Multi-dimensional Quadratic BSDEs of Diagonally-Quadratic Generators

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Outline

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1. BSDEs and Quadratic BSDEs

BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0, T].$$

Existence and uniqueness theorem (Pardoux & Peng, 1990). Assume that the generator $g(s, y, z)$ is uniformly Lipschitz continuous in the unknown pair of variables (y, z) and $E\int_0^T |g(\cdot, 0, 0)|^2\, ds < \infty$ and $\xi \in L^2(\Omega, \mathscr{F}_T, P; R^n)$. Then the preceding BSDE has a unique adapted solution (Y, Z) such that

$$E\left[\max_{s\in[0,T]} |Y_s|^2\right] + E\int_0^T |Z_s|^2\, ds \le C\left(E|\xi|^2 + E\int_0^T |g(s, 0, 0)|^2\, ds\right).$$

Quadratic BSDEs refer to the case where the generator $g(s, y, z)$ grows quadratically in the second unknown variable z, i.e.

$$|g(s, y, z)| \le C(1 + |y| + |z|^2), \quad (s, y, z) \in [0, T] \times R^n \times R^{n\times m}.$$

One asks the existence and uniqueness in suitable spaces.

2. One-dimensional quadratic BSDEs.

BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0, T].$$

Kobylansky (2000, Ann. Prob.):
Backward stochastic differential equations and partial differential equations with quadratic growth.

Briand and Hu (2006 & 2008, PTRF):

Barrieu and El Karoui (2013, Ann. Prob.)

$$E\left[\exp(\gamma|\xi|)\right] < \infty.$$

Literature on Systems of Quadratic BSDEs

El Karoui and Hamadene (SPA, 2003): special quadratic BSDEs arising from non-zero sum risk sensitive games

Tevzadze (SPA, 2008): Solvability on the assumption of small terminal value

Frei and dos Reis (Math. Fin. Econ., 2011): a counterexample

Cheridito and Nam (2014): Subquadratic or Markovian Structure

Kramkov and Pulido (2014a,b): special generator with small quadratic part

Kardaras, Xing and Zitkovic (2015)

Jamneshan, Kupper and Luo (2015)

Counterexample to the global existence result for two-dimensional quadratic BSDEs:

$$y_1(t) = \xi_1 - \int_t^T z_1(s)\,dW_s, \quad t \in [0,T];$$

$$y_2(t) = \int_t^T [\gamma|z_1(s)|^2 + \frac{1}{2}|z_2(s)|^2]\,ds - \int_t^T z_2(s)\,dW_s, \quad t \in [0,T]. \tag{1}$$

When γ is sufficiently large such that

$$E\exp\left(\gamma\int_0^T |z_1(s)|^2\,ds\right) = \infty,$$

then the preceding one-dimensional quadratic BSDE (1) of the following form (with $Y_2(t) := y_2(t) + \int_0^t |z_1(s)|^2\,ds$):

$$Y_2(t) = \int_0^T \gamma|z_1(s)|^2\,ds + \frac{1}{2}\int_t^T |z_2(s)|^2\,ds - \int_t^T z_2(s)\,dW_s, \quad t \in [0,T]$$

has no global solution on $[0,T]$. Otherwise, we have (noting $\exp(Y_2(t))$ is a super-martingale) the contradiction:

$$E\exp\left(\gamma\int_0^T |z_1(s)|^2\,ds\right) \le E\exp(Y_2(0)) = \exp(Y_2(0)) < \infty$$

Stability Theorem

Assume that (i) the function $f : \Omega \times [0,T] \times R^d \to R$ has the following quadratic growth and locally Lipschitz continuity in the last variable:

$$\begin{aligned} |f(s,z)| &\le \frac{\gamma}{2}|z|^2, \quad z \in R^d; \\ |f(s,z_1) - f(s,z_2)| &\le C(1+|z_1|+|z_2|)|z_1-z_2|, \quad z_1, z_2 \in R^d; \end{aligned} \tag{2}$$

(ii) the process $f(\cdot, z)$ is $\mathscr{F}_t$-adapted for each $z \in R^d$; and (iii) the process $h : \Omega \times [0,T] \to R$ is $\mathscr{F}_t$-adapted and $|h_s| \le |H_s|^{1+\alpha}$ such that the stochastic integral $H \cdot W$ is a BMO martingale. Then for bounded ξ, the following BSDE

$$Y_t = \xi + \int_t^T [f(s, Z_s) + h_s]\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0,T] \tag{3}$$

has a unique solution (Y, Z) such that Y is (essentially) bounded and $Z \cdot W$ is a BMO martingale. Furthermore, we have

$$e^{\gamma |Y_t|} \le E^{\mathscr{F}_t}\left[e^{\gamma \xi + \gamma \int_t^T (C + |h_s|)\, ds} \right].$$

Comparison Theorem

Let $(\widetilde{Y}, \widetilde{Z})$ solve the following BSDE:

$$Y_t = \widetilde{\xi} + \int_t^T [\widetilde{f}(s, Z_s) + h_s]\, ds - \int_t^T Z_s\, dW_s, \quad t \in [0, T]$$

where the pair $(\widetilde{f}, \widetilde{\xi})$ has the same above-mentioned properties as (f, ξ), $\widetilde{f} \geq f$, and $\widetilde{\xi} \geq \xi$. Then we have $\widetilde{Y}_t \geq Y_t$.

3 Multi-dimensional Diagonally-Quadratic BSDEs

For $i = 1, \ldots, n$, we denote by z^i the ith row (component) of matrix (vector) z. Consider the multi-dimensional BSDE of the following structured quadratic generator $g := (g^1, \cdots, g^n)^*$ with

$$|g^i(t,y,z)| \le C(1 + |y| + |z|^{1+\alpha} + |z^i|^2), \quad \alpha \in [0,1), \quad i = 1, \ldots, n.$$

This kind of structured generator g is said to be "diagonally" quadratic. Assuming some additional (locally or globally) Lipschitz continuity in both unknowns of the generator g, we prove that for $\alpha \in [-1, 1)$ and a given bounded terminal value ξ, multi-dimensional diagonally quadratic BSDE admits a unique local solution (Y, Z) on $[T - \varepsilon, T]$ $(\varepsilon > 0)$ such that Y is bounded and $Z \cdot M$ is a BMO martingale, and moreover, it has a unique global solution on $[0, T]$ in the case of $\alpha = -1$.

We make the following three assumptions.

$(\mathscr{A}1)$ The function $f := (f^1, \ldots, f^n)^* : \Omega \times [0,T] \times R^d \to R^n$ has the following quadratic growth and locally Lipschitz continuity in the last variable:

$$\begin{aligned} |f(s,z)| &\leq \frac{\gamma}{2}|z|^2, \quad z \in R^d; \\ |f(s,z_1) - f(s,z_2)| &\leq C(1+|z_1|+|z_2|)|z_1-z_2|, \quad z_1, z_2 \in R^d. \end{aligned} \tag{4}$$

For each $z \in R^d$, the process $f(\cdot, z)$ is $\mathscr{F}_t$-adapted.

$(\mathscr{A}2)$ There is $\alpha \in [0,1)$ such that the function $h := (h^1, \ldots, h^n)^* : \Omega \times [0,T] \times R^n \times R^{n\times d} \to R^n$ has the following subquadratic growth and Lipschitz continuity in the last two variables:

$$\begin{aligned} &|h(t,y,z)| \leq C(1+|y|+|z|^{1+\alpha}), \quad (y,z) \in R^n \times R^{n\times d}; \\ &|h(t,y_1,z_1) - h(t,y_2,z_2)| \leq C|y_1-y_2| + C(1+|z_1|^\alpha+|z_2|^\alpha)|z_1-z_2|, \end{aligned} \tag{5}$$

for $(y_j, z_j) \in R^n \times R^{n\times d}$ with $j = 1, 2$. For each $(y,z) \in R^n \times R^{n\times d}$, the process $g(\cdot, y, z)$ is $\mathscr{F}_t$-adapted.

$(\mathscr{A}3)$ The terminal condition $\xi := (\xi^1, \ldots, \xi^n)^*$ is uniformly bounded.

Theorem 1 (Local solution)

Let assumptions $(\mathscr{A}1), (\mathscr{A}2)$, and $(\mathscr{A}3)$ be satisfied with $\alpha \in [0,1)$. Then, for any bounded ξ, the following BSDE

$$Y_t^i = \xi^i + \int_t^T [f^i(s, Z_s^i) + h^i(s, Y_s, Z_s)]\, ds - \int_t^T Z_s^i\, dW_s, \quad t \in [0,T]; \quad i = 1, \ldots, n$$

has a unique local solution (Y, Z).

Theorem 2 (Global solution)

Let assumptions $(\mathscr{A}1)$ and $(\mathscr{A}3)$ be satisfied. Moreover, assume that there is a positive constant C such that for $(s, y, z) \in [0,T] \times R^n \times R^{n\times d}$ and $(\bar{y}, \bar{z}) \in R^n \times R^{n\times d}$,

$$|h(s,y,z)| \le C(1+|y|), \qquad |h(s,y,z) - h(s,\bar{y},\bar{z})| \le C(|y-\bar{y}| + |z-\bar{z}|).$$

Then, the following BSDE

$$Y_t^i = \xi^i + \int_t^T [f^i(s, Z_s^i) + h^i(s, Y_s, Z_s)]\, ds - \int_t^T Z_s^i\, dW_s, \quad t \in [0,T]; \quad i = 1, \ldots, n$$

has a unique adapted solution (Y, Z) on $[0,T]$ such that Y is bounded. Furthermore, $Z \cdot W$ is a $BMO(P)$ martingale.

Features of the proof

Key Observation: any solution (if it exists) of the multi-dimensional diagonally quadratic BSDE is a fixed point of the quadratic solution map $\Gamma : (U, V) \mapsto \Gamma(U, V)$ which is defined as the adapted solution to the following *decoupled* system of quadratic BSDEs:

$$Y_t^i = \xi^i + \int_t^T \left[f^i(s, Z_s^i) + h^i(s, U_s, V_s)\right] ds - \int_t^T Z_s^i \, dW_s, \ t \in [0, T]; i = 1, \ldots, n. \tag{6}$$

We have to consider unbounded V, and we are typically facing with the crucial issue of solution of one-dimensional quadratic BSDEs with unbounded data (even if the terminal condition ξ is assumed to be bounded!). Fortunately, it can be proved that for a pair of bounded adapted process U and BMO martingale $V \cdot W$, the system (6) has a unique adapted solution (Y, Z) such that Y is bounded and $Z \cdot W$ is a BMO martingale. It means that in the Banach space $\mathscr{S}^\infty(R^n) \times (BMO_2(P))^n$, the quadratic solution map Γ is well-defined and is further a transformation.

Γ is stable within a centered ball of a suitably chosen radius (whose choice is *delicate but rather technical*), where we use the well-known sharp result of John-Nirenberg inequality for BMO martingales. In the proof of the contraction of Γ, we use the reverse Hölder inequality for BMO martingales.

Construction of the map

For a pair of bounded adapted process U and BMO martingale $V \cdot W$, we consider the following *decoupled* system of quadratic BSDEs:

$$Y_t^i = \xi^i + \int_t^T \left[f^i(s, Z_s^i) + h^i(s, U_s, V_s)\right] ds - \int_t^T Z_s^i \, dW_s, \; t \in [0, T]; i = 1, \ldots, n. \tag{7}$$

It has a unique adapted solution (Y, Z) such that Y is bounded and $Z \cdot W$ is a BMO martingale. Define the quadratic solution map $\Gamma : (U, V) \mapsto \Gamma(U, V)$ as follows:

$$\Gamma(U, V) := (Y, Z), \quad \forall (U, V \cdot W) \in \mathscr{S}^\infty(R^n) \times (BMO_2(P))^n.$$

It is a transformation in the Banach space $\mathscr{S}^\infty(R^n) \times (BMO_2(P))^n$.

Notations (1)

$$
\begin{aligned}
C_\delta :=& e^{\frac{6}{1-\alpha}\gamma C T + \frac{1-\alpha}{2}\left(\frac{3}{1-\alpha}\gamma C\left(\frac{n}{\delta}\frac{1+\alpha}{2}\right)^{\frac{1+\alpha}{2}}\right)^{\frac{2}{1-\alpha}} T},\\
\beta :=& \frac{1}{2}(1-\alpha)C^{\frac{2}{1-\alpha}}(2(1+\alpha))^{\frac{1+\alpha}{1-\alpha}},\\
\mu_1 :=& (1-\alpha)\left(1+\frac{1-\alpha}{(1+\alpha)\gamma}\right) = 1-\alpha+\frac{(1-\alpha)^2}{(1+\alpha)\gamma},\\
\mu_2 :=& (1+\alpha)\left(1+\frac{1-\alpha}{(1+\alpha)\gamma}\right) = 1+\alpha+\frac{1-\alpha}{\gamma},\\
\mu :=& n\left(\beta + C\mu_1\right)\gamma^{\frac{2}{\alpha-1}} + C\mu_2.
\end{aligned}
\tag{8}
$$

Consider the following standard quadratic equation of A and the discriminant:

$$
\delta A^2 - \left(1 + 4n\gamma^{-2}e^{\gamma|\xi|_\infty}\delta\right)A + 4n\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}\varepsilon = 0,
$$

$$
\begin{aligned}
\Delta :=& \left(1 + 4n\gamma^{-2}e^{\gamma|\xi|_\infty}\delta\right)^2 - 4\delta\left[4n\gamma^{-2}e^{\gamma|\xi|_\infty} + 4\mu C_\delta e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}\varepsilon\right]\\
=& \left(1 - 4n\gamma^{-2}e^{\gamma|\xi|_\infty}\delta\right)^2 - 16\mu\delta C_\delta e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}\varepsilon.
\end{aligned}
\tag{9}
$$

Notations (2)

$$\delta := \frac{1}{8n}\gamma^2 e^{-\gamma|\xi|_\infty}, \quad \varepsilon \le \min\left\{\frac{1}{3nC}, \frac{n}{8\mu C_\delta}\gamma^{-2} e^{(1-\frac{3n}{1-\alpha})\gamma|\xi|_\infty}\right\},$$
$$A := \frac{1 + 4n\gamma^{-2}e^{\gamma|\xi|_\infty}\delta - \sqrt{\Delta}}{2\delta} = \frac{3 - 2\sqrt{\Delta}}{4\delta} \le \frac{3}{4\delta} = 6n\gamma^{-2}e^{\gamma|\xi|_\infty}. \tag{10}$$

$$\Delta \ge 0, \quad 1 - \delta A = \frac{1 + 2\sqrt{\Delta}}{4} > 0,$$
$$\frac{n}{\gamma^2} e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1 - \delta A}\varepsilon + \frac{1}{4}A = \frac{1}{2}A. \tag{11}$$

Theorem 1 is proved if the quadratic solution map Γ is a contraction on the closed convex set $\mathscr{B}_\varepsilon$ defined by

$$\mathscr{B}_\varepsilon := \Big\{ (U, V): \quad U \in \mathscr{S}^\infty(R^n), \quad V \cdot W \in (BMO_2(P))^n,$$
$$\|V \cdot W\|^2_{BMO_2} \le A, \quad e^{\frac{2}{1-\alpha}\gamma|U|_\infty} \le \frac{C_\delta e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A} \Big\} \tag{12}$$

for a positive constant ε (to be determined later).

Estimation of the quadratic solution map

Assertion: $\Gamma(\mathscr{B}_\varepsilon) \subset \mathscr{B}_\varepsilon$, that is,

$$\Gamma(U,V) \in \mathscr{B}_\varepsilon, \qquad \forall\, (U,V) \in \mathscr{B}_\varepsilon. \tag{13}$$

Step 1. Exponential transformation.

$$\begin{aligned} &\phi(Y_t^i) + \frac{1}{2} E_t \left[\int_t^T |Z_s^i|^2 \, ds \right] \\ &\le \phi(|\xi^i|_\infty) + C E_t \left[\int_t^T |\phi'(Y_s^i)| \left(2 + |U_s| + |V_s|^{1+\alpha}\right) \, ds \right]. \end{aligned} \tag{14}$$

For $t \in [T-\varepsilon, T]$, we have

$$\begin{aligned} &\sum_{i=1}^n \phi(Y_t^i) + \frac{1}{2} E_t \left[\int_t^T |Z_s|^2 \, ds \right] \\ &\le \sum_{i=1}^n \phi(|\xi^i|_\infty) + C \sum_{i=1}^n E_t \left[\int_t^T |\phi'(Y_s^i)| \left(2 + |U_s| + |V_s|^{1+\alpha}\right) \, ds \right] \end{aligned} \tag{15}$$

Since (in view of the definition of notation β)

$$C|\phi'(Y_s^i)||V_s|^{1+\alpha} \leq \frac{1}{4}|V_s|^2 + \beta|\phi'(Y_s^i)|^{\frac{2}{1-\alpha}},$$

we have

$$\begin{aligned}
&\sum_{i=1}^n \phi(Y_t^i) + \frac{1}{2}E_t\left[\int_t^T |Z_s|^2\, ds\right] \\
&\leq \sum_{i=1}^n \phi(|\xi^i|_\infty) + C\sum_{i=1}^n E_t\left[\int_t^T |\phi'(Y_s^i)|\,(2+|U_s|)\ ds\right] \\
&\quad + \beta\sum_{i=1}^n E_t\left[\int_t^T |\phi'(Y_s^i)|^{\frac{2}{1-\alpha}}\, ds\right] + \frac{1}{4}E_t\left[\int_t^T |V_s|^2\, ds\right] \\
&\leq \sum_{i=1}^n \phi(|\xi^i|_\infty) + 2C\sum_{i=1}^n E_t\left[\int_t^T |\phi'(Y_s^i)|\left(1+\frac{1}{2}|U_s|\right)\ ds\right] \\
&\quad + \beta\sum_{i=1}^n E_t\left[\int_t^T |\phi'(Y_s^i)|^{\frac{2}{1-\alpha}}\, ds\right] + \frac{1}{4}E_t\left[\int_t^T |V_s|^2\, ds\right].
\end{aligned} \tag{16}$$

Since for $x > 0$,

$$1 + x \leq \left(1 + \frac{1-\alpha}{\gamma(1+\alpha)}\right) e^{\frac{\gamma(1+\alpha)}{1-\alpha}x},$$

we have

$$\begin{aligned} & 2C \sum_{i=1}^{n} E_t \left[\int_t^T |\phi'(Y_s^i)| \left(1 + |U_s|\right) \, ds \right] \\ \leq & 2C \sum_{i=1}^{n} E_t \left[\int_t^T |\phi'(Y_s^i)| \left(1 + \frac{1-\alpha}{\gamma(1+\alpha)}\right) e^{\gamma \frac{1+\alpha}{1-\alpha}|U_s|} \, ds \right]. \end{aligned} \tag{17}$$

Since (by Young's inequality)

$$|\phi'(Y_s^i)| e^{\gamma \frac{1+\alpha}{1-\alpha}|U_s|} \leq \frac{1-\alpha}{2} |\phi'(Y_s^i)|^{\frac{2}{1-\alpha}} + \frac{1+\alpha}{2} e^{\frac{2}{1-\alpha}\gamma|U_s|},$$

in view of the definition of the notations μ_1 and μ_2, we have

$$\begin{aligned} & 2C \sum_{i=1}^{n} E_t \left[\int_t^T |\phi'(Y_s^i)| \left(1 + |U_s|\right) \, ds \right] \\ \leq & C\mu_1 E_t \left[\int_t^T \sum_{i=1}^{n} |\phi'(Y_s^i)|^{\frac{2}{1-\alpha}} \, ds \right] + C\mu_2 E_t \left[\int_t^T e^{\frac{2\gamma}{1-\alpha}|U_s|} \, ds \right]. \end{aligned} \tag{18}$$

In view of inequality (16), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\phi(Y_t^i)+\frac{1}{2}E_t\left[\int_t^T|Z_s|^2\,ds\right]\\
\le&\sum_{i=1}^{n}\phi(|\xi^i|_\infty)+(\beta+C\mu_1)\sum_{i=1}^{n}E_t\left[\int_t^T|\phi'(Y_s^i)|^{\frac{2}{1-\alpha}}\,ds\right]\\
&+C\mu_2E_t\left[\int_t^T e^{\frac{2\gamma}{1-\alpha}|U_s|}\,ds\right]+\frac{1}{4}E_t\left[\int_t^T|V_s|^2\,ds\right]\\
\le&\sum_{i=1}^{n}\phi(|\xi^i|_\infty)+\gamma^{\frac{2}{\alpha-1}}(\beta+C\mu_1)\sum_{i=1}^{n}E_t\left[\int_t^T e^{\frac{2\gamma}{1-\alpha}|Y_s^i|}\,ds\right]\\
&+C\mu_2E_t\left[\int_t^T e^{\frac{2\gamma}{1-\alpha}|U_s|}\,ds\right]+\frac{1}{4}E_t\left[\int_t^T|V_s|^2\,ds\right].
\end{aligned}\tag{19}$$

Step 2. Estimate of $e^{\gamma|Y|_\infty}$.

For $i = 1, \ldots, n$,

$$\begin{aligned} e^{\frac{3}{1-\alpha}\gamma|Y_t^i|} &\le \quad E_t\left[e^{\frac{3}{1-\alpha}\gamma\left(|\xi^i|+\int_t^T (C+|h^i(U_s,V_s)|)\,ds\right)}\right] \\ &\le \; E_t\left[e^{\frac{3}{1-\alpha}\gamma\left(|\xi|_\infty+C\int_t^T (2+|U_s|+|V_s|^{1+\alpha})\,ds\right)}\right]. \end{aligned} \tag{20}$$

Since (by Young's inequality)

$$\frac{3}{1-\alpha}\gamma C|V_s|^{1+\alpha} \le \frac{1-\alpha}{2}\left(\frac{3}{1-\alpha}\gamma C\left(\frac{n}{\delta}\frac{1+\alpha}{2}\right)^{\frac{1+\alpha}{2}}\right)^{\frac{2}{1-\alpha}} + \frac{\delta}{n}|V_s|^2, \tag{21}$$

in view of the definition of notation C_δ, we have for $i = 1, \ldots, n$,

$$e^{\frac{3}{1-\alpha}\gamma|Y_t^i|} \le C_\delta E_t\left[e^{\left(\frac{3}{1-\alpha}\gamma|\xi|_\infty+\frac{3}{1-\alpha}\gamma C\varepsilon|U|_\infty+\frac{\delta}{n}\int_t^T |V_s|^2\,ds\right)}\right]; \tag{22}$$

and therefore,

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \le C_\delta\, e^{\left(\frac{3}{1-\alpha}n\gamma|\xi|_\infty+\frac{3}{1-\alpha}n\gamma C\varepsilon|U|_\infty\right)} E_t\left[e^{\delta\int_t^T |V_s|^2\,ds}\right]. \tag{23}$$

It follows from the definition of $\mathscr{B}_\varepsilon$ that $\|\sqrt{\delta}V \cdot W\|^2_{BMO_2(P)} \leq \delta A < 1$, applying the John-Nirenberg inequality to the BMO martingale $\sqrt{\delta}V \cdot W$, we have

$$e^{\frac{3}{1-\alpha}\gamma|Y_t|} \leq \frac{C_\delta\, e^{\left(\frac{3}{1-\alpha}n\gamma|\xi|_\infty + \frac{3}{1-\alpha}Cn\gamma\varepsilon|U|_\infty\right)}}{1-\delta\|V \cdot W\|^2_{BMO_2}} \leq \frac{C_\delta\, e^{\left(\frac{3}{1-\alpha}n\gamma|\xi|_\infty + \frac{3}{1-\alpha}Cn\gamma\varepsilon|U|_\infty\right)}}{1-\delta A}. \tag{24}$$

Since $3nC\varepsilon \leq 1$ (see the choice of ε and $(U, V) \in \mathscr{B}_\varepsilon$, we have

$$\begin{aligned} e^{\left(\frac{3}{1-\alpha}\gamma|Y|_\infty\right)} &\leq \frac{C_\delta e^{\left(\frac{3}{1-\alpha}n\gamma|\xi|_\infty + \frac{1}{1-\alpha}\gamma|U|_\infty\right)}}{1-\delta A} \\ &\leq C_\delta \frac{e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A}\left(C_\delta \frac{e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A}\right)^{\frac{1}{2}} \\ &\leq \left(\frac{C_\delta e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A}\right)^{\frac{3}{2}}, \end{aligned} \tag{25}$$

which gives a half of the desired result (13).

Step 3. Estimate of $\|Z \cdot W\|^2_{BMO_2}$.

From inequality (19) and the definition of notation μ, we have

$$\frac{1}{2}E_t\left[\int_t^T |Z_s|^2\,ds\right] \le \frac{n}{\gamma^2}e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A}\varepsilon + \frac{1}{4}A. \tag{26}$$

In view of (11), we have

$$\frac{1}{2}\|Z \cdot W\|^2_{BMO_2} \le \frac{n}{\gamma^2}e^{\gamma|\xi|_\infty} + \mu C_\delta \frac{e^{\frac{3}{1-\alpha}n\gamma|\xi|_\infty}}{1-\delta A}\varepsilon + \frac{1}{4}A = \frac{1}{2}A, \tag{27}$$

The other half of the desired result (13) is then proved.

Contraction of the quadratic solution map

For $(U,V) \in \mathscr{B}_\varepsilon$ and $(\widetilde{U},\widetilde{V}) \in \mathscr{B}_\varepsilon$, set

$$(Y,Z) := \Gamma(U,V), \quad (\widetilde{Y},\widetilde{Z}) := \Gamma(\widetilde{U},\widetilde{V}).$$

That is, for $i = 1, \ldots, n$,

$$\begin{aligned} Y_t^i &= \xi^i + \int_t^T \left[f^i(s, Z_s^i) + h^i(s, U_s, V_s) \right] ds - \int_t^T Z_s^i dW_s, \\ \widetilde{Y}_t^i &= \xi^i + \int_t^T \left[f^i(s, \widetilde{Z}_s^i) + h^i(s, \widetilde{U}_s, \widetilde{V}_s) \right] ds - \int_t^T \widetilde{Z}_s^i dW_s. \end{aligned} \tag{28}$$

Fix i, we can define the vector process $\beta(i)$ in an obvious way such that

$$\begin{aligned} &|\beta_s(i)| \le C(1 + |Z_s^i| + |\widetilde{Z}_s^i|), \\ &f^i(s, Z_s^i) - f^i(s, \widetilde{Z}_s^i) = (Z_s^i - \widetilde{Z}_s^i)\beta_s(i). \end{aligned} \tag{29}$$

Then $\widetilde{W}_t(i) := W_t - \int_0^t \beta_s(i)\, ds$ is a Brownian motion under the equivalent probability measure P^i defined by

$$dP^i := \mathscr{E}(\beta(i) \cdot W)_0^T \, dP,$$

and from the above-established a priori estimate, there is $K > 0$ such that $\|\beta(i) \cdot W\|^2_{BMO_2} \le K^2 := 3C^2T + 6C^2A.$

Since

$$Y_t^i - \widetilde{Y}_t^i + \int_t^T (Z_s^i - \widetilde{Z}_s^i)\, d\widetilde{W}_s(i) = \int_t^T \left[h^i(U_s, V_s) - h^i(\widetilde{U}_s, \widetilde{V}_s)\right] ds, \tag{30}$$

taking square and then the conditional expectation W.R.T. P^i (denoted by E_t^i), we have

$$\begin{aligned}
&|Y_t^i - \widetilde{Y}_t^i|^2 + E_t^i\left[\int_t^T |Z_s^i - \widetilde{Z}_s^i|^2\, ds\right] \\
&= E_t^i\left[\left(\int_t^T \left(h^i(U_s, V_s) - h^i(\widetilde{U}_s, \widetilde{V}_s)\right) ds\right)^2\right] \\
&\leq C^2 E_t^i\left[\left(\int_t^T \left(|U_s - \widetilde{U}_s| + (1 + |V_s|^\alpha + |\widetilde{V}_s|^\alpha)|V_s - \widetilde{V}_s|\right) ds\right)^2\right] \\
&\leq 2C^2(T-t)^2|U - \widetilde{U}|_\infty^2 \\
&\quad + 2C^2 E_t^i\left[\int_t^T (1 + |V_s|^\alpha + |\widetilde{V}_s|^\alpha)^2\, ds \int_t^T |V_s - \widetilde{V}_s|^2\, ds\right] \\
&\leq 2C^2(T-t)^2|U - \widetilde{U}|_\infty^2 \\
&\quad + 6C^2 E_t^i\left[\left(\int_t^T (1 + |V_s|^{2\alpha} + |\widetilde{V}_s|^{2\alpha})\, ds\right)^2\right]^{\frac{1}{2}} E_t^i\left[\left(\int_t^T |V_s - \widetilde{V}_s|^2\, ds\right)^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{31}$$

Let L_4 be a generic constant for the following dominance:

$$\sup_\tau E_\tau\left[(\langle M\rangle)^2\right] \leq L_4^4 E\|M\|_{BMO_2}^4 := L_4^4 \sup_\tau E_\tau\left[\langle M\rangle\right]^2$$

for any BMO martingale M. We have

$$\begin{aligned} E_t^i\left[\left(\int_t^T |V_s - \widetilde{V}_s|^2\, ds\right)^2\right]^{\frac{1}{2}} &\leq L_4^2\|(V-\widetilde{V})\cdot W\|_{BMO_2(P^i)} ^2 \\ &\leq L_4^2 c_2^2\|(V-\widetilde{V})\cdot W\|_{BMO_2(P)}^2 \end{aligned} \tag{32}$$

and for $t \in [T-\varepsilon, T]$,

$$
\begin{aligned}
& E_t^i\left[\left(\int_t^T (1+|V_s|^{2\alpha}+|\widetilde{V}_s|^{2\alpha})\,ds\right)^2\right]^{\frac{1}{2}} \\
\leq\; & E_t^i\left[\left(\varepsilon+\varepsilon^{1-\alpha}\left(\int_t^T |V_s|^2\,ds\right)^{\alpha}+\varepsilon^{1-\alpha}\left(\int_t^T |\widetilde{V}_s|^2\,ds\right)^{\alpha}\right)^2\right]^{\frac{1}{2}} \\
\leq\; & \varepsilon^{1-\alpha}E_t^i\left[\left(\varepsilon^{\alpha}+\left(\int_t^T |V_s|^2\,ds\right)^{\alpha}+\left(\int_t^T |\widetilde{V}_s|^2\,ds\right)^{\alpha}\right)^2\right]^{\frac{1}{2}} \\
\leq\; & \varepsilon^{1-\alpha}E_t^i\left[\left(T^{\alpha}+2-2\alpha+\alpha\int_t^T |V_s|^2\,ds+\alpha\int_t^T |\widetilde{V}_s|^2\,ds\right)^2\right]^{\frac{1}{2}} \\
\leq\; & \varepsilon^{1-\alpha}\left[T^{\alpha}+2+\alpha E_t^i\left[\left(\int_t^T |V_s|^2\,ds\right)^2\right]^{\frac{1}{2}}+\alpha E_t^i\left[\left(\int_t^T |\widetilde{V}_s|^2\,ds\right)^2\right]^{\frac{1}{2}}\right] \\
\leq\; & \varepsilon^{1-\alpha}\left[T^{\alpha}+2+\alpha L_4^2\|V\cdot W\|^2_{BMO_2(P^i)}+\alpha L_4^2\|\widetilde{V}\cdot W\|^2_{BMO_2(P^i)}\right] \\
\leq\; & \varepsilon^{1-\alpha}\left[T^{\alpha}+2+\alpha L_4^2c_2^2\|V\cdot W\|^2_{BMO_2(P)}+\alpha L_4^2c_2^2\|\widetilde{V}\cdot W\|^2_{BMO_2(P)}\right] \\
\leq\; & \varepsilon^{1-\alpha}\left(T^{\alpha}+2+2\alpha L_4^2c_2^2A\right).
\end{aligned}
\tag{33}
$$

Concluding the above estimates, we have for $t \in [T-\varepsilon, T]$,

$$
\begin{aligned}
&|Y_t^i - \widetilde{Y}_t^i|^2 + E_t^i\left[\int_t^T |Z_s^i - \widetilde{Z}_s^i|^2\, ds\right] \\
\le\, &2C^2\varepsilon^2|U - \widetilde{U}|_\infty^2 \\
&+ 6C^2L_4^2c_2^2\left(T^\alpha + 2 + 2\alpha L_4^2c_2^2A\right)\varepsilon^{1-\alpha}\|(V - \widetilde{V})\cdot W\|_{BMO_2(P)}^2.
\end{aligned}
\tag{34}
$$

We have for $t \in [T-\varepsilon, T]$,

$$
\begin{aligned}
&|Y - \widetilde{Y}|_\infty^2 + c_1^2\|(Z - \widetilde{Z})\cdot W\|_{BMO_2(P)}^2 \\
\le\, &4C^2n\,\varepsilon^2|U - \widetilde{U}|_\infty^2 + 12C^2L_4^2c_2^2n\left(T^\alpha + 2 + 2\alpha L_4^2c_2^2A\right)\varepsilon^{1-\alpha}\|(V - \widetilde{V})\cdot W\|_{BMO_2(P)}^2.
\end{aligned}
$$

Global solution: The length of small interval depends upon only on the uniform norm of Y. Then it suffices to give an a priori estimate of Y, which can be done by induction.

Possible application: non-zero sum risk sensitive game (El Karoui and Hamadene 2003).

Thank you!