ON OPTIMAL STOPPING WITH EXPECTATION CONSTRAINTS

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Optimal stopping

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Optimal stopping
Optimal stopping in continuous time

- risk factor:
\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \]

- payoff function: \( f \)

- optimal stopping problem
\[ E[f(X_{\tau})] \rightarrow \text{max!} \]
ODE approach

▶ value function

\[ v(x) = \sup \{ E[f(X^x_\tau)] : \tau \text{ stopping time} \} \]

▶ \( v(x) \geq f(x) \) and \( \mathcal{L}v(x) \leq 0 \)

▶ \( v(x) = f(x) \) or \( \mathcal{L}v(x) = 0 \)

\[ \implies \min\{ -\mathcal{L}v(x), v(x) - f(x) \} = 0 \]
Adding constraints

1. **constraint:** \( \tau \leq T \) (deterministic bound)

\[ \implies \text{value function depends also on time} \]

HJB becomes

\[
\min\{-\frac{\partial v}{\partial t}(t, x) - \mathcal{L}v(t, x), v(t, x) - f(x)\} = 0
\]
Expectation constraints

2. constraint: $E[\tau] \leq T$

Why interesting?
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- repeated stopping
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Why difficult?
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**Why interesting?**
- repeated stopping

**Why difficult?**
- no simple dependence of the constraint on time
- constraint has to be turned into a *scenario-dependent constraint*
The basic idea

- extend state space by
  \[ M_t = E[\tau | \mathcal{F}_t] \]

- In a Brownian setting we have
  \[ M_t = E[\tau] + \int_0^t \alpha_s dW_s \]

Interpret \((\alpha_s)\) as a control.
Why it works

We can recover $\tau$ from $M$:

$$\tau = \inf \{ t \geq 0 : M_t - t \leq 0 \}$$

Proposition

*There is a one-to-one correspondence between*

1) The set of stopping times $\tau$ with $E[\tau] = m$ and
2) The set of controlled processes with

$$dM_t = \alpha_t 1_{\{M_t > t\}} dW_t, \quad M_0 = m. \quad (1)$$
Reformulating the stopping problem

\[ \sup \{ E[f(X_\tau)] : E[\tau] \leq T \} \]

\[ = \sup \{ E[f(X_{M^\alpha,m})] : \alpha \in L^2_{loc}, m \in [0, T] \} \]
Dynamic Programming principle

**Definition:**

\[
U(T, x) = \sup \{ E[f(X^x_T)] : \tau \text{ with } E[\tau] = T \}.
\]

**DDP:** for stopping times \((\vartheta^\alpha)_{\alpha \in L^2_{loc}}\) we have

\[
U(T, x) = \sup_{\alpha \in L^2_{loc}} E[1\{M_{\vartheta^\alpha}^\alpha - \vartheta^\alpha \leq 0\} f(X_{M_{\vartheta^\alpha}^\alpha}^x) + 1\{M_{\vartheta^\alpha}^\alpha - \vartheta^\alpha > 0\} U(M_{\vartheta^\alpha}^\alpha - \vartheta^\alpha, X_{\vartheta^\alpha})]
\]

where

\[
M_t^\alpha = T + \int_0^t \alpha_s 1\{M_s > s\} \, dW_s.
\]
If \( U \in C^2((0, \infty) \times I) \) and ...., then

\[
U_T(T, x) - \mathcal{L}U(T, x) + \frac{\sigma^2(x)}{2} \frac{U_{Tx}^2(T, x)}{U_{TT}(T, x)} = 0, \quad (T, x) \in (0, \infty) \times I
\]

(2)

with initial condition \( U(0, x) = f(x) \).
Verification theorem

- $u \in C^2((0, \infty) \times I) \cap C([0, \infty) \times I)$
- $u$ solves the HJB equation
- $u$ is concave in $T$
- the processes are well defined:

$$\alpha_s^* := -\sigma(X_s^x) \left( \frac{u_{T|X}}{u_{TT}} \right) (M_s^* - s, X_s^x) \in L^2_{loc}(W)$$

and

$$dM_s^* = \alpha_s^* 1\{M_s^* > s\} dW_s, \quad M_0^* = T.$$ 

Then $u(T, x) = U(T, x)$ and $(\alpha_s^*)$ is an optimal control.
\[ V(T, x) = \sup \{ E[f(X^x_T)] : \tau \text{ with } E[\tau] \leq T \}. \]

- \( V \) is non-decreasing
- \( V \) is concave

\[ \implies V(T, x) = \sup_{S \leq T} U(S, x) = U(T \wedge \tilde{T}(x), x), \]

where \( \tilde{T}(x) = \inf \{ t \geq 0 : \frac{\partial}{\partial T} U(t, x) = 0 \} \).
PDE for $V$

$$V(T, x) = \sup \{ E[f(X^x_\tau)] : \tau \text{ with } E[\tau] \leq T \}.$$ 

$$\min \left\{ V_T(T, x), V_T(T, x) - \mathcal{L}V(T, x) + \frac{\sigma^2(x)}{2} \frac{V_{TT}^2(T, x)}{V_{TT}(T, x)} \right\} = 0$$ 

with initial condition $V(0, x) = f(x)$. 
Generalizations

- multidimensional case: ✓
- More general constraints: 
  $$E \left[ \int_0^T h(X_s) ds \right] \leq C.$$

$$\min \left\{ V_T(T, x), h(x) V_T(T, x) - \mathcal{L} V(T, x) + \frac{\sigma^2(x)}{2} \frac{V_{Tx}^2(T, x)}{V_{TT}(T, x)} \right\} = 0$$
Example

- $X = W$, a 1-dimensional Brownian motion
- $f(x) = |x|$
- $u(T, x) := \sqrt{T + x^2}$

$u$ satisfies the assumption of the verification theorem $\implies$

$$U(T, x) = V(T, x) = \sqrt{T + x^2}$$

and $\tau^* = \text{first exit time of } (-\sqrt{T + x^2}, \sqrt{T + x^2})$
Lagrange approach

Comparison with the Lagrange approach

\[ E[f(X_\tau)] - \lambda (E[\tau] - T) \rightarrow \max! \]

If \( \tau^* \) optimal for \( \lambda \) and \( E[\tau^*] = T \), then \( \tau^* \) is optimal in the constrained problem. HOWEVER: for one \( \lambda \) there may be many \( \tau^* \)!

Reference for an approach with the Lagrange method: Kennedy 1982.
We describe a new solution method for stopping problems with expectation constraints.

Idea: rewrite the problem as a control problem where one chooses the integrand in the martingale representation of the stopping time.

There is a DPP and a HJB equation.

We can formulate a verification theorem.

Next steps: viscosity solutions, numerical solution of the PDEs.
Thank you!