)_{t\geq 0}$ be a spectrally negative Lévy process (SNLP), that is a Markov process with stationary and independent increments and no positive jumps.

Its Laplace transform is given by

$$\mathbb{E}\left[\mathrm{e}^{\theta X_t}\right] = \mathrm{e}^{t\psi(\theta)},$$

where

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty \left(\mathrm{e}^{-\theta z} - 1 + \theta z \mathbf{1}_{(0,1]}(z)\right)\Pi(\mathrm{d}z),$$

where Π is the Lévy measure, i.e. a measure on $(0,\infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

One can define $\Phi(q) = \sup\{\theta \geq 0 \mid \psi(\theta) = q\}$ such that $\psi(\Phi(q)) = q$, for $q \geq 0$.

Brownian motion with drift

If $X = (X_t)_{t \geq 0}$ is a Brownian motion with drift:

$$X_t = \gamma t + \sigma B_t,$$

then $\Pi(\mathrm{d}z) \equiv 0$ and

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2, \quad \Phi(q) = \frac{\sqrt{\gamma^2 + 2\sigma^2 q} - \gamma}{\sigma^2}.$$

Jump-diffusion process with hyper-exponential jumps

Let X be a compound Poisson process plus drift, and possibly perturbed by Brownian motion, with a hyperexponential jump distribution:

$$X_t = \mathsf{c}t + \sigma B_t - \sum_{i=1}^{N_t} Y_i,$$

where $Y_1, Y_2, \ldots$ are iid with common p.d.f. given by

$$z \mapsto \sum_{i=1}^{k} a_i \alpha_i \mathrm{e}^{-\alpha_i z}, \quad z > 0,$$

where k is a positive integer, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$ and $\sum_{i=1}^{k} a_i = 1$, where $a_i > 0$ for all $i = 1, \ldots, k$.

Jump-diffusion process with hyper-exponential jumps

In this case, the Laplace exponent of X is given by

$$\psi(\theta) = \mathsf{c}\theta + \frac{1}{2}\sigma^2\theta^2 - \lambda + \lambda \sum_{i=1}^{k} \frac{a_i \alpha_i}{\theta + \alpha_i}, \quad \theta > -\alpha_1,$$

and

$$\Phi(q) = \sup\{\theta \geq 0 \mid \psi(\theta) - q = 0\}.$$

Pricing step options in an exponential spectrally negative Lévy model

We now consider $S_t = S_0 \mathrm{e}^{X_t}$, where $X = \{X_t, t \geq 0\}$ is now a SNLP.

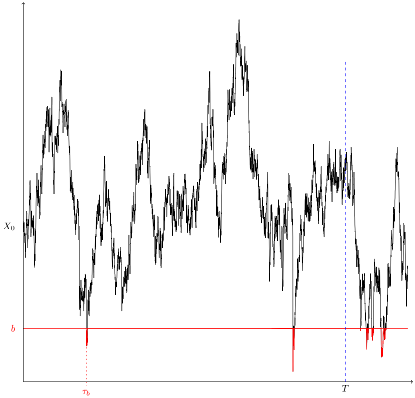
Under a risk-neutral measure $\mathbb{P}$, the price of a step call option is

$$C(T) = \mathrm{e}^{-rT} \int_{\ln(K/S_0)}^{\infty} (S_0 \mathrm{e}^y - K)\, \mathbb{E}\left[\mathrm{e}^{-\rho \int_0^T \mathbf{1}_{(-\infty, \ln(L/S_0))}(X_s) \mathrm{d}s} ; X_T \in \mathrm{d}y\right]$$

Thus, pricing step options boils down to identyfing the following distribution:

$$\mathbb{E}\left[\mathrm{e}^{-q \int_0^T \mathbf{1}_{(-\infty, b)}(X_s) \mathrm{d}s} ; X_T \in \mathrm{d}y\right],$$

for a given value $b \in \mathbb{R}$ and a parameter $q > 0$.


X_0
b
τ_b
T

Scale functions

All our results will be explicitly stated in terms of the scale functions of the SNLP.

Recall that $\Phi(q) = \sup\{\theta \geq 0 \mid \psi(\theta) = q\}$.

For $q \geq 0$, there exists a unique nonnegative continuous function $W^{(q)}$ strictly increasing on $\mathbb{R}^+$ such that such $W^{(q)}(x) = 0$ for $x < 0$ and

$$\int_0^\infty \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q).$$

Those functions are called scale functions.

We will also use the following function:

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y.$$

Brownian motion

In particular, for a Brownian motion with drift $X_t = \gamma t + \sigma B_t$ with $\gamma = \psi'(0+) \geq 0$ and $\sigma > 0$, we have

$$W^{(q)}(x) = \frac{2}{\sqrt{\gamma^2 + 2q\sigma^2}} \mathrm{e}^{-\frac{\gamma}{\sigma^2}x} \sinh\left(\frac{x}{\sigma^2}\sqrt{\gamma^2 + 2q\sigma^2}\right).$$

For standard Brownian motion,

$$W^{(q)}(x) = \sqrt{\frac{2}{q}} \sinh(x\sqrt{2q})$$

and then

$$Z^{(q)}(x) = \cosh(x\sqrt{2q}).$$

Jump-diffusion process with hyper-exponential jumps

Recall that for the jump-diffusion process with hyperexponential jumps:

$$\psi(\theta) = \mathrm{c}\theta + \frac{1}{2}\sigma^2\theta^2 - \lambda + \lambda \sum_{i=1}^{k} \frac{a_i \alpha_i}{\theta + \alpha_i}, \quad \theta > -\alpha_1.$$

If we assume that $q > 0$ or $\psi'(0) \neq 0$, then, for $x \geq 0$,

$$W^{(q)}(x) = \sum_{i=1}^{N} \frac{\mathrm{e}^{\theta_i^{(q)} x}}{\psi'\left(\theta_i^{(q)}\right)}, \tag{1}$$

$$Z^{(q)}(x) = \begin{cases} q \sum_{i=1}^{N} \frac{\mathrm{e}^{\theta_i^{(q)} x}}{\theta_i^{(q)} \psi'\left(\theta_i^{(q)}\right)} & \text{if } q > 0, \\ 1 & \text{if } q = 0, \end{cases} \tag{2}$$

where $N = (n+1) + \mathbf{1}_{\{\sigma > 0\}}$ and where $\theta_1^{(q)} = \Phi(q) \geq 0 > \theta_2^{(q)} > \ldots > \theta_N^{(q)}$ are the roots of $\theta \mapsto \psi(\theta) - q$.

Distribution of passage times

For $a < b$, we define

$$\tau_b^+ = \inf\{t > 0 : X_t > b\},$$
$$\tau_a^- = \inf\{t > 0 : X_t < a\}.$$

Then

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\tau_b^+<\infty}\right] = e^{-\Phi(q)b},$$
$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\tau_b^+<\tau_a^-}\right] = \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)},$$
$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^-}\mathbf{1}_{\tau_a^-<\tau_b^+}\right] = Z^{(q)}(x-a) - Z^{(q)}(b-a)\frac{W^{(q)}(x-a)}{W^{(q)}(b-a)}.$$

Main result

To state our result, we will need the following 2nd-generation scale functions:

$$\mathcal{W}_a^{(p,q)}(x) = W^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-z)W^{(p)}(z)\mathrm{d}z,$$

$$\mathcal{H}^{(p,q)}(x) = \mathrm{e}^{\Phi(p)x}\left[1 + q\int_0^x \mathrm{e}^{-\Phi(p)z}W^{(p+q)}(z)\mathrm{d}z\right].$$

Using a probabilistic decomposition:

Theorem (G. & Renaud)
Fix $b \in \mathbb{R}$. For all $x \in \mathbb{R}$ and $q \geq 0$, we have

$$p \mapsto \int_0^\infty \mathrm{e}^{-pT}\mathbb{E}_x\left[\mathrm{e}^{-q\int_0^T \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s}; X_T \in \mathrm{d}y\right]\mathrm{d}T$$
$$= \left\{\left(\frac{\Phi(p+q)-\Phi(p)}{q}\right)\mathcal{H}^{(p+q,-q)}(x-b)\mathcal{H}^{(p,q)}(b-y) - \mathcal{W}_{x-b}^{(p,q)}(x-y)\right\}\mathrm{d}y.$$

Then the Laplace transform of the down-and-out step call option price is given by: assuming $r = 0$ and $S_0, K > L$, we have

$$\int_0^\infty \mathrm{e}^{-pT} C(T)\mathrm{d}T$$
$$= \frac{\Phi(p+\rho) - \Phi(p)}{\rho\Phi(p)(\Phi(p)-1)} \mathcal{H}^{(p+\rho,-\rho)}(\ln(S_0/L)) K \left(\frac{L}{K}\right)^{\Phi(p)} - \int_0^{\ln(S_0/K)} \left(S_0 \mathrm{e}^{-y} - K\right) W^{(p)}(y)\mathrm{d}y,$$

with

$$\mathcal{H}^{(p+\rho,-\rho)}(\ln(S_0/L)) = \left(\frac{S_0}{L}\right)^{\Phi(p+\rho)} \left(1 - \rho \int_0^{\ln(S_0/L)} \mathrm{e}^{-\Phi(p+\rho)y} W^{(p)}(y)\mathrm{d}y\right).$$

In a Lévy jump-diffusion model

Further, for a Lévy jump-diffusion model with hyper-exponential jumps, since for $x \geq 0$ we have

$$\mathcal{H}^{(p,q)}(x) = q \sum_{i=1}^{N} \frac{e^{\theta_i^{(p+q)} x}}{\left(\theta_i^{(p+q)} - \theta_1^{(p)}\right) \psi'\left(\theta_i^{(p+q)}\right)},$$

and, for $x < 0$, $\mathcal{H}^{(p,q)}(x) = \exp\left(\theta_1^{(p)} x\right)$, and, for $a < x$,

$$\mathcal{W}_a^{(p,q)}(x) = q \sum_{i,j=1}^{N} \frac{\exp\left(\theta_i^{(p+q)} x + \left(\theta_j^{(p)} - \theta_i^{(p+q)}\right) a\right)}{\left(\theta_i^{(p+q)} - \theta_j^{(p)}\right) \psi'\left(\theta_i^{(p+q)}\right) \psi'\left(\theta_j^{(p)}\right)},$$

then we have

$$\int_0^\infty \mathrm{e}^{-pT} C(T)\mathrm{d}T$$

$$= K \sum_{i=2}^{N} \frac{1}{\psi'\left(\theta_i^{(p)}\right)} \left[\frac{\Phi(p+\rho)-\Phi(p)}{\left(\Phi(p+\rho)-\theta_i^{(p)}\right)\Phi(p)(\Phi(p)-1)} \left(\frac{L}{K}\right)^{\Phi(p)} \left(\frac{S_0}{L}\right)^{\theta_i^{(p)}} - \frac{1}{\theta_i^{(p)}\left(\theta_i^{(p)}-1\right)} \left(\frac{S_0}{K}\right)^{\theta_i^{(p)}} \right],$$

when $S_0 > K > L$,

and

$$\int_0^\infty \mathrm{e}^{-pT} C(T)\mathrm{d}T$$
$$= K \frac{\Phi(p+\rho) - \Phi(p)}{\Phi(p)(\Phi(p)-1)} \sum_{i=1}^{N} \frac{1}{\left(\Phi(p+\rho) - \theta_i^{(p)}\right)\psi'\left(\theta_i^{(p)}\right)} \left(\frac{L}{K}\right)^{\Phi(p)} \left(\frac{S_0}{L}\right)^{\theta_i^{(p)}},$$

when $K \geq S_0 > L$.

Other applications

Our main result gives an expression in term of scale functions of the Laplace transform with respect to t of

$$\mathbb{E}_x \left[\mathrm{e}^{-q \int_0^t \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s} ; X_t \in \mathrm{d}y \right].$$

We can then deduce the distribution of the occupation time of the process X under the level b

$$\int_0^t \mathbf{1}_{(-\infty,b)}(X_s)\mathrm{d}s.$$

Brownian motion with drift

We recover the generalized Arcsinus law obtained by Akahori (1995).

For a fixed $t > 0$, the distribution of the occupation time of the negative half-line of the drifted Brownian motion $X_t = ct + \sigma B_t$, is given by

$$\mathbb{P}_0\left(\int_0^t \mathbf{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right) = \frac{2}{\sigma^2}\left\{\frac{\sigma \mathrm{e}^{-(c^2/2\sigma^2)s}}{\sqrt{2\pi s}} - c\overline{N}\left(\frac{c\sqrt{s}}{\sigma}\right)\right\} \times \left\{c + \frac{\sigma \mathrm{e}^{-(c^2/2\sigma^2)(t-s)}}{\sqrt{2\pi(t-s)}} - c\overline{N}\left(\frac{c\sqrt{t-s}}{\sigma}\right)\right\}\mathrm{d}s,$$

for $0 \le s \le t$, where $\overline{N}$ denotes the tail of the standard normal distribution, that is

$$\overline{N}(x) = \int_x^\infty \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y.$$

Cramér-Lundberg model with exponential claims

Let us consider

$$X_t = \mathsf{c}t - \sum_{i=1}^{N_t} C_i, \tag{3}$$

where $\mathsf{c} > 0$ denotes the constant premium rate, $N = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and the C_i are i.i.d. exponential variables with rate $\alpha > 0$.

The characteristics are explicits

$$\psi(\theta) = \mathsf{c}\theta - \frac{\lambda\theta}{\alpha+\theta}, \quad \Phi(q) = \frac{1}{2\mathsf{c}}\left(q + \lambda - c\alpha + \sqrt{\Delta_q}\right)$$

$$W^{(q)}(x) = \frac{\alpha + \Phi(q)}{\sqrt{\Delta_q}} \mathrm{e}^{\Phi(q)x} - \frac{\alpha + \theta(q)}{\sqrt{\Delta_q}} \mathrm{e}^{\theta(q)x}$$

with $\Delta_q = (q + \lambda - \mathsf{c}\alpha)^2 + 4\mathsf{c}\alpha q$ and $\theta(q) = \frac{1}{2\mathsf{c}}\left(q + \lambda - \mathsf{c}\alpha - \sqrt{\Delta_q}\right)$.

For a fixed $t > 0$, we have

$$\mathbb{P}_0\left(\int_0^t \mathbf{1}_{(-\infty,0)}(X_u)\mathrm{d}u \in \mathrm{d}s\right) = a_t\delta_0\left(\mathrm{d}s\right) + a_{t-s}\left(\lambda - c\alpha(1-a_s)\right)\mathbf{1}_{(0,t)}(s)\mathrm{d}s,$$

with

$$\begin{aligned} a_t &:= \mathbb{P}(\tau_0^- > t) \\ &= \left(1 - \frac{\lambda}{c\alpha}\right)_+ + \frac{2\lambda}{\pi}\mathrm{e}^{-(\lambda+c\alpha)t}\int_{-1}^1 \frac{\sqrt{1-u^2}}{\lambda + c\alpha + 2\sqrt{c\alpha\lambda}\,u}\mathrm{e}^{-2\sqrt{c\alpha\lambda}\,t\,u}\mathrm{d}u. \end{aligned}$$

Thank you for your attention!

this is a joint work with Jean-François Renaud (UQÀM)