Degenerate Backward SPDE with Singular Terminal Value and Related Applications in Mathematical Finance

Qi Zhang

Fudan University

Jointed with Ulrich Horst, Jinniao Qiu

Advanced Methods in Mathematical Finance, Angers

September 4^{th}, 2015

Portfolio Liquidation Problem

$$\inf_{\xi,\rho} E\Big[\int_0^T \Big(\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2\Big)ds + \int_0^T \int_{\mathcal{Z}} \gamma_s(y_s,z)|\rho_s(z)|^2 \mu(dz)ds\Big]$$

subject to

$$\begin{cases} x_t = x - \int_0^t \xi_s \, ds - \int_0^t \int_{\mathcal{Z}} \rho_s(z)\pi(dz,ds), \qquad t \in [0,T] \\ x_T = 0 \\ y_t = y + \int_0^t b_s(y_s)ds + \int_0^t \sigma_s(y_s)dW_s. \end{cases}$$

ξ: the rates of the portfolio liquidated in the primary market.

ρ: the block trades placed in the dark pool.

Value Function

Admissible control set:
$\mathcal{A} = \{(\xi, \rho) \in \mathcal{L}^2_{\bar{\mathscr{F}}}(0, T; \mathbb{R}^1) \times \mathcal{L}^2_{\bar{\mathscr{F}}}(0, T; L^2(\mathcal{Z}))$ such that $x_T = 0\}$.
Quadratic cost functional:

$$
\begin{aligned}
J_t(x_t, y_t; \xi, \rho) \quad = \quad & E\Big[\int_t^T \Big(\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2 \Big) ds \\
& + \int_t^T \int_{\mathcal{Z}} \gamma_s(y_s, z)|\rho_s(z)|^2 \, \mu(dz) ds \Big| \mathscr{F}_t \Big].
\end{aligned}
$$

Value function:

$$
V_t(x, y) \triangleq \operatorname*{ess\,inf}_{\xi, \rho \in \mathcal{A}} J_t(x_t, y_t; \xi, \rho)\Big|_{x_t = x, y_t = y}.
$$

Equivalent Stochastic Optimal Control Problem without Constraints

$$\inf_{\xi,\rho} E\Big[\int_0^T \Big(\eta_s(y_s)|\xi_s|^2 + \lambda_s(y_s)|x_s|^2\Big)ds + \int_0^T\int_{\mathcal{Z}} \gamma_s(y_s,z)|\rho_s(z)|^2\mu(dz)ds + (+\infty)|x_T|^2 1_{\{x_T\neq 0\}}\Big]$$

subject to

$$\begin{cases} x_t = x - \int_0^t \xi_s\, ds - \int_0^t\int_{\mathcal{Z}} \rho_s(z)\,\pi(dz,ds), \quad t\in[0,T]; \\ y_t = y + \int_0^t b_s(y_s)\, ds + \int_0^t \sigma_s(y_s)\, dW_s. \end{cases}$$

$\mu(dz)$: characteristic measure of point process on $(\mathcal{Z},\mathscr{Z})$.

$\pi(dt,dz)$: associated Poisson random measure.

Associated BSPDE with Singular Terminal Value

Inspired by Peng 1992 SICON and linear-quadratic structure, the dynamic programming principle suggests

$$V_t(x,y) = u_t(y)x^2.$$

Here (u,ψ) satisfies BSPDE with singular terminal value:

$$\begin{cases} -du_t(y) = \Big[\mathrm{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 u_t(y) + D\psi_t(y)\sigma_t^*(y)\right) + b_t^*(y)Du_t(y) \\ \qquad\qquad + F(s,y,u_t(y))\Big]\, dt - \psi_t(y)dW_t, \quad (t,y)\in[0,T]\times\mathbb{R}^d, \\ u_T(y) = +\infty, \quad y\in\mathbb{R}^d, \end{cases}$$

where

$$F(t,y,r) \triangleq -\int_{\mathcal{Z}} \frac{r^2}{\gamma(t,y,z)+r}\mu(dz) - \frac{r^2}{\eta_t(y)} + \lambda_t(y),\ (t,y,r)\in\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^1$$

Assumptions for BSPDE with Singular Terminal Value

(1) $b, \sigma, \eta, \lambda : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{d\times m} \times \mathbb{R}^+ \times \mathbb{R}^+$ are $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d)$ measurable and essentially bounded by $\Lambda > 0$, $\gamma : \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{Z} \to [0,+\infty]$ is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^d) \times \mathscr{Z}$ measurable. Moreover, there exists a constant $\kappa > 0$ s.t. for any $(y,s) \in \mathbb{R}^d \times [0,T]$, $\eta_s(y) \geqslant \kappa$.

(2) The first derivatives of b, η, λ and the up to second-order derivatives of σ exist and are uniformly bounded by $L > 0$.

(3) There exists $(T_0, p_0) \in [0,T) \times (2,\infty)$ s.t.

$$\kappa_1 \triangleq \operatorname*{ess\,inf}_{(\omega,t,y)\in\Omega\times[T_0,T]\times\mathbb{R}^d} \eta_t(y) \geqslant \left(1 - \frac{1}{2p_0}\right) \operatorname*{ess\,sup}_{(\omega,t,y)\in\Omega\times[T_0,T]\times\mathbb{R}^d} \eta_t(y).$$

Definition of Solution of BSPDE Whose Terminal Value Could Be Infinite

A pair of processes (u, ψ) is a solution to the BSPDE

$$\begin{cases} -du_t(y) = f(t, y, Du, D^2u, \psi, D\psi)dt - \psi_t(y)dW_t, & (t, y) \in [0, T] \times \mathbb{R}^d \\ u_T(y) = G(y), \quad y \in \mathbb{R}^d, \end{cases}$$

if $(u, \psi) \in \mathcal{L}^2_{\mathscr{F}}(0, \tau; H^{1,2}_{loc}) \times \mathcal{L}^2_{\mathscr{F}}(0, \tau; H^{0,2}_{loc})$ for any $\tau \in (0, T)$, and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, $0 \leqslant t \leqslant \tau < T$,

$$\langle \varphi, u_t \rangle = \langle \varphi, u_\tau \rangle + \int_t^\tau \langle \varphi, f(s, y, Du, D^2u, \psi, D\psi) \rangle ds - \int_t^\tau \langle \varphi, \psi_s dW_s \rangle$$

and

$$\lim_{\tau \to T-} u_\tau(y) = G(y).$$

Existing Results

Stochastic optimal control problems with constrains in non-Markovian framework:

- Ankirchner-Jeanblanc-Kruse 2014 SICON:
  BSDE with singular terminal value and open-loop form control problem independent of benchmark price
- Graewe-Horst-Qiu 2015 SICON:
  non-degenerate BSPDE with singular terminal value and feedback form control problem dependent on benchmark price

Non-Degenerate condition is not satisfied for many diffusion models and absolutely continuous factors driving liquidation costs.

Degenerate BSPDE:

- Du-Tang-Zhang 2013 JDE:
  regularity of linear degenerate BSPDE
- Du-Zhang 2013 SPA:
  regularity of semilinear degenerate BSPDE

Equivalent Equation

For a given $q > d$ we define $\theta(y) = (1+|y|^2)^{-q}$ for $y \in \mathbb{R}^d$ and $\theta F(\cdot,\cdot,0) \in \mathcal{L}^p(0,T;H^{1,p})$ for any $p \in [1,\infty)$.
(u,ψ) solves original BSPDE if and only if $(v,\zeta) \triangleq (\theta u, \theta\psi)$ solves

$$\begin{cases} -dv_t(y) = \Big[\operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t(y) + D\zeta_t\sigma_t^*(y)\right) + \tilde{b}_t^* D v_t(y) + \beta_t^*\zeta_t(y) \\ \qquad\qquad + c_t v_t(y) + \theta(y) F(t,y,\theta^{-1}(y)v_t(y))\Big] dt - \zeta_t(y)dW_t, \\ v_T(y) = +\infty \end{cases}$$

with
$\tilde{b}_t^i(y) \triangleq b_t^i(y) + 2q(1+|y|^2)^{-1}\sum_{j=1}^d (\sigma_t\sigma_t^*)^{ij}(y)y^j, \quad i = 1,\dots,d,$
$\beta_t^r(y) \triangleq 2q(1+|y|^2)^{-1}\sum_{j=1}^d \sigma_t^{jr}(y)y^j, \quad r = 1,\dots,m,$
$c_t(y) \triangleq q(1+|y|^2)^{-1}\big(\operatorname{tr}(\sigma_t\sigma_t^*(y)) + \sum_{i=1}^d 2y^i b_t^i(y)$
$\qquad +2(q-1)(1+|y|^2)^{-1}\sum_{i,j=1}^d (\sigma_t\sigma_t^*)^{ij}(y)y^i y^j\big).$

Truncated BSPDEs

$$\begin{cases} -dv_t^N(y) = & \Big[\mathrm{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t^N(y) + D\zeta_t^N \sigma_t^*(y)\right) + \tilde{b}_t^* D v_t^N(y) \\ & +\beta_t^* \zeta_t^N(y) + c_t v_t^N(y) + \theta(y)\hat{F}(t,y,\theta^{-1}(y)v_t^N(y)) \Big] dt - \zeta_t^N(y) dW_t, \\ v_T^N(y) = & N\theta(y), \end{cases}$$

where for each $N \in \mathbb{N}$, $\hat{F}(t,y,\phi(y)) \triangleq F(t,y,|\phi(y)|)$ for any $(t,y,\phi) \in \mathbb{R}^+ \times \mathbb{R}^d \times L^0(\mathbb{R}^d)$.

Proposition Assume (1)-(2). For $p \in [2,\infty)$, truncated BSPDE has a unique solution (v^N, ζ^N) with $(v^N, \zeta^N + \sigma^* D v^N) \in \left(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p})\right) \times L^2_{\mathscr{F}}(0,T; H^{1,2})$ and $\theta^{-1} v^N \in \mathcal{L}^\infty_{\mathscr{F}}(0,T; L^\infty(\mathbb{R}^d))$.

Sketch of Proof

Truncate the quadratic term in $\hat{F}$ by a smooth function of sign function h_M with support $[-M, M]$:

$$\begin{cases} -dv_t^{N,M}(y) = \Big[\operatorname{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 v_t^{N,M}(y) + D\zeta_t^{N,M}\sigma_t^*(y)\right) \\ \qquad + \tilde{b}_t^* Dv_t^{N,M}(y) + \beta_t^*\zeta_t^{N,M}(y) + c_t v_t^{N,M}(y) + \theta\lambda_t(y) \\ \qquad - \int_{\mathcal{Z}} \frac{\theta^{-1}|v_t^{N,M}|^2(y)}{\gamma_t(y,z) + |\theta^{-1}v_t^{N,M}(y)|}\mu(dz) \\ \qquad - \frac{h_M(\theta^{-1}v_t^{N,M}(y))|v_t^{N,M}(y)|^2}{\eta_t(y)}\Big] dt - \zeta_t^{N,M}(y)dW_t, \\ v_T^{N,M}(y) = N\theta(y). \end{cases}$$

By Du-Zhang 2013 SPA, truncated BSPDE has a unique solution $(v^{N,M}, \zeta^{N,M} + \sigma^* Dv^{N,M}) \in$
$(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p})) \times L^2_{\mathscr{F}}(0,T; H^{1,2})$.

Comparison Theorem

Assume that b, σ satisfy (1)-(2), $\varrho : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ satisfies the same conditions as b and $G \in L^2(\Omega, \mathscr{F}_T; H^{1,2})$. Let $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^1 \to \mathbb{R}^1$ satisfy that (i) $\partial_y f$, $\partial_v f$ exist; (ii) $f(\cdot,\cdot,0) \in \mathcal{L}^2_{\mathscr{F}}(0,T; H^{1,2})$; (iii) $f(t,y,v)$, $\partial_y f(t,y,v)$ is Lipschitz w.r.t. v. Consider BSPDE

$$\begin{cases} -du_t(y) = & \Big[\mathrm{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2 u_t(y) + D\psi_t\sigma_t^*(y)\right) + b_t^* Du_t(y) \\ & \quad + \varrho_t^*\left(\psi_t + \sigma_t^* Du_t\right)(y) + f(t,y,u_t) \Big]\, dt - \psi_t(y)\, dW_t, \\ u_T(y) = & G(y). \end{cases}$$

Let (G', f') be another driver with solution (u', ψ'). If $f(\omega,t,y,u_t(y)) \leqslant f'(\omega,t,y,u_t(y))$ and $G(\omega,y) \leqslant G'(\omega,y)$ for a.e. $\Omega \times [0,T] \times \mathbb{R}^d$, then we have a.s. $u \leqslant u'$ for a.e. $\mathbb{R}^d$, all $t \in [0,T]$.

Sketch of Proof (Continued)

Changing (λ, γ, M) in truncated BSPDE to $(\Lambda, +\infty, 0)$. For this new equation, $(\hat{v}_t(y), 0) \triangleq (\theta(y)\,(N + \Lambda(T-t))\,, 0)$ is a solution, and the comparison theorem yields:

$$0 \leqslant v_t^{N,M} \leqslant \hat{v}_t \quad \text{for a.e. } y \in \mathbb{R}^d, \text{ all } t \in [0,T], \text{ a.s.}$$

Thus, for the sufficiently large $M \in \mathbb{N}$, $(v^{N,M}, \zeta^{N,M})$ is independent of M and is a solution of truncated BSPDE.

The uniqueness of solution is due to the uniqueness of solution to the truncated BSPDE.

Main Results for Singular BSPDE

Theorem Assume (1)-(3). For any $p \in [2, p_0)$, BSPDE with singular terminal value has a solution (u, ψ) s.t. for some $\alpha \in (1, 2)$, $\{(T-t)^\alpha(\theta u_t, \theta\psi_t + \sigma^* D(\theta u_t))(y);\ (t, y) \in [0, T] \times \mathbb{R}^d\}$ belongs to

$$\left(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p})\right) \times L^2_{\mathscr{F}}(0,T; H^{1,2}),$$

and

$$\frac{c_0}{T-t} \leqslant u_t(y) \leqslant \frac{c_1}{T-t} \quad \text{for a.e. } y \in \mathbb{R}^d\text{, all } t \in [0,T)\text{, a.s.}$$

with two constants $c_0 > 0$ and $c_1 > 0$.

Monotonicity of Solutions of Truncated BSPDEs

Proposition Assume (1)-(2) and (v^N, ζ^N) is solution of truncated BSPDE. If $(\tilde{\lambda}, \tilde{\gamma}, \tilde{\eta})$ satisfies (1)-(2), $G \in L^2(\Omega, \mathscr{F}_T; H^{1,2})$ and $(\tilde{v}, \tilde{\zeta}) \in S^w_{\mathscr{F}}([0,T]; H^{1,2}) \times L^2_{\mathscr{F}}(0,T; L^2)$ with $\theta^{-1}\tilde{v} \in \mathcal{L}^\infty_{\mathscr{F}}(0,T; L^\infty(\mathbb{R}^d))$ is a solution of BSPDE:

$$\begin{cases} -d\tilde{v}_t(y) = & \Big[\text{tr}\left(\frac{1}{2}\sigma_t\sigma_t^* D^2\tilde{v}_t(y) + D\tilde{\zeta}_t\sigma_t^*(y)\right) + \tilde{b}_t^* D\tilde{v}_t(y) + \beta_t^*\tilde{\zeta}_t(y) \\ & + c_t\tilde{v}_t(y) + \theta\tilde{\lambda}_t(y) - \int_{\mathcal{Z}} \frac{\theta^{-1}(y)|\tilde{v}_t(y)|^2}{\tilde{\gamma}_t(y,z) + \theta^{-1}|\tilde{v}_t(y)|}\mu(dz) \\ & - \frac{\theta^{-1}(y)\,|\tilde{v}_t(y)|^2}{\tilde{\eta}_t(y)}\Big]dt - \tilde{\zeta}_t(y)dW_t, \\ \tilde{v}_T(y) = & G(y). \end{cases}$$

If $G \geqslant N\theta$, $\tilde{\lambda} \geqslant \lambda$, $\tilde{\gamma} \geqslant \gamma$, $\tilde{\eta} \geqslant \eta$, we have $\tilde{v}_t(y) \geqslant v_t^N(y)$ for a.e. $y \in \mathbb{R}^d$, all $t \in [0,T]$, a.s.

The Limit of Solutions of Truncated BSPDEs

The solutions $\{v^N\}$ increase to some limit v, which satisfies the growth condition. Replace the coefficients (λ, γ, η) by their lower bound $(0, 0, \kappa)$ and upper bound $(\Lambda, +\infty, \Lambda)$, respectively. You can verify that $(\dot{v}^N, 0)$ and $(\ddot{v}^N, 0)$ are the corresponding BSPDEs, respectively, where $\dot{v}_t^N(y) \triangleq \frac{\kappa\mu(\mathcal{Z})\theta(y)}{\left(1+\frac{\kappa_1\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1}$ and $\ddot{v}_t^N(y) \triangleq \frac{2\Lambda\theta(y)}{1-\frac{N-\Lambda}{N+\Lambda}\cdot e^{-2(T-t)}} - \Lambda\theta(y)$.

By comparison theorem,
$\dot{v}_t^N(y) \leqslant v_t^N(y) \leqslant \ddot{v}_t^N(y)$ for a.e. $y \in \mathbb{R}^d$, all $t \in [0, T)$, a.s., which yields

$$\frac{\kappa\mu(\mathcal{Z})\theta(y)}{\left(1+\frac{\kappa\mu(\mathcal{Z})}{N}\right)e^{\mu(\mathcal{Z})(T-t)}-1} \leqslant v_t(y) \leqslant \frac{\theta(y)e^{2T}}{\frac{1}{N+\Lambda}+\frac{T-t}{\Lambda}}.$$

Estimates of Solution of Semilinear BSPDE by Lipschitz Condition

By Du-Zhang 2013 SPA, to the solution (u,ψ) of semilinear BSPDE, we know for $p\geqslant 2$,

$$\begin{aligned}
& E\sup_{t\in[0,T]}\|u_t\|^p_{H^{1,p}}+E\int_0^T\|\psi_t+\sigma_t^*Du_t\|^2_{H^{1,2}}dt\\
\leqslant\ & C(E\|G\|^p_{H^{1,p}}+E\int_0^T\|f(t,\cdot,0)\|^p_{H^{0,p}}dt),
\end{aligned}$$

where G is the terminal value, f is the nonlinear term and C is a constant depending on the Lipschitz constant of $f(t,x,y)$ w.r.t. y.

Estimates of Solution of Semilinear BSPDE by Monotonic Condition

If there exist constant C_0 and function $g \in \mathcal{L}^p_{\mathscr{F}}(0,T;H^{0,p})$ s.t.

$$u_s(y)f(s,y,u_s(y)) + \sum_{i=1}^{d} \partial_{y^i} u_s(y) \left(\partial_{y^i} + \partial_{y^i} u_s(y)\partial_u\right) f(s,y,u_s(y))$$

$$\leqslant |g_s(y)|^2 + C_0\Big(|u_s(y)|^2 + \sum_{i=1}^{d} |\partial_{y^i} u_s(y)|^2\Big),$$

we have

$$E \sup_{t\in[0,T]} \|u_t\|^p_{H^{1,p}} + E\int_0^T \|\psi_t + \sigma_t^* D u_t\|^2_{H^{1,2}} dt$$

$$\leqslant \quad CE\Big[\|G\|^p_{H^{1,p}} + \int_0^T \|g_t\|^p_{H^{0,p}} dt\Big],$$

where C depends on C_0 but independent of the Lipschtiz constant.

Transform of Truncated BSPDEs

For p_0 in (3), let $\alpha_0 \triangleq 1 - \frac{1}{2p_0}$, and choose $\alpha_1, \alpha_2 \in (1, \infty)$ and $p_1 \in [2, p_0)$ s.t. $2\alpha_0 = \alpha_1\alpha_2$ and $(2-\alpha_2)p_1 < 1$. Let $T_1 \in [T_0, T)$ and $N_0 > 2\Lambda + \kappa\mu(\mathcal{Z})$ s.t. $\big(1 + \frac{\kappa_1\mu(\mathcal{Z})}{N_0}\big)e^{\mu(\mathcal{Z})(T-T_1)} < \alpha_1$, and for each $N > N_0$, set $\delta^N \triangleq \big(1 + \frac{\kappa_1\mu(\mathcal{Z})}{N}\big)e^{\mu(\mathcal{Z})(T-T_1)}$. Then $(Q_t^N, \xi_t^N) \triangleq \big(\frac{\kappa_1}{N} + \delta^N(T-t)\big)^{\alpha_2}(v_t^N, \zeta_t^N)$, $t \in [T_1, T]$, satisfies

$$\begin{cases} -dQ_t^N(y) = \Big[\mathrm{tr}\Big(\frac{1}{2}\sigma_t\sigma_t^* D^2 Q_t^N(y) + D\xi_t^N\sigma_t^*(y)\Big) + \tilde{b}_t^* DQ_t^N(y) \\ \qquad\qquad + \beta_t^*\xi_t^N(y) + c_t Q_t^N(y) + \Big(\frac{\kappa_1}{N} + \delta^N(T-t)\Big)^{\alpha_2} \\ \qquad\qquad \times\Big(\theta\lambda_t(y) - \int_{\mathcal{Z}} \frac{\theta^{-1}|v_t^N(y)|^2}{\gamma_t(y,z) + \theta^{-1}|v_t^N(y)|}\mu(dz) - \frac{\theta^{-1}|v_t^N(y)|^2}{\eta_t(y)}\Big) \\ \qquad\qquad + \alpha_2\delta^N\Big(\frac{\kappa_1}{N} + \delta^N(T-t)\Big)^{\alpha_2-1} v_t^N(y)\Big]dt - \xi_t^N(y)dW_t, \\ Q_T^N(y) = \kappa_1^{\alpha_2}N^{1-\alpha_2}\theta(y). \end{cases}$$

Gradient Estimate of the Transform BSPDE

Proposition Assume (1)-(3). (Q_t^N, ξ_t^N) satisfies

$$\sup_{N>N_0}\Big\{E\Big[\sup_{t\in[T_1,T]}\big(\|Q_t^N\|_{H^{1,2}}^2+\|Q_t^N\|_{H^{1,p_1}}^{p_1}\big)\Big] +\|\sigma^* DQ^N+\xi^N\|^2_{\mathcal{L}^2(T_1,T;H^{1,2})}\Big\}<\infty.$$

The estimate is derived by verifying that the nonlinear term satisfies the monotonic condition, in which (3) is applied.

Corollary Above Proposition, together with the estimate to the time interval $[0,T_1]$, leads to the gradient estimate on $[0,T]$.

The gradient estimate allows us to extract a subsequence (Q^{N_k},ξ^{N_k}) s.t. Q^{N_k} converges to Q weakly in $\mathcal{L}^p(0,T;H^{1,p})$ as well as weak-star in $\mathcal{L}^\infty(0,T;H^{1,p})$ for any $p\in\{2,p_1\}$, and $(\xi^{N_k},\xi^{N_k}+\sigma^*DQ^{N_k})$ converges weakly to $(\xi,\xi+\sigma^*DQ)$ in $\mathcal{L}^2(0,T;L^2)\times\mathcal{L}^2(0,T;H^{1,2})$.

Existence of Solution of the Transform BSPDE

Since $\{v^N\}$ increases to v a.e. in $\mathbb{R}^d$ for all $t \in [0, T]$, passing to the limit we get $Q_t(y) = e^{\alpha_2 \mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2} v_t(y)$. By Mazur's Lemma, we choose a sequence of convex combinations of $(Q^{N_k}, \xi^{N_k}, \xi^{N_k} + \sigma^* DQ^{N_k})$ which converges strongly in corresponding spaces. Therefore, (Q, ξ) solves

$$
\begin{cases}
-dQ_t(y) = \Big[\text{tr}\Big(\frac{1}{2}\sigma_t \sigma_t^* D^2 Q_t(y) + D\xi_t^N \sigma_t^*(y) \Big) + \tilde{b}_t^* DQ_t(y) \\
\qquad + \beta_t^* \xi_t(y) + c_t Q_t(y) + e^{\alpha_2 \mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2} \\
\qquad \times \Big(\theta \lambda_t(y) - \int_{\mathcal{Z}} \frac{\theta^{-1}|v_t(y)|^2}{\gamma_t(y,z) + \theta^{-1}|v_t(y)|} \mu(dz) - \frac{\theta^{-1}|v_t(y)|^2}{\eta_t(y)} \Big) \\
\qquad + \alpha_2 e^{\alpha_2 \mu(\mathcal{Z})(T-T_1)}(T-t)^{\alpha_2 - 1} v_t(y) \Big] dt - \xi_t(y) dW_t, \\
Q_T(y) = 0.
\end{cases}
$$

Existence of Solution of Singular BSPDE

(Q, ξ) admits a version, still denoted by (Q, ξ), s.t.

$$(Q, \xi + \sigma^* DQ) \in \left(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p_1})\right) \times L^2_{\mathscr{F}}(0,T; H^{1,2}).$$

Recovering (v, ζ) from (Q, ξ) and setting $(u, \psi) \triangleq \theta^{-1}(v, \zeta)$ we can verify (u, ψ) solves original singular BSPDE and for $p \in [2, p_0)$,

$$\{(T-t)^{\alpha_2}(\theta u_t, \theta \psi_t + \sigma^* D(\theta u_t))(y);\ (t,y) \in [0,T] \times \mathbb{R}^d\}$$
$$\in \left(S^w_{\mathscr{F}}([0,T]; H^{1,2}) \cap C^w_{\mathscr{F}}([0,T]; H^{1,p})\right) \times L^2_{\mathscr{F}}(0,T; H^{1,2}).$$

Moreover,

$$\frac{c_0}{T-t} \leqslant u_t(y) \leqslant \frac{c_1}{T-t} \quad \text{for a.e. } y \in \mathbb{R}^d, \text{ all } t \in [0,T), \text{ a.s.}$$

holds with $c_0 = \kappa e^{-\mu(\mathscr{Z})T}$ and $c_1 = \Lambda e^{2T}$.

Minimality and Uniqueness of Solution of Singular BSPDE

The solution (u, ψ) constructed above is the minimal solution of singular BSPDE. Subsequently, we derive the uniqueness of the nonnegative solution in a certain class.

Theorem Assume (1)-(3). If $(\tilde{u}, \tilde{\psi})$ is another solution satisfying $(\theta\tilde{u}, \theta\tilde{\psi} + \sigma^* D(\theta\tilde{\psi})) \in S^w_{\mathscr{F}}([0,t]; H^{1,2}) \times L^2_{\mathscr{F}}(0,t; H^{1,2}), \quad \forall\, t \in (0,T)$ and if $\tilde{u}_t(y) \geqslant 0$ a.e. in $\Omega \times [0,T) \times \mathbb{R}^d$, then for all $t \in [0,T)$, $\tilde{u}_t \geqslant u_t$ for a.e. $y \in \mathbb{R}^d$, a.s.

Moreover, if we further have $p_0 > 2d+2$ and $\theta\tilde{u} \in C^w_{\mathscr{F}}([0,t]; H^{1,p})$ for some $p \in (2d+2, p_0)$, then for all $t \in [0,T)$, $\tilde{u}_t = u_t$ for a.e. $y \in \mathbb{R}^d$, a.s.

Sketch of Proof for Minimality: Let (v^N, ζ^N) be the unique solution of truncated BSPDE and set $(\tilde{v}, \tilde{\zeta}) = \theta\,(\tilde{u}, \tilde{\psi})$. Since v^N increases to v as $N \to \infty$, we only need to prove for all $t \in [0,T]$,

$$\tilde{v}_t \geqslant v_t^N \qquad \text{for a.e. } y \in \mathbb{R}^d, \text{ a.s.}$$

Verification Theorem

The uniqueness of the nonnegative solution in a certain class is a result from the following verification theorem. In the proof of verification theorem, $p > 2d + 2$ is needed to establish the relationship between BSPDE and non-Markovian BSDE.

Theorem Assume (1)–(2). If (u, ψ) is a solution to the singular BSPDE s.t. $\theta u \in C^w(0, t; H^{1,p}) \cap S^w(0, t; H^{1,2})$, $\forall\, t \in (0, T)$, for some $p > 2d + 2$, and a.s.

$$\frac{c_0}{T - t} \leqslant u_t(y) \leqslant \frac{c_1}{T - t}, \quad \forall\, (t, y) \in [0, T) \times \mathbb{R}^d$$

with two constants $c_0 > 0$ and $c_1 > 0$, then

$$V(t, y, x) \triangleq u_t(y) x^2, \quad \forall\, (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,$$

coincides with the value function. Moreover, the optimal feedback control is given by $(\xi_t^*,\ \rho_t^*(z)) = \left(\frac{u_t(y_t) x_t}{\eta_t(y_t)},\ \frac{u_t(y_t) x_{t-}}{\gamma_t(z, y_t) + u_t(y_t)} \right)$.

Sketch of Proof for Uniqueness

The minimality arguments have given the lower bound. To establish the upper bound $\tilde{u}_t(y) \leqslant \frac{c_1}{T-t}$, consider the deterministic function $\hat{u}_t \triangleq \frac{2\Lambda}{1-e^{-2(T-t)}} - \Lambda \leqslant \frac{\Lambda e^{2T}}{T-t}$. Then, $(\hat{u}, 0)$ is a solution of singular BSPDE with the triple (λ, γ, η) replaced by $(\Lambda, +\infty, \Lambda)$. Moreover, for $\delta \in [0, T)$ $(\hat{u}_{\cdot+\delta}, 0)$ is the solution of singular BSPDE with $(\Lambda, +\infty, \Lambda)$ and the singularity at $t = T - \delta$.

By calculus, we conclude

$$\int_{[0,T-\delta]\times\mathbb{R}^d} E|(\theta\tilde{u}_t - \theta\hat{u}_{t+\delta})^+(y)|^2 dydt = 0.$$

This yields, for all $t \in [0, T-\delta]$

$$\tilde{u}_t \leqslant \frac{\Lambda e^{2T}}{T-\delta-t} \qquad \text{for a.e. } y \in \mathbb{R}^d, \text{ a.s.}$$

Finally, letting $\delta \to 0$ we obtain the desired upper bound as well as the uniqueness by the verification theorem.

Thank You