# On Viscosity solutions of second order integral-partial differential equations

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# 0. Outlines

1. Standard IPDEs ;
2. Link with BSDEs with jumps ;
3. Another framework for IPDEs: The IPDE without monotonicity condition ;
4. Existence and uniqueness of the solution ;
5. Extensions. □

# 1. Standard IPDEs

Let us consider the following IPDE:

$$\begin{cases} -\partial_t u(t,x) - b(t,x)^\top D_x u(t,x) - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(t,x)D^2_{xx}u(t,x)\right) - Ku(t,x) \\ \qquad -h(t,x,u(t,x),(\sigma^\top D_x u)(t,x),Bu(t,x)) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^k; \\ u(T,x) = g(x) \end{cases} \quad (1)$$

where:

(i) the operators $B$ and $K$ are given by

$$Bu(t,x) := \int_E \gamma(t,x,e)\,(u(t,x+\beta(t,x,e)) - u(t,x))\,\lambda(de)\,;$$

$$Ku(t,x) := \int_E \Big(u(t,x+\beta(t,x,e)) - u(t,x) - \beta(t,x,e)^\top D_x u(t,x)\Big)\,\lambda(de). \tag{2}$$

(ii) $\lambda$ is a Lévy measure on $E = \mathbb{R}^\ell - \{0\}$ ;

(iii) $\beta(t,x,e)$ , $\gamma(t,x,e)$ given functions ;

(iv) $b(t,x)$, $\sigma(t,x)$, $h(t,x,y,z,\zeta)$ and $g(x)$ are also given functions ;

Remarks:

(i) This IPDE is of non local type due to the operators $Bu$ and $Ku$ ;

(ii) $Bu$ and $Ku$ are defined only if $u$ is regular enough. $\square$

For $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$, we set:

$$\mathcal{L}^X \phi(t,x) :=$$

$$\frac{1}{2}\,\mathrm{Tr}[(\sigma\sigma^\top)(t,x) D^2_{xx}\phi(t,x)] + b(t,x)^\top D_x\phi(t,x) + K\phi(t,x).$$

# Barles-Buckdahn-Pardoux's definition of a viscosity solution of (1)

### Definition

A continuous function $u(t,x)$ is a viscosity sub-solution (resp. super-solution) of the IPDE (1) if:

(i) $\forall x \in \mathbb{R}^k$, $u(T,x) \leq g(x)$ (resp. $u(T,x) \geq g(x)$) ;

(ii) For any $(t,x) \in (0,T) \times \mathbb{R}^k$ and any function $\phi$ of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ such that $(t,x)$ is a global maximum (resp. minimum) point of $u - \phi$ and $(u-\phi)(t,x) = 0$, one has

$$
\begin{aligned}
&-\partial_t \phi(t,x) - \mathcal{L}^X \phi(t,x) \\
&\qquad -h(t,x,u(t,x),\sigma^\top(t,x) D_x \phi(t,x), B\phi(t,x)) \leq \ (\textit{resp.}\ \geq) 0\,;
\end{aligned}
$$

(iii) The function $u(t,x)$ is a viscosity solution of (1) if it is both a viscosity sub-solution and viscosity super-solution.

# 2. Link with BSDEs with jumps

Let $(t, x) \in [0, T] \times \mathbb{R}^k$ and $(X_s^{t,x})_{s \leq T}$ be the solution of the following standard SDE of diffusion-jump type:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r$$
$$+ \int_t^s \int_E \beta(r, X_{r-}^{t,x}, e)\tilde{\mu}(dr, de),\ s \in [t, T];\ X_s^{t,x} = x \text{ if } s \leq t \quad (3)$$

where:

(i) $B := (B_t)_{t \geq 0}$ a $d$-dimensional Brownian motion ;

(ii) $\mu$ is an independant Poisson random measure on $\mathbb{R}^+ \times E$ with compensator $\nu(dt, de) = dt\lambda(de)$ ; $\lambda$ is $\sigma$-finite measure on $(E, \mathcal{E})$ and integrating the function $(1 \wedge |e|^2)_{e \in E}$.

(iii) $\beta : (t, x, e) \in [0, T] \times \mathbb{R}^k \times E \to \beta(t, x, e) \in \mathbb{R}^k$ a function such that,

$$
\begin{aligned}
&\text{(a)} \quad |\beta(t, x, e)| \leq C(1 \wedge |e|); \\
&\text{(b)} \quad |\beta(t, x, e) - \beta(t, x', e)| \leq C|x - x'|(1 \wedge |e|); \\
&\text{(c)} \quad \text{the mapping } (t, x) \in [0, T] \times \mathbb{R}^k \to \beta(t, x, e) \in \mathbb{R}^k \\
&\quad \text{is continuous uniformly wrt } e.
\end{aligned}
\tag{4}
$$

Next let us consider the following BSDE with jumps:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr \\ \qquad - \int_s^T Z_r^{t,x} dB_r - \int_s^T \int_E U_r^{t,x}(e)\tilde{\mu}(dr, de), \;\; s \leq T. \end{cases} \quad (5)$$

**Theorem (Tang-Li '94, Barles et al. '97)**

*If :*

*(i) $f(t, x, y, z, v)$ is measurable and Lipschitz in $(y, z, v) \in \mathbb{R}^{1+d} \times L^2(E, \lambda)$ uniformly wrt to $(t, x)$ ;*

*(ii) $g(X_T^{t,x})$ is square integrable.*

*Then the solution $(Y^{t,x}, Z^{t,x}, U^{t,x})$ of (5) exists and is unique such that*

$$E[\sup_{s \leq T} |Y_s^{t,x}|^2 + \int_0^T |Z_s^{t,x}|^2 ds + \int_0^T ds \int_E |U^{t,x}(s, e)|^2 \lambda(de)] < \infty. \; \square$$

## Assumption [A1]

(i) (a) the function $h$ is continuous and

$$(y, z, v) \in \mathbb{R}^{1+d} \times L^2(E, \lambda) \mapsto h(t, x, y, z, \int_E \gamma(t, x, e) v(e) \lambda(de))$$

is Lipschitz ;

(b) the mapping $\zeta \mapsto h(t, x, y, z, \zeta)$ is non-decreasing ;

(c) the maping $(t, x) \mapsto h(t, x, 0, 0, 0)$ is of polynomial growth ;

(ii) $g$ is continuous and of polynomial growth ;

(iii) the function $\gamma(t, x, e)$ verifies:

$$\begin{aligned} &(a) \quad |\gamma(t, x, e)| \leq C(1 \wedge |e|); \\ &(b) \quad |\gamma(t, x, e) - \gamma(t, x', e)| \leq C|x - x'|(1 \wedge |e|); \\ &(c) \quad \text{the mapping } (t, x) \in [0, T] \times \mathbb{R}^k \to \gamma(t, x, e) \in \mathbb{R}^k \\ &\quad \text{is continuous uniformly wrt } e; \\ &(d) \quad \gamma(t, x, e) \geq 0. \end{aligned} \tag{6}$$

The link between IPDE (1) and BSDEs with jumps of type (5) is given by:

**Theorem (BBP '97)**

*Assume [A1] and let $(Y^{t,x}, Z^{t,x}, U^{t,x})$ be the solution of*

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \\ \int_s^T h(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_E \gamma(r, X_r^{t,x}, e) U_r^{t,x}(e)\lambda(de))dr \\ \qquad - \int_s^T Z_r^{t,x} dB_r - \int_s^T \int_E U_r^{t,x}(e)\tilde{\mu}(dr, de), \ s \leq T. \end{cases} \tag{7}$$

*Then there exists a deterministic continuous function with polynomial growth $u(t,x)$ such that*

$$\forall s \in [t, T], \ Y_s^{t,x} = u(s, X_s^{t,x}). \tag{8}$$

*Moreover $u(t,x)$ is the unique viscosity solution of the IPDE (5) in the class of continuous functions with polynomial growth.*

# 3. The IPDE (1) without monotonicity condition

Objective: Weaken the monotonicity condition of $h$ wrt to $\zeta$ and $\gamma \geq 0$. Hereafter we do not make those assumptions.

**Lemma**

*Assume that the Lévy measure $\lambda$ is finite i.e. $\lambda(E) < \infty$ and let $(Y^{t,x}, Z^{t,x}, U^{t,x})$ be the solution of the BSDE associated with $(h(...), g(X_T^{t,x}))$ and $u(t,x)$ such that*

$$\forall s \in [t, T], \; Y_s^{t,x} = u(s, X_s^{t,x}). \tag{9}$$

*Then $ds \otimes d\mathbb{P} \otimes d\lambda$ on $[t, T] \times \Omega \times E$,*

$$U_s^{t,x}(e) = u(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x}). \tag{10}$$

## Sketch of the proof

Since $u$ is of polynomial growth and $\beta$ bounded then

$$\mathbf{E}[\int_0^T\int_E\{|U_s^{t,x}(e)|^2+|u(s,X_{s-}^{t,x}+\beta(s,X_{s-}^{t,x},e)-u(s,X_{s-}^{t,x})|^2\}\lambda(de)ds]<\infty$$

since $X^{t,x}$ has uniform moments of any order. As $\lambda(E)$ is finite then

$$\mathbf{E}[\int_0^T\int_E\{|U_s^{t,x}(e)|+|u(s,X_{s-}^{t,x}+\beta(s,X_{s-}^{t,x},e)-u(s,X_{s-}^{t,x})|\}\lambda(de)ds]<\infty.$$

Therefore $\forall s\in[t,T]$,

$$\int_t^s\int_E U_r^{t,x}(e)\tilde{\mu}(dr,de)=\int_t^s\int_E U_r^{t,x}(e)\mu(dr,de)-\int_t^s\int_E U_r^{t,x}(e)\lambda(de)dr.$$

But $Y^{t,x}$ is solution of the BSDJ (7) then for any $s\in[t,T]$,

$$\sum_{t<r\leq s}\{Y_r^{t,x}-Y_{r-}^{t,x}\}=\int_t^s\int_E U_r^{t,x}(e)\mu(dr,de).$$

On the other hand $Y_s^{t,x} = u(s, X_s^{t,x})$ and $u$ is continuous then

$$\int_t^s \int_E (u(r, X_{r-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(r, X_{r-}^{t,x}))\mu(dr, de)$$

$$= \sum_{t<r\leq s}\{Y_r^{t,x} - Y_{r-}^{t,x}\}, \ s \in [t, T].$$

It follows that $\forall s \in [t, T]$,

$$\int_t^s \int_E (u(r, X_{r-}^{t,x} + \beta(r, X_{r-}^{t,x}, e)) - u(r, X_{r-}^{t,x}) - U_r^{t,x}(e))\mu(dr, de) = 0.$$

Taking the quadratic variation of this last process and then expectation to obtain

$$\mathbf{E}[\int_t^T dr \int_E |u(r, X_{r-}^{t,x} + \beta(r, X_{r-}^{t,x}, e)) - u(r, X_{r-}^{t,x}) - U_r^{t,x}(e)|^2 \lambda(de)] = 0. \ \square$$

# Second definition of a viscosity solution

Let $\mathcal{U}$ be the set of deterministic functions $u(t,x)$ such that

$$|u(t,x) - u(t,x')| \leq C(1 + |x|^p + |x'|^p)|x - x'|, \ \forall t, x, x'. \quad (11)$$

**Definition**

A deterministic continuous function $u(t,x)$ of $\mathcal{U}$ is a viscosity sub-solution (resp. super-solution) of the IPDE (1) if:

(i) $\forall x \in \mathbb{R}^k$, $u(T,x) \leq g(x)$ (resp. $u(T,x) \geq g(x)$) ;

(ii) For any $(t,x) \in (0,T) \times \mathbb{R}^k$ and any function $\phi$ of class $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ such that $(t,x)$ is a global maximum (resp. minimum) point of $u - \phi$ and $(u - \phi)(t,x) = 0$, one has

$$-\partial_t \phi(t,x) - \mathcal{L}^X \phi(t,x)$$

$$-h(t,x,u(t,x),\sigma^\top(t,x)D_x\phi(t,x), Bu(t,x)) \leq 0 \text{ (resp. } \geq 0).$$

**Definition (continued)**

(iii) The function $u(t, x)$ is a viscosity solution of (1) if it is both a viscosity sub-solution and viscosity super-solution.

Remark: When $h$ does not depend on $Bu$, the two definitions coincide.

Assumption [A2]

(i) $g$ belongs to $\mathcal{U}$ ;

(ii) $(t,x) \mapsto h(t,x,y,z,\zeta)$ belongs uniformly to $\mathcal{U}$.

(iii) $\lambda(E) < \infty$ is no longer required.

**Proposition**

*Let $(Y^{t,x}, Z^{t,x}, U^{t,x})$ be the solution of (7). Under [A2], $u(t,x)$ belongs to $\mathcal{U}$.*

**Proof:** Sketch.

Step 1: $\forall p \geq 1$, $\exists \rho \geq 0$,

$$\mathbf{E}[\{\int_0^T ds \int_E |U^{t,x}(e)|^2 \lambda(de)\}^{\frac{p}{2}}] \leq C(1+|x|^\rho).$$

Step 2: By Itô's formula with

$$(Y_s^{t,x} - Y_s^{t,x'})^2, s \in [t,T].$$

**Proposition**

*Let $(Y^{t,x}, Z^{t,x}, U^{t,x})$ be the solution of the BSDE associated with $(h(\ldots), g(X_T^{t,x}))$ and $u(t,x)$ such that*

$$\forall s \in [t, T],\ \ Y_s^{t,x} = u(s, X_s^{t,x}). \tag{12}$$

*Under [A2], $ds \otimes d\mathbb{P} \otimes d\lambda$ on $[t, T] \times \Omega \times E$,*

$$U_s^{t,x}(e) = u(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x}). \tag{13}$$

## Main steps of the proof

(i) Troncation: For $k \geq 1$ let

$$\mu_k(ds, de) = 1_{|e| \geq \frac{1}{k}} \mu(ds, de)$$

which still a Poisson random measure. Its componsator is

$$ds\lambda_k(de) := 1_{\{|e| \geq \frac{1}{k}\}} ds\lambda(de).$$

Morever

$$\lambda_k(E) < \infty.$$

We set:

$$\tilde{\mu}_k(ds, de) = \mu_k(ds, de) - ds\lambda_k(de).$$

(ii) Approximation and convergence

For $k \geq 1$, let ${}^kX^{t,x}$, $({}^kY^{t,x}, {}^kZ^{t,x}, {}^kU^{t,x})$ be the processes verifying:
(a)

$$
{}^kX_s^{t,x} = x + \int_t^s b(r, {}^kX_r^{t,x})dr + \int_t^s \sigma(r, {}^kX_r^{t,x})dB_r
$$
$$
+ \int_t^s \int_E \beta(r, {}^kX_{r-}^{t,x}, e)\tilde{\mu}_k(dr, de),\ s \in [t, T];\ {}^kX_s^{t,x} = x \text{ if } s \leq t \tag{14}
$$

$$
\begin{cases}
{}^kY_s^{t,x} = g({}^kX_T^{t,x}) + \\
\int_s^T h(r, {}^kX_r^{t,x}, {}^kY_r^{t,x}, {}^kZ_r^{t,x}, \int_E \gamma(r, {}^kX_r^{t,x}, e), {}^kU_r^{t,x}(e)\lambda_k(de))dr \\
\qquad - \int_s^T {}^kZ_r^{t,x} dB_r - \int_s^T \int_E {}^kU_r^{t,x}(e)\tilde{\mu}_k(dr, de),\ s \leq T.
\end{cases} \tag{15}
$$

Then:

(i)

$$E[\sup_{s\le T}|X_s^{t,x} - {}^kX_s^{t,x}|^2] \to_k 0;$$

(ii)

$${}^kY_s^{t,x} = {}^ku(s, {}^kX_s^{t,x}), s \in [t, T].$$

(iii)

$$E[\sup_{s\le T}|Y_s^{t,x} - {}^kY_s^{t,x}|^2] + \int_0^T |Z_s^{t,x} - {}^kZ_s^{t,x}|^2 ds$$

$$+ \int_0^T ds \int_E \lambda(de)|U_s^{t,x}(e) - {}^kU_s^{t,x}(e)1_{\{|e|\ge\frac{1}{k}\}}|^2] \to_k 0.$$

But

$$ {}^kU_s^{t,x}(e) = {}^ku(s, {}^kX_{s-}^{t,x} + \beta(s, {}^kX_{s-}^{t,x}, e)) - {}^ku(s, {}^kX_{s-}^{t,x}), \ \ ds \otimes d\mathbb{P} \otimes d\lambda_k $$

and then $ds \otimes d\mathbb{P} \otimes d\lambda$

$$ 1_{\{|e| \geq \frac{1}{k}\}} \, {}^kU_s^{t,x}(e) = 1_{\{|e| \geq \frac{1}{k}\}}({}^ku(s, {}^kX_{s-}^{t,x} + \beta(s, {}^kX_{s-}^{t,x}, e)) - {}^ku(s, {}^kX_{s-}^{t,x})). $$

But

(a)

$$ {}^ku(t,x) \rightarrow_k u(t,x) $$

(b) ${}^ku$ verifies uniformly (11) (not needed).

Taking the limit wrt $k$ to obtain: $ds \otimes d\mathbb{P} \otimes d\lambda$,

$$ U_s^{t,x}(e) = u(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x}). \qquad (16) $$

# 4. Existence and uniqueness of the solution

**Theorem**

*Assume [A2]. Then the function $u(t,x)$ of (12) is a viscosity solution of (1). Moreover it is unique in the class of continuous functions with polynomial growth.*

Sketch of the proof: It is a solution.

Let us consider the following BSDEJ:

$$
\begin{cases}
\underline{Y}_T^{t,x} = g(X_T^{t,x}) \text{ and } \forall s \le T, \\
d\underline{Y}_s^{t,x} = \underline{Z}_s^{t,x} dB_s + \int_E \underline{U}_s^{t,x}(e)\tilde{\mu}(ds,de) \\
-h(s, X_s^{t,x}, \underline{Y}_s^{t,x}, \underline{Z}_s^{t,x}, \\
\int_E \gamma(s, X_s^{t,x}, e)\{u(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x})\}\lambda(de))ds.
\end{cases}
\tag{17}
$$

The solution of this equation exists and is unique.

The generator of this BSDEJ does not depend on the jump part and since

$$(t,x,y,z) \longmapsto h(t,x,y,z,\int_E \gamma(t,x,e)\{u(t,x+\beta(t,x,e))-u(t,x)\}\lambda(de))$$

verify [A1] then there exists a deterministic continuous function of polynomial growth $\underline{u}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\forall s \in [t,T],\ \underline{Y}_s^{t,x} = \underline{u}(s, X_s^{t,x}).$$

Then $\underline{u}$ is the unique viscosity solution of

$$\begin{cases} -\partial_t \underline{u}(t,x) - b(t,x)^\top D_x \underline{u}(t,x) - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(t,x) D^2_{xx}\underline{u}(t,x)\right) \\ K\underline{u}(t,x) - h(t,x,\underline{u}(t,x),(\sigma^\top D_x \underline{u})(t,x), Bu(t,x)) = 0,\ (t,x) \in [0,T] \\ \underline{u}(T,x) = g(x) \end{cases} \tag{18}$$

Next $(Y^{t,x}, Z^{t,x}, U^{t,x})$ verify:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \\ \int_s^T h(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_E \gamma(r, X_r^{t,x}, e) U_r^{t,x}(e)\lambda(de))dr \\ \quad - \int_s^T Z_r^{t,x} dB_r - \int_s^T \int_E U_r^{t,x}(e)\tilde{\mu}(dr, de), \ s \leq T. \end{cases} \quad (19)$$

and

$$U_s^{t,x}(e) = u(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x})$$

By plugging this relation in the second term of the previuos BSDEJ and by uniqueness of the solution of this BSDEJ one gets

$$\forall s \in [t, T], \ \underline{Y}_s^{t,x} = Y_s^{t,x}.$$

and then

$$u = \underline{u}$$

which implies that $u$ is a viscosity solution of (1) in the sense of the second definition. □

## Uniqueness in $\mathcal{U}$

Let $\bar{u} \in \mathcal{U}$ be another function solution of the IPDE (1) in the sense of the second definition.

$$\begin{cases} -\partial_t \bar{u}(t,x) - b(t,x)^\top D_x \bar{u}(t,x) - \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^\top(t,x) D^2_{xx}\bar{u}(t,x)\right) \\ -K\bar{u}(t,x) - h(t,x,\bar{u}(t,x),(\sigma^\top D_x \bar{u})(t,x), B\bar{u}(t,x)) = 0; \\ \bar{u}(T,x) = g(x) \end{cases} \tag{20}$$

Let us consider the following BSDEJ:

$$
\begin{cases}
\bar{Y}_T^{t,x} = g(X_T^{t,x}) \text{ and } \forall s \leq T, \\
d\bar{Y}_s^{t,x} = -h(s, X_s^{t,x}, \bar{Y}_s^{t,x}, \bar{Z}_s^{t,x}, \\
\int_E \gamma(s, X_s^{t,x}, e)\{\bar{u}(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}(s, X_{s-}^{t,x})\}\lambda(de))ds \\
\qquad\qquad +\bar{Z}_s^{t,x} dB_s + \int_E \bar{U}_s^{t,x}(e)\tilde{\mu}(ds, de).
\end{cases}
\tag{21}
$$

Therefore there exists a deterministic continuous function $v(t,x)$ of PG such that

$$\forall s \in [t,T], \;\; \bar{Y}_s^{t,x} = v(s, X_s^{t,x}).$$

and

$$\bar{U}_s^{t,x}(e) = v(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v(s, X_{s-}^{t,x})$$

Additionally by Barles et al.'s result $v(t,x)$ is the unique solution in the class of continuous functions of PG of:

$$\begin{cases} -\partial_t v(t,x) - b(t,x)^\top D_x v(t,x) - \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^\top(t,x) D^2_{xx} v(t,x)\big) \\ -Kv(t,x) - h(t,x,v(t,x),(\sigma^\top D_x v)(t,x), B\bar{u}(t,x)) = 0; \\ v(T,x) = g(x) \end{cases} \tag{22}$$

Finally by the uniqueness of the solution of this latter we have $v = \bar{u}$ and then

$$\bar{U}_s^{t,x}(e) = v(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - v(s, X_{s-}^{t,x})$$

$$= \bar{u}(s, X_{s-}^{t,x} + \beta(s, X_{s-}^{t,x}, e)) - \bar{u}(s, X_{s-}^{t,x}).$$

Plug this equality in BSDEJ (21) and by the uniqueness of the solution of the BSDEJ we have

$$\bar{Y}^{t,x} = Y^{t,x}$$

and then $u = \bar{u}$. $\square$

# 5. Extensions

(i) Some multidimensional cases ;

(ii) IPDEs with one or two obstacles ;

(iii) IPDEs with generators of type

$$h(t, x, y, z, \|v\|). \square$$

Thanks for your attention.