No-arbitrage conditions in HJM multi-curve term structure models

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Introduction

A number of anomalies has appeared in the financial markets due to the recent financial crisis. In particular, in interest rate markets the following issues have emerged:

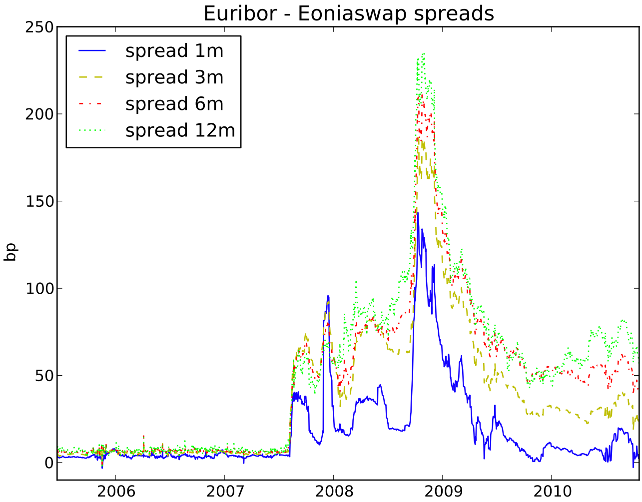
- Counterparty risk – risk of non-payment of promised cash-flows due to default of the counterparty in a bilateral OTC interest rate derivative transaction
- funding (liquidity) issues – cost of borrowing and lending for funding of a position in a contract: CVA, DVA, FVA... adjustments

Introduction

A fundamental interest rate is the Libor/Euribor. Its value is determined by a panel that takes into account various risk factors, principally credit and liquidity risk in the interbank sector. It has therefore been strongly affected by the crisis and, among the consequences we have that:

- The standard no-arbitrage relations between Libors of different maturities have broken down and significant spreads have been observed between LIBORs of different tenors as well as between LIBOR and OIS swap rates.
  - $\rightarrow$ This leads to the multi curve issue: one curve for discounting and different Libor rates, one for each tenor.

Introduction



Introduction

Main Goal: obtain arbitrage-free dynamics for the bonds (*discount curve*) and the forward Libors.

- For simplicity, here only a two curve model with one discounting curve and a single Libor curve corresponding to a fixed tenor.

Outline

- Discounting curve and absence of arbitrage for the basic traded assets
- Arbitrage-free dynamic models in an HJM setup
- Possible approaches for the Libor dynamics
  - A hybrid LMM-HJM approach
  - An approach based on fictitious bonds
    - i) "Intrinsic" no-arbitrage conditions (*in relation to the basic assets*)
    - ii) "Pseudo" no-arbitrage conditions (*based on possible interpretations of the fictitious bonds*)

Discounting curve

First some comments concerning the pure discount curve and the related martingale measure.

- To avoid issues of arbitrage, one should possibly have a common discount curve to be applied to all future cash flows independently of the tenor.
- A common choice is the OIS curve, where OIS stands for overnight indexed swap – a swap in which the floating leg is obtained by discretely compounding the overnight rate (Eonia/FF)
  - $\rightarrow$ Denote simply by $p(t, T) = p^{OIS}(t, T)$ the discount factors (*zero-coupon bonds*) that can be stripped from the quoted OIS rates.

Discounting curve

Motivations for this choice are

- Being based on overnight rates, the OIS rates bear little risk
- The OIS rates are commonly used in collateralization, which by now became standard for all typical interest rate derivative OTC contracts (in particular, swaps)

Absence of arbitrage

When developing an arbitrage-free model for the term structure, a first issue is to identify the basic traded assets in the market:

- they are the OIS bonds and FRA (forward rate agreement) contracts, where the latter are basic linear interest rate derivatives and building blocks for all other linear derivatives.

$\rightarrow$ Before the crisis there existed a classical relationship between FRA rates and OIS bonds, namely

$$F(t; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$$

Absence of arbitrage

After the crisis, the FRA prices became disconnected from the OIS bond prices.

→ The issue of absence of arbitrage translates then into the question of existence of a martingale measure, under which the discounted prices of OIS bonds and of FRA contracts are local martingales (*a first systematic discussion of this issue in the recent literature has appeared in Cuchiero, Fontana and Gnoatto (2015)*)

Absence of arbitrage

We thus need arbitrage-free models for the OIS bonds and the FRA contracts. For this purpose we proceed as follows:

- First the OIS bond prices are modeled directly under a martingale measure, denoted by Q.
  - $\rightarrow$ This also gives explicit expressions for the forward measures, namely e.g. the forward measure $Q^{T+\Delta}$ is given by the following Radon-Nikodym density

$$\frac{dQ^{T+\Delta}}{dQ}\bigg|_{\mathcal{F}_t} = \frac{p(t, T+\Delta)}{B_t p(0, T+\Delta)},$$

where $B_t = \exp\left(\int_0^t r_s ds\right)$ is a bank account related to the short rate r, and which is used as a numeraire for Q.

Absence of arbitrage

- Next we consider suitable quantities connected to the FRA contracts and shall develop for them a dynamic model so that the complete model is free of arbitrage.

- To this effect consider a FRA contract for the time interval $[T, T+\Delta]$ where the fixed a rate is R and the underlying risky rate is the Libor rate $L(T; T, T+\Delta)$, determined at the (inception) time T by the LIBOR panel.

Absence of arbitrage

The FRA value at time t is given by

$$P^{FRA}(t; T, T+\Delta, K) = \Delta p(t, T+\Delta) E^{Q^{T+\Delta}}[L(T; T, T+\Delta) - R \mid \mathcal{F}_t]$$

where $Q^{T+\Delta}$ denotes the forward martingale measure associated to the date $T+\Delta$, with bond price $p(t, T+\Delta)$ as a numeraire.

→ The implied FRA rate is hence

$$R_t = E^{Q^{T+\Delta}}[L(T; T, T+\Delta) \mid \mathcal{F}_t] \neq \frac{1}{\Delta}\left(\frac{p(t,T)}{p(t,T+\Delta)} - 1\right)$$

→ We call this rate the Libor FRA rate or the forward Libor rate and denote it by $L(t; T, T+\Delta)$.

Dynamic models

For simplicity we consider in the sequel dynamic models driven by Brownian motion only; however the reasoning remains analogous for more general drivers, only the no-arbitrage conditions become more complicated.

For these dynamic models we shall put ourselves into an HJM setup and obtain no-arbitrage conditions that extend the classical HJM drift condition.

OIS prices

Since we intend to follow a HJM methodology, for the modeling of the OIS prices we start from the Q-dynamics of the instantaneous forward rates:

$$df(t,T) = a(t,T)dt + \sigma(t,T)dW_t$$

It follows, recalling that $p(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$,

$$dp(t,T) = p(t,T)\left((r_t - A(t,T) + \frac{1}{2} \mid \Sigma(t,T) \mid^2)dt - \Sigma(t,T)dW_t\right)$$

where W is a multidimensional Q-Brownian motion and

$$r_t = f(t,t), \quad A(t,T) := \int_t^T a(t,u)du, \quad \Sigma(t,T) := \int_t^T \sigma(t,u)du$$

OIS prices

- → The OIS bonds are traded assets and so their discounted values have to be Q-local martingales, which implies the standard HJM drift condition

$$A(t, T) = \frac{1}{2} \mid \Sigma(t, T) \mid^2$$

What about the forward Libor rates (FRA rates):

$$L(t; T, T + \Delta) = E^{T+\Delta} \{L(T; T, T + \Delta) \mid \mathcal{F}_t\}?$$

- → Various approaches are possible and we shall discuss them next.

I. Hybrid LMM-HJM approach

One idea is to mimic the Libor market model (top-down approach) and model directly the forward Libor rates in a suitable fashion: we refer to such models as hybrid LMM-HJM models

In Crépey, Grbac, Ngor and Skovmand (2014) (see also Moreni and Pallavicini (2014)) the following observable quantity is modeled

$$G(t; T, T+\Delta) = \Delta E^{T+\Delta}\left\{L(T; T, T+\Delta) \mid \mathcal{F}_t\right\} = \Delta L(t; T, T+\Delta)$$

- $\rightarrow$ Recall that no-arbitrage for FRA contracts on the Libor requires $L(t; T, T+\Delta)$ to be a martingale under the forward measure $Q^{T+\Delta}$
- $\rightarrow$ No arbitrage thus translates into a martingale condition also on $G(t; T, T+\Delta)$

I. Hybrid LMM-HJM approach

We model now directly the dynamics of the process $G(t; T, T + \Delta)$, however not under the forward measure $Q^{T+\Delta}$ as in the LMM setup, but under the martingale measure Q as in the HJM setup

$$G(t; T, T + \Delta) = G(0; T, T + \Delta)$$

$$\cdot \exp\left(\int_0^t \alpha(s, T, T + \Delta)ds + \int_0^t \sigma(s, T, T + \Delta)dW_s\right)$$

I. Hybrid LMM-HJM approach

The drift $\alpha(s, T, T+\Delta)$ has to be such that $G(t; T, T+\Delta)$ is a martingale under $Q^{T+\Delta}$, namely

$$\alpha(s, T, T+\Delta) = -\frac{1}{2} \mid \sigma(s, T, T+\Delta) \mid^2 + \langle \sigma(s, T, T+\Delta), \Sigma(s, T+\Delta) \rangle$$

and this follows by the fact that the measure change $\frac{dQ^{T+\Delta}}{dQ} \mid_{\mathcal{F}_t} = \frac{p(t, T+\Delta)}{B_t p(0, T+\Delta)}$ yields that

$$dW_t^{T+\Delta} := dW_t + \Sigma(t, T+\Delta)dt$$

is a $Q^{T+\Delta}$-Brownian motion.

II. Approach based on fictitious bond prices

A second idea is to introduce fictitious bonds $\bar{p}(t,T)$ in order to reproduce the classical relationship between the Libor rates and the bond prices:

$$L(T;T,T+\Delta)=\frac{1}{\Delta}\left(\frac{1}{\bar{p}(T,T+\Delta)}-1\right)$$

thereby supposing that $\bar{p}(t,T)$ are affected by the same factors as the Libor rates. Then

$$\begin{aligned} L(t;T,T+\Delta) &= E^{T+\Delta}\{L(T;T,T+\Delta)\mid\mathcal{F}_t\} \\ &= \tfrac{1}{\Delta}E^{T+\Delta}\left\{\left(\tfrac{1}{\bar{p}(T,T+\Delta)}-1\right)\mid\mathcal{F}_t\right\} \end{aligned}$$

II. Approach based on fictitious bond prices

- We assume $\bar{p}(t, T)$ are defined for all t and that $\bar{p}(T, T) = 1$.
- The $\bar{p}(t, T)$-bonds are not observable in the market.
- These bonds are referred to by some authors as Libor bonds or interbank bonds (*can also be seen as average bonds, issued by a representative bank from the Libor group*).
- Just as the Libor rate, also $\bar{p}(t, T)$ depends on the tenor (*here we had assumed just one tenor*).

II. Approach based on fictitious bond prices

We shall consider two alternative ways to obtain arbitrage-free dynamic models for the $\bar{p}(t,T)$- prices. We first describe them briefly and then we work them out in an HJM setup.

(a) The first alternative, in which we denote $\bar{p}(t,T)$ more specifically as $p^{\Delta}(t,T)$, is to impose the classical relationship not only at the level of the spot Libor rates, but also for the forward Libor rates by assuming

$$L(t;T,T+\Delta) = \begin{cases} E^{T+\Delta}\left\{L(T;T,T+\Delta) \mid \mathcal{F}_t\right\} \\ \frac{1}{\Delta}\left(\frac{p^{\Delta}(t,T)}{p^{\Delta}(t,T+\Delta)} - 1\right) \end{cases}$$

$\rightarrow$ Dynamic models for $p^{\Delta}(t,T)$ then imply dynamic models also for $L(t;T,T+\Delta)$.

II. Approach based on fictitious bond prices

Recall again that the no-arbitrage for FRA contracts on the Libor implies that $L(t;T,T+\Delta)$ has to be a $Q^{T+\Delta}-$martingale.

$\rightarrow$ The no-arbitrage condition thus translates here into a martingale condition on the ratio $\frac{p^{\Delta}(t,T)}{p^{\Delta}(t,T+\Delta)}$ under $Q^{T+\Delta}$.

II. Approach based on fictitious bond prices

(b) The second alternative is to interpret the fictitious bonds $\bar{p}(t, T)$ on the basis of either a credit risk analogy or a foreign exchange analogy.

- This will lead to pseudo no-arbitrage conditions, which are not implied by the absence of arbitrage in the market for FRAs and OIS bonds, but rather by each specific analogy (*they are not intrinsic no-arbitrage conditions in relation to the basic traded assets*).

$\rightarrow$ *Details below*

HJM model for the fictitious bonds (both alternatives)

Following a common practice of deriving multi-curve quantities by adding a spread over the corresponding one-curve risk-free quantities, we introduce an HJM model for $\bar{p}(t,T)$ by extending the one-curve model for $p(t,T)$, thereby introducing the instantaneous forward rates $\bar{f}(t,T)$ such that

$$\bar{f}(t,T) = f(t,T) + g(t,T),$$

where $f(t,T)$ are the forward rates corresponding to the OIS bonds and $g(t,T)$ are the forward rate spreads.

- Analogously to the relationship between $p(t,T)$ and $f(t,T)$ here we assume that

$$\bar{f}(t,T) := -\frac{\partial}{\partial T} \log \bar{p}(t,T) \;\; \Leftrightarrow \;\; \bar{p}(t,T) = \exp\left(-\int_t^T \bar{f}(t,u)du \right)$$

HJM model for the fictitious bonds (both alternatives)

Under the martingale measure Q we now assume the following dynamics for the spreads:

$$dg(t,T) = a^*(t,T)dt + \sigma^*(t,T)dW_t$$

and so for $\bar{f}(t,T) = f(t,T) + g(t,T)$ we get

$$d\bar{f}(t,T) = \bar{a}(t,T)dt + \bar{\sigma}(t,T)dW_t$$

with $\bar{a}(t,T) = a(t,T) + a^*(t,T)$; similarly for $\bar{\sigma}$.

HJM model for the fictitious bondes (both alternatives)

Recalling $\bar{p}(t,T) = \exp\left(-\int_t^T \bar{f}(t,u)du\right)$, we thus get

$$d\bar{p}(t,T) = \bar{p}(t,T)\left((\bar{r}_t - \bar{A}(t,T) + \frac{1}{2} \mid \bar{\Sigma}(t,T) \mid^2)dt - \bar{\Sigma}(t,T)dW_t\right)$$

where W is the same Q-Brownian motion as before and

$$\bar{r}_t = \bar{f}(t,t), \quad \bar{A}(t,T) := \int_t^T \bar{a}(t,u)du, \quad \bar{\Sigma}(t,T) := \int_t^T \bar{\sigma}(t,u)du$$

HJM model for the fictitious bondes (1st alternative)

Coming then to the first alternative **II.a** for the fictitious bonds, expressed as $p^{\Delta}(t,T)$, we recall their $Q-$dynamics

$$dp^{\Delta}(t,T) = p^{\Delta}(t,T)$$

$$\cdot\left((r_t^{\Delta} - A^{\Delta}(t,T) + \frac{1}{2} \mid \Sigma^{\Delta}(t,T) \mid^2)dt - \Sigma^{\Delta}(t,T)dW_t\right)$$

$\rightarrow$ The drift $A^{\Delta}(t,T)$ has to be chosen in such a way that the ratio $\frac{p^{\Delta}(t,T)}{p^{\Delta}(t,T+\Delta)}$ becomes a $Q^{T+\Delta}$ martingale.

HJM model for the fictitious bonds (1st alternative)

This yields the following drift condition

$$A^\Delta(t, T+\Delta) - A^\Delta(t, T) = -\tfrac{1}{2} \mid \Sigma^\Delta(t, T+\Delta) - \Sigma^\Delta(t, T) \mid^2$$

$$+\langle \Sigma(t, T+\Delta), \Sigma^\Delta(t, T+\Delta) - \Sigma^\Delta(t, T)\rangle$$

$\rightarrow$ This the drift condition is expressed only on the difference $A^\Delta(t, T+\Delta) - A^\Delta(t, T)$ and does not uniquely define the coefficient $A^\Delta(t, T)$; this is a consequence of a non-unique definition of $p^\Delta(t, T)$ via

$$L(t; T, T+\Delta) = \frac{1}{\Delta}\left(\frac{p^\Delta(t, T)}{p^\Delta(t, T+\Delta)} - 1\right)$$

HJM model for the fictitious bonds (2nd alternative)

Coming then to the second alternative **II.b** for the fictitious bonds, we emphasize the following:

- the drift conditions on $\bar{p}(t, T)$ are derived on the basis of different interpretations of the fictitious bonds and they are not directly implied by the no-arbitrage conditions for the FRA contracts, as it was the case before.
- We call these conditions pseudo no-arbitrage conditions as they are not strictly necessary to ensure the absence of arbitrage in the model for FRAs and OIS bonds.
- Exploiting the existing HJM approaches with more than one curve, one may consider a credit risk and a foreign exchange analogy

Credit risk analogy

We base ourselves here on the approach developed in Crépey, Grbac and Nguyen (2012) (see also Morini (2009), Filipović and Trolle (2013) and Ametrano and Bianchetti (2013))):

- $\bar{p}(t, T)$ are interpreted here as pre-default values of bonds issued by an average representative of the Libor panel, which may possibly default at a hypothetical random time τ^*.
- The defaultable bonds are now traded assets and hence their discounted values have to be Q-martingales, which yields a drift condition

Credit risk analogy

Using a reduced form approach, where the default is modeled via a default intensity process and for a specific choice of the recovery scheme this leads to a particularly nice form of the drift condition on $\bar{p}(t, T)$ given by

$$\bar{A}(t, T) = \frac{1}{2} \mid \bar{\Sigma}(t, T) \mid^2$$

$\rightarrow$ This is a classical HJM condition (as for the OIS bonds), where all the risk-free quantities $A(\cdot)$ and $\Sigma(\cdot)$ are replaced by the corresponding "risky" ones $\bar{A}(\cdot)$ and $\bar{\Sigma}(\cdot)$

Foreign exchange analogy

The foreign exchange analogy has been first suggested in Bianchetti (2010) and exploited in Cuchiero, Fontana and Gnoatto (2015) and Miglietta (2015). The specific HJM setup presented here is from Grbac and Runggaldier (2015).

Foreign exchange analogy

- We denote the fictitious bonds more specifically by $p^f(t, T)$ to distinguish them from the previous case and interpret them as bonds denominated in a different, foreign currency (*one may consider a different foreign country for each tenor; here only one.*)
- The OIS bonds $p(t, T)$ are seen as domestic bonds.
- The fictitious bonds $p^f(t, T)$ can now be considered as traded assets in the foreign economy and the drift conditions on $p^f(t, T)$ stem from the absence of arbitrage in the foreign market.
- The two markets are linked via an exchange rate, which allows to derive corresponding conditions in the domestic market ensuring the absence of arbitrage in the foreign market .

Foreign exchange analogy

For $p^f(t,T)$ we consider a model, under Q, analogous to that for $p^\Delta(t,T)$ and perform an additional measure transformation from the forward measure $Q^{T+\Delta}$ to a further suitable forward measure $Q^{f,T+\Delta}$ so that the absence of arbitrage in the foreign market can be expressed in terms of a martingality condition for $\frac{p^f(t,T)}{p^f(t,T+\Delta)}$ under $Q^{f,T+\Delta}$.

$\rightarrow$ This yields the following drift condition

$$A^f(t,T+\Delta) - A^f(t,T) = -\frac{1}{2} \mid \Sigma^f(t,T+\Delta) - \Sigma^f(t,T) \mid^2$$

$$+\langle \Sigma^f(t,T+\Delta) - \beta_t, \Sigma^f(t,T+\Delta) - \Sigma^f(t,T) \rangle$$

Foreign exchange analogy

- As for $p^{\Delta}(t, T)$, the drift condition is again only on the difference $A^f(t, T + \Delta) - A^f(t, T)$, due to the fact that the ratio $\frac{p^f(t,T)}{p(t,T+\Delta)}$ has to be a martingale under $Q^{f,T+\Delta}$.
- The drift conditions on $p^{\Delta}(t, T)$ and $p^f(t, T)$ look similar, but are not the same: besides the appearance of the volatility β_t from the dynamics of the spot exchange rate, the second condition results from the further measure transformation from $Q^{T+\Delta}$ to $Q^{f,T+\Delta}$ (*the difference being justified also by the different interpretations of the fictitious bonds*).

Concluding remarks

- We have studied various possible modeling choices in the HJM setup for multiple curves.
- different choices lead to different types of martingale (drift) conditions, which are in general not interconnected.
- We distinguish between “true” no-arbitrage conditions implied by absence of arbitrage between basic traded quantities (*OIS bonds and FRA contracts*) in the multiple curve markets and pseudo no-arbitrage conditions implied by specific interpretations of the fictitious bonds.

Concluding remarks

- Apart from the forward Libor rates, one possibility is to model various forward discretely compounded spreads in addition to the OIS bond prices (cf. Cuchiero et al. (2015)) – this is again a hybrid HJM-LMM approach with "true" no-arbitrage conditions.
- Regarding implementation and calibration of these models, the choice might depend on a specific task at hand, however we note that in some of the mentioned alternatives the initial term structure of the modeling quantities may not be observable.

Thank you for your attention