A measure-valued SDE with applications to interest rates and stochastic volatility

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Modelling goals

- No arbitrage
- Match data well
- Tractable for quick calibration

Factor models

- short rate $r_t = R(Z_t)$
- bond prices $P(t, T) = H(T - t, Z_t)$
- $Z = (Z_t)_{t \geq 0}$ factor process

Model the factor

- $dZ = b(Z)dt + \sigma(Z)dW$

No arbitrage

- $H(0, z) = 1$
- $\partial_x H = b\ \partial_z H + \frac{1}{2}\sigma^2\ \partial_{zz} H - RH$

Suppose H is not degenerate in a certain sense and that

$$H(x, \cdot)$$

is a polynomial of degree n.

Theorem (Si Cheng, MT 2014)

Suppose $n \geq 2$

- $\mathrm{degree}(R) \leq 2$
- $\mathrm{degree}(b) \leq 3$
- $\mathrm{degree}(\sigma^2) \leq 4$

Note n is arbitrary!

Theorem

If $n = 1$

- $\mathrm{degree}(R) = 1$
- $\mathrm{degree}(b) \leq 2$
- σ *arbitrary!*

Computations

Suppose

- $R(z) = R_0 + R_1 z + R_2 z^2$
- $b(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3$
- $\sigma^2(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$
- $H(x, z) = \sum_{k=0}^{n} g_k(x) z^k$

Theorem (continued)

- $R_2 = \frac{n}{2} b_3 = -\frac{n(n-1)}{2} a_4$
- $R_1 = n b_2 + \frac{n(n-1)}{2} a_3.$

$$\begin{aligned} g_i{}' = & g_{i-2} \left((i-2) b_3 + \frac{(i-2)(i-3)}{2} a_4 - R_2 \right) \\ & + g_{i-1} \left((i-1) b_2 + \frac{(i-1)(i-2)}{2} a_3 - R_1 \right) \\ & + g_i \left(i b_1 + \frac{i(i-1)}{2} a_2 - R_0 \right) \\ & + g_{i+1} \left((i+1) b_0 + \frac{i(i+1)}{2} a_1 \right) \\ & + g_{i+2} \frac{(i+2)(i+1)}{2} a_0 \end{aligned}$$

If $S =$

$$\begin{pmatrix} -R_0 & b_0 & a_0 & & & \\ -R_1 & b_1 - R_0 & 2b_0 + a_1 & 3a_0 & & \\ -R_2 & b_2 - R_1 & 2b_1 + a_2 - R_0 & 3b_0 + 3a_1 & 6a_0 & \\ & b_3 - R_2 & 2b_2 + a_3 - R_1 & 3b_1 + 3a_2 - R_0 & 4b_0 + 6a_1 & \ddots \\ & & 2b_3 + a_4 - R_2 & 3b_2 + 3a_3 - R_1 & 4b_1 + 6a_2 - R_0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

then

$$G(x) = \begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix}$$

solves

$$G' = SG.$$

Boundary condition

$$H(0,z) = 1 \Rightarrow G(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Spectral representation

- $\{r_0, \ldots, r_n\}$ eigenvalues of $-S$
- $G_k(x) = \sum_{i=0}^n q_{i,k} e^{-r_i x}$
- $Q_i(z) = \sum_{k=0}^n q_{i,k} z^k$

$$H(x, z) = \sum_{i=0}^{n} Q_i(z) e^{-r_i x}$$

equivalently

$$P(t, T) = \sum_{i=0}^{n} Q_i(Z_t) e^{-r_i(T-t)}$$

Related work

- The $n = 1$ case is due to Siegel (2014).
- Cuchiero, Keller-Ressel & Teichmann (2012) study Markov Y such that

$$\mathbb{E}[Y_t^k | Y_0 = y] = \text{ polynomial of degree } \le k$$

for all k

- we study Markov Z such that

$$\mathbb{E}[e^{-\int_0^t R(Z_s)ds} | Z_0 = z] = \text{ polynomial of degree } \le n$$

for fixed n.

Examples when $n \geq 2$

- Suppose $R(z) = z$
- $dr = (b_0 + b_1 r + \frac{c}{n} r^2)dt + \sqrt{a_0 + a_1 r + a_2 r^2 + \frac{2(1-c)}{n(n-1)} r^3} dW_t.$
- seven parameters
- 3/2-type model family

- Now suppose $R(z) = R_0 + z^2$
- $dZ = (b_0 + b_1 Z + \frac{1}{n} Z^2(2Z - c))dt + \sqrt{a_0 + a_1 Z + a_2 Z^2 - \frac{2}{n(n-1)} Z^3(Z - c)} dW_t$

These are essentially the only examples

Generalisation: spectral models

- $P(t, T) = \int e^{-r(T-t)} \mu_t(dr)$
- $R_t = \int r \; \mu_t(dr)$
- $\mu_t(\mathbb{R}) = 1$

$$P(t,T) = \mathbb{E}[e^{-\int_t^T R_s ds} | \mathcal{F}_s]$$

"if and only if"

$$d\,[\mu_t(dr)] = (R_t - r)\mu_t(dr)dt + M(dr \times dt) \qquad (*)$$

meaning

$$M_t^{\varphi} = \int \varphi(r)\mu_t(dr) - \int \varphi(r)\mu_0(dr) - \int_0^t \left(\int \varphi(r)(R_s - r)\mu_s(dr) \right) ds$$

is a local martingale for test functions φ

A convenient feature

Suppose $(\mu_t)_{t\geq 0}$ solves the SDE (*).

- If μ_t is a non-negative measure supported on $[0,\infty)$ then

$$R_t = \int r \ \mu_t(dr) \geq 0$$

- More generally, if μ_t^+ is supported on $[0,\infty)$ and μ_t^- supported on $(-\infty, 0]$ then $R_t \geq 0$.

An inconvenient feature

Suppose $(\mu_t)_{t\geq 0}$ solves the SDE (*) and takes values in the space of probability measures. Then the spot rate

$$R_t = \int r \ \mu_t(dr)$$

is a supermartingale.

A Hilbert space treatment

- $\|\varphi\|_H^2 = \varphi(0)^2 + \int_{[-N,N]} \varphi'(r)^2 dr$
- $H = \{\varphi : [-N, N] \to \mathbb{R}, \|\varphi\|_H < \infty\}$
- For signed-measure μ, let

$$\|\mu\|_{H^*} = \sup_{\varphi \in H, \|\phi\|_H = 1} \int_{[-N,N]} \varphi(r)\mu(dr)$$

- H^* = completion of signed measures

Assumptions

- $\sigma : H^* \to L(\mathbb{R}^d, H^*)$

Centering $\sigma(\mu)^* \mathbf{1} = 0.$

Lipschitz $\|\sigma(\mu) - \sigma(\nu)\|_{HS} \leq C\|\mu - \nu\|_{H^*}.$

continuity $\sigma(\mu)(dr) = g(\mu, r)\mu(dr)$ where g is bounded

Theorem (Si Cheng, MT 2015)

- *μ_0 a probability measure on $[-N, N]$*
- *W is $\mathbb{R}^d$-valued Brownian motion (or cylindrical Brownian motion)*

then there is a unique strong probability measure valued strong solution to the SDE (*) where

$$M_t = \int_0^t \sigma(\mu_s) dW_s$$

Challenges

- quadratic non-linearity in drift
- H^* contains distributions which are wilder than signed measures

Method of proof

Approximate

$$\mu_t \approx \sum_i X_t^i \delta_{r_i}$$

then $(X_t^1, \ldots, X_t^n)$ solves a finite-dimensional SDE (Siegel's model) then take limits

Representation of the solution

Let

$$\nu_t(dr) = e^{-rt}\mu_0(dr) + \int_0^t e^{-r(t-s)} M(ds \times dr)$$

Then

$$\mu_t(dr) = \frac{\nu_t(dr)}{\nu_t(\mathbb{R})}$$

Stochastic vol $\Leftrightarrow$ interest rate

- $dS = S\sigma dW$
-

$$\mathbb{E}[S_T^\theta | \mathcal{F}_t] = S_t^\theta \mathbb{E}\left[e^{\theta \int_t^T \sigma_s dW_s - \frac{1}{2}\theta \int_t^T \sigma_s^2 ds} | \mathcal{F}_t \right]$$
$$= S_t^\theta \mathbb{E}^\theta \left[e^{-\frac{1}{2}\theta(1-\theta) \int_t^T \theta \sigma_s^2 ds} | \mathcal{F}_t \right]$$

- $\frac{d\mathbb{Q}^\theta}{d\mathbb{Q}} = e^{\theta \int_0^T \sigma_s dW_s - \frac{1}{2}\theta^2 \int_0^T \sigma_s^2 ds}$
- $dW^\theta = dW - \theta\sigma dt$

- Let $\sigma_t = \Sigma(Z_t)$
- $S_t^\theta H(T - t, Z_t; \theta)$ local martingale
- $H(x, z; \theta) = \sum_{k=0}^{n(\theta)} g_k(x; \theta) z^k$

Note degree depends on θ