ON OPTIMAL STOPPING WITH EXPECTATION CONSTRAINTS

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Optimal stopping

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Optimal stopping in continuous time

- risk factor:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

- payoff function: f
- optimal stopping problem

$$E[f(X_\tau)] \longrightarrow \max!$$

ODE approach

- value function

$$v(x) = \sup\{E[f(X_\tau^x)] : \tau \text{ stopping time}\}$$

- $v(x) \geq f(x)$ and $\mathcal{L}v(x) \leq 0$
- $v(x) = f(x)$ **or** $\mathcal{L}v(x) = 0$

$$\Longrightarrow \qquad \boxed{\min\{-\mathcal{L}v(x), v(x) - f(x)\} = 0}$$

Adding constraints

1. constraint: $\tau \leq T$ (deterministic bound)

$\Longrightarrow$ value function depends also on time

HJB becomes

$$\boxed{\min\{-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}v(t,x), v(t,x) - f(x)\} = 0}$$

Expectation constraints

2. constraint: $E[\tau] \leq T$

Why interesting?

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- repeated stopping

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Why difficult?

Expectation constraints

2. constraint: $E[\tau] \leq T$

Why interesting?

- repeated stopping

Why difficult?

- no simple dependence of the constraint on time
- constraint has to be turned into a **scenario-dependent constraint**

The basic idea

- extend state space by

$$M_t = E[\tau|\mathcal{F}_t]$$

- In a Brownian setting we have

$$M_t = E[\tau] + \int_0^t \alpha_s dW_s$$

Interpret (α_s) as a control.

Why it works

$$M_t = E[\tau|\mathcal{F}_t] = E[\tau] + \int_0^t \alpha_s dW_s$$

We can recover τ from M:

$$\tau = \inf\{t \geq 0 : M_t - t \leq 0\}$$

Proposition

There is a one-to-one correspondence between
1) the set of stopping times τ with $E[\tau] = m$ and
2) the set of controlled processes with

$$dM_t = \alpha_t 1_{\{M_t > t\}} dW_t, \quad M_0 = m. \tag{1}$$

Reformulating the stopping problem

$$\begin{aligned} & \sup\{E[f(X_\tau)] : E[\tau] \leq T\} \\ = \ & \sup\{E[f(X_{M_\infty^{\alpha,m}})] : \alpha \in L^2_{loc}, m \in [0,T]\} \end{aligned}$$

Dynamic Programming principle

Definition:

$$U(T,x) = \sup\{E[f(X^x_\tau)] : \tau \text{ with } E[\tau] = T\}.$$

DDP: for stopping times $(\vartheta^\alpha)_{\alpha \in L^2_{loc}}$ we have

$$U(T,x) = \sup_{\alpha \in L^2_{loc}} E[1_{\{M^\alpha_{\vartheta^\alpha} - \vartheta^\alpha \le 0\}} f(X^x_{M^\alpha_{\vartheta^\alpha}}) + 1_{\{M^\alpha_{\vartheta^\alpha} - \vartheta^\alpha > 0\}} U(M^\alpha_{\vartheta^\alpha} - \vartheta^\alpha, X^x_{\vartheta^\alpha})]$$

where

$$M^\alpha_t = T + \int_0^t \alpha_s 1_{\{M_s > s\}} dW_s.$$

HJB Equation

If $U \in \mathcal{C}^2((0,\infty) \times \mathcal{I})$ and ...., then

$$U_T(T,x) - \mathcal{L}U(T,x) + \frac{\sigma^2(x)}{2} \frac{U_{Tx}^2(T,x)}{U_{TT}(T,x)} = 0, \qquad (T,x) \in (0,\infty) \times \mathcal{I} \tag{2}$$

with initial condition $U(0,x) = f(x)$.

Verification theorem

- $u \in \mathcal{C}^2((0,\infty)\times\mathcal{I}) \cap \mathcal{C}([0,\infty)\times\mathcal{I})$
- u solves the HJB equation
- u is concave in T
- the processes are well defined:

$$\alpha_s^* := -\sigma(X_s^x)\left(\frac{u_{Tx}}{u_{TT}}\right)(M_s^* - s, X_s^x) \in L^2_{loc}(W)$$

and

$$dM_s^* = \alpha_s^* 1_{\{M_s^* > s\}} dW_s, \quad M_0^* = T.$$

Then $u(T,x) = U(T,x)$ and (α_s^*) is an optimal control.

Back to V

$$V(T,x) = \sup\{E[f(X_\tau^x)] : \tau \text{ with } E[\tau] \le T\}.$$

- V is non-decreasing
- V is concave

$\Longrightarrow$

$$V(T,x) = \sup_{S \le T} U(S,x) = U(T \wedge \tilde{T}(x), x),$$

where $\tilde{T}(x) = \inf\{t \ge 0 : \frac{\partial}{\partial T} U(t,x) = 0\}.$

PDE for V

$$V(T,x) = \sup\{E[f(X_\tau^x)] : \tau \text{ with } E[\tau] \leq T\}.$$

$$\min\left\{V_T(T,x), V_T(T,x) - \mathcal{L}V(T,x) + \frac{\sigma^2(x)}{2}\frac{V_{Tx}^2(T,x)}{V_{TT}(T,x)}\right\} = 0$$

with initial condition $V(0,x) = f(x)$.

Generalizations

- multidimensional case: ✓
- More general constraints: $E\left[\int_0^\tau h(X_s)ds\right] \leq C$.

$$\min\left\{V_T(T,x), h(x)V_T(T,x) - \mathcal{L}V(T,x) + \frac{\sigma^2(x)}{2}\frac{V_{Tx}^2(T,x)}{V_{TT}(T,x)}\right\} = 0$$

Example

- $X = W$, a 1-dimensional Brownian motion
- $f(x) = |x|$
- $u(T, x) := \sqrt{T + x^2}$

u satisfies the assumption of the verification theorem $\Longrightarrow$

$$\boxed{U(T, x) = V(T, x) = \sqrt{T + x^2}}$$

and $\tau^* =$ first exit time of $(-\sqrt{T + x^2}, \sqrt{T + x^2})$

Lagrange approach

Comparison with the Lagrange approach

$$E[f(X_\tau)] - \lambda(E[\tau] - T) \longrightarrow \text{max!}$$

If τ^* optimal for λ and $E[\tau^*] = T$, then τ^* is optimal in the constrained problem. HOWEVER: for **one** λ there may be **many** τ^*!

Reference for an approach with the Lagrange method: Kennedy 1982.

Conclusion

- We describe a new solution method for stopping problems with expectation constraints
- Idea: rewrite the problem as a control problem where one chooses the integrand in the martingale representation of the stopping time
- There is a DPP and a HJB equation
- We can formulate a verification theorem
- Next steps: viscosity solutions, numerical solution of the PDEs

Thank you!