Risk Minimisation under Mortality and its Stochastics

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Talk is based on joint works with

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Ghent University, Belgium

The plan of the talk

- **Our Framework?**

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- **Literature and Main Problems**
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- **Optional Martingale Representation Theorem**

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- $$m := G + D^{o,\mathbb{F}}, \quad R := \inf\{t \geq 0 \mid G_t = 0\}. \qquad (2.2)$$

Quadratic hedging

. Precisely, the Risk-minimisation method of Föllmer-Sonderman. We work under the risk neutral measure, and assume that is P. $S \in \mathcal{M}^2_{loc}(\mathbb{H}, P)$. For a liability process A, and for $\rho = (\xi, \eta)$,

$$V(\rho) = \xi^{tr} S + \eta, \qquad C(\rho) = V(\rho) - \xi \cdot S + A.$$

Risk-Minimising strategy ρ^* is the strategy ρ that minimises

$$R(\rho)_t := E((C_T(\rho) - C_t(\rho))^2 | \mathcal{H}_t)$$

among those $V_T(\rho) = 0$ and $V(\rho)$ is square integrable process. Galtchouk-Kunita-Watanabe of $E(A_T|\mathcal{H}_t)$ with respect to S

$$E(A_T|\mathcal{H}_t) = E(A_T) + \xi^A \cdot S + L^A,$$

L^A = remaining risk, and ξ^A = the risk minimising strategy (See Möller and Schweizer papers).

- **Assumptions:**
  There exists $Q \sim P$ such that $S \in \mathcal{M}^2_{loc}(\mathbb{G}, Q)$ and

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- **Goal:** For $h \in L^2(\widetilde{\Omega}, \mathcal{O}(\mathbb{F}), P \otimes D)$, we would like to describe as explicit as possible the risk-minimising strategy for h_τ.
  a) Prove that

$$\xi^{(h,\mathbb{G})} = f\left(\xi^{(1)}, ..., \xi^{(k)}\right)$$
$$L^{(h,\mathbb{G})} = \kappa\left(L^{(1)}, ..., L^{(k)}, L^{(\tau,1)}, ..., L^{(\tau,l)}\right)$$

  b) **Which (intrinsic) mortality Risk that can not be hedged in $(S, \mathbb{G})$?**
  c) **How mortality risk and financial risk interplay?**

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- Barbarin(2008): Predictable h, relatively general initial market model, CONDITIONS ON τ:
  1) $G > 0$ (existence of hazard rate)
  2) either τ avoids $\mathbb{F}$-stopping times of all $\mathbb{F}$-martingales are continuous.
  3) Independence between S and information related to τ, and the recovery benefit is independent of τ.

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- **Gerber/Shiu/Yang(2012,2013)**: market model is driven by Brownian and Poisson: They priced the Equity-linked death benefits:
  1)$e^{-\delta\tau}b(S_\tau)$, where $b(s)$ is a benefit function .
  2) Guarantee: $Max(S_\tau, K)$
  3) Fixed-strike lookback call option:
  $\left(\max\left(H, \sup_{0\leq t\leq\tau} S_t\right) - K\right)^+$.
  4) Fixed-strike lookback put option:
  $\left(K - \min\left(H, \inf_{0\leq t\leq\tau} S_t\right)\right)^+$.
  5) Floting-strike lookback option: $\max\left(H, \sup_{0\leq t\leq\tau} S_t\right) - S_\tau$.
  6) Barrrier option: $I_{\{\sup_{0\leq t\leq\tau} S_t\geq L\}}b(S_\tau)$.

Most of the recent works above, in a way or another, use a martingale representation result of Blanchet/Jeanblanc(2004).

$$E(h_\tau \big| \mathcal{G}_t) = H_0 + \xi^{(h)} \centerdot \overline{M^h} + \theta^{(h)} \centerdot \overline{m} + \zeta^{(h)} \centerdot \overline{N}^{\mathbb{G}}.$$

Here,

$$\overline{N}^{\mathbb{G}} := D - (G_-)^{-1} I_{[\![0,\tau[\![} \centerdot D^{p,\mathbb{F}},$$

M^h is an $\mathbb{F}$-martingale, and for any $\mathbb{F}$-martingale, M,

$$\overline{M} := M^\tau - (G_-)^{-1} I_{[\![0,\tau[\![} \centerdot \langle M, m\rangle^{\mathbb{F}}.$$

$$\widehat{M} = M^{\tau} - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet [M, m] + I_{[\![0,\tau]\!]} \bullet \left(\Delta M_{\widetilde{R}} I_{[\![\widetilde{R},+\infty[\![}\right)^{p,\mathbb{F}}, \quad (5.1)$$

$$\mathcal{M}^{(1)}(\mathbb{G}) := \left\{ \widehat{M} \text{ defined in (5.1)} \mid \widehat{M}_{\infty} \in L^1(P),\ M \in \mathcal{M}_{0,loc}(\mathbb{F}) \right\}. \quad (5.2)$$

Theorem 1:

Consider the following process

$$N^{\mathbb{G}} := D - \left(\widetilde{G}\right)^{-1} I_{[\![0,\tau]\!]} \cdot D^{o,\mathbb{F}}. \tag{5.3}$$

Then, the following assertions hold.

- $N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation.

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- $N^{\mathbb{G}}$ is a $\mathbb{G}$-martingale with integrable variation.
- Let K be an $\mathbb{F}$-optional process, which is Lebesgue-Stieljes integrable with respect to $N^{\mathbb{G}}$. Then, $K \centerdot N^{\mathbb{G}} \in \mathcal{A}(\mathbb{G})$ if and only if K belongs to $\mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G})$, where

$$\mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G}) := \left\{ K \in \mathcal{O}(\mathbb{F}) \;\middle|\; E\left(|K| G \widetilde{G}^{-1} 1_{\{\widetilde{G}>0\}} \centerdot D_\infty \right) < +\infty \right\} \tag{5.4}$$

- The elements of

$$\mathcal{M}^{(2)}(\mathbb{G}) := \left\{ K \bullet N^{\mathbb{G}} \;\middle|\; K \in \mathcal{I}^o(N^{\mathbb{G}}, \mathbb{G}) \right\}, \tag{5.5}$$

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- If all $\mathbb{F}$-martingales are continuous, then $N^{\mathbb{G}} = \overline{N}^{\mathbb{G}}$, as in this case $\widetilde{G} = G_-$ and $D^{o,\mathbb{F}} = D^{p,\mathbb{F}}$ continuous.

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- It is not possible to go further in weakening the space of integrands for $N^{\mathbb{G}}$!!

on $(\Omega\times[0,+\infty),\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+))$, we consider $\mu := P\otimes D$ and on $(\Omega,\mathcal{F})$, we consider the following σ-fields

$$\mathcal{F}_{\tau-} := \sigma\left(X_\tau \;\Big|\; X \text{ is } \mathbb{F}\text{-predictable}\right),$$

$$\mathcal{F}_{\tau} := \sigma\left(X_\tau \;\Big|\; X \text{ is } \mathbb{F}\text{-optional}\right),$$

$$\mathcal{F}_{\tau+} := \sigma\left(X_\tau \;\Big|\; X \text{ is } \mathbb{F}\text{-progressively measurable}\right).$$

For $\mathcal{H}\in\{\mathcal{P}(\mathbb{F}),\mathcal{O}(\mathbb{F}),\mathcal{P}_{rog}(\mathbb{F})\}$, we define

$$L^1(\mathcal{H},P\otimes D) := \left\{ X\ \mathcal{H}-\text{measurable} \;\Big|\; \begin{array}{l} E(|X_\tau| I_{\{\tau<+\infty\}}) \\ =: E_{P\otimes D}(|X|) < +\infty \end{array} \right\}$$

Lemma

Let X be an $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^+)$-measurable process such that $X\geq 0$ $\mu-$ a.e. (recall that $\mu := P\otimes D$) or $X\in L^1(\mu)$. Then, the following equalities hold.

$$\mathbb{E}_\mu\left(X\middle|\mathcal{P}(\mathbb{F})\right)=\mathbb{E}\left(X_\tau\middle|\mathcal{F}_{\tau-}\right)I_{[\![\tau]\!]},\quad \mathbb{E}_\mu\left(X\middle|\mathcal{O}(\mathbb{F})\right)=\mathbb{E}\left(X_\tau\middle|\mathcal{F}_{\tau}\right)I_{[\![\tau]\!]},$$
$$\mathbb{E}_\mu\left(X\middle|\mathcal{P}_{rog}(\mathbb{F})\right)=\mathbb{E}\left(X_\tau\middle|\mathcal{F}_{\tau+}\right)I_{[\![\tau]\!]}.$$

Here $\mathbb{E}_\mu(.|.)$ is the conditional expectation under the finite measure μ.

Theorem 2:

The following assertions hold.

- For any $k \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right)$, there exists a unique $h \in L^1\left(\mathcal{O}(\mathbb{F}), P \otimes D\right)$ satisfying

$$E\left(k_\tau \,\middle|\, \mathcal{F}_\tau\right) = h_\tau \quad P - a.s. \quad \text{on} \quad \{\tau < +\infty\}. \tag{5.7}$$

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- The elements of the set

$$\mathcal{M}^{(3)}(\mathbb{G}) := \left\{ k \centerdot D \,\middle|\, k \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P\otimes D\right) \;\&\; E(k_\tau|\mathcal{F}_\tau) = 0 \right\}$$

are $\mathbb{G}$-martingales that are orthogonal to locally bounded elements of both $\mathcal{M}^{(1)}(\mathbb{G})$ and $\mathcal{M}^{(2)}(\mathbb{G})$.

A Jeulin's Martingale Space:

$$\mathcal{M}^{(4)}(\mathbb{G}) := \left\{ k \centerdot D \;\middle|\; \begin{array}{l} k \in L^1\left(\mathcal{P}_{rog}(\mathbb{F}), P \otimes D\right) \\ \text{and } E(k_\tau | \mathcal{F}_{\tau-}) = 0 \ \ P - a.s \end{array} \right\}$$

Then,

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Then,

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- The elements of this space are NOT orthogonal to those of $\mathcal{M}^{(i)}(\mathbb{G})$, $i = 1, 2$.

Theorem 3:

Consider $h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$. Then, the $\mathbb{G}$-martingale $H_t := \mathbb{E}[h_\tau | \mathcal{G}_t]$ admits the following representation

$$H := {}^{o,\mathbb{G}}(h_\tau) = H_0 + G_-^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{M^h} - J_- G_-^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{m} + (h - J) \bullet N^{\mathbb{G}}.$$

Here M^h and J are given by

$$M^h = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dD_u^{o,\mathbb{F}} \right) \quad \text{and} \quad J = \left(M^h - h \bullet D^{o,\mathbb{F}} \right) G^{-1} I_{[\![0,R[\![}. \tag{6.1}$$

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$$\begin{aligned} H := {}^{o,\mathbb{G}}(h_\tau) = \mathbb{E}[h_\tau | \mathcal{G}_t] &= H_0 + \frac{1}{G_-} I_{[\![0,\tau]\!]} \centerdot \widehat{m^h} \\ &+ \frac{h - J_-}{G_-} I_{[\![0,\tau]\!]} \centerdot \widehat{m} + (h - J) \centerdot N^{\mathbb{G}}. \end{aligned}$$

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Let $h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$, and consider M^h and J, defined in (6.1). Then, the following hold

- If τ is an $\mathbb{F}$-pseudo stopping time, then it holds that

$$H = H_0 + {G_-}^{-1} I_{[\![0,\tau]\!]} \centerdot (M^h)^\tau + (h - J) \centerdot N^{\mathbb{G}}.$$

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- In case τ avoids $\mathbb{F}$-stopping times, we get

$$H = H_0 + {G_-}^{-1} I_{[\![0,\tau]\!]} \bullet \overline{M^h} - J_-(G_-)^{-1} I_{[\![0,\tau]\!]} \bullet \overline{m} + (h - J_-) \bullet \overline{N}^{\mathbb{G}}.$$

Here $\overline{N}^{\mathbb{G}}$ is given by (5.6), and for any $\mathbb{F}$-local martingale, M, $\bar{M}$ is defined by

$$\overline{M} := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet [M, m]. \quad (6.3)$$

- If all $\mathbb{F}$-martingales be continuous, then

$$H = H_0 + \frac{I_{[\![0,\tau]\!]}}{G_-} \bullet \overline{M^h} - \frac{J_-}{G_-} I_{[\![0,\tau]\!]} \bullet \overline{m} + (h - J_-) \bullet \overline{N}^{\mathbb{G}},$$

where $\overline{N}^{\mathbb{G}}$ and $\overline{M}$ are given by (5.6) and (6.3) respectively.

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- If G is strictly positive, then

$$H = H_0 + {G_-}^{-1} I_{[\![0,\tau]\!]} \bullet \overline{M^h}^{\mathbb{G}} - J_- {G_-}^{-1} I_{[\![0,\tau]\!]} \bullet \overline{m} + (h - J) \bullet N^{\mathbb{G}},$$

where for any $\mathbb{F}$-local martingale M,

$$\overline{M}^{\mathbb{G}} := M^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet [M, m].$$

Comparing with Blanchet/Jeanblanc(2004):

Suppose that $h \in L^1(\widetilde{\Omega}, \mathcal{P}(\mathbb{F}), P \otimes D)$, and either τ avoids $\mathbb{F}$-stopping times or all $\mathbb{F}$-martingales are continuous. Then

$$\begin{aligned} H &= H_0 + G_-{}^{-1} I_{[\![0,\tau]\!]} \cdot \overline{m^h} + h - J_- G_-{}^{-1} I_{[\![0,\tau]\!]} \cdot \overline{m} \\ &+ (h - J_-) G_- G^{-1} I_{[\![0,\tau]\!]} \cdot \overline{N}^{\mathbb{G}}. \end{aligned}$$

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$$m^h = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dF_u\right) \quad \text{and} \quad J = \left(m^h - h \cdot F\right)(G^\tau)^{-1}.$$

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- $D^{o,\mathbb{F}} = D^{p,\mathbb{F}}$ implies that $m = 1 - Z$, such that

$$\begin{aligned} \overline{M} &= M^{\tau} - G^{-1} I_{[\![0,\tau]\!]} \cdot [M, m] = M^{\tau} + G^{-1} I_{[\![0,\tau]\!]} \cdot [M, Z], \\ \bar{m} &= 1 - Z^{\tau} - G^{-1} I_{[\![0,\tau]\!]} \cdot [Z, Z] = 1 - \bar{Z}. \end{aligned}$$

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- The previous martingale representation coincides with the one established in Theorem 1 of Blanchet/Jeanblanc(2004). However, in our case, G is allowed to vanish as opposed to the setting considered by Blanchet/Jeanblanc(2004).

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- If τ is an $\mathbb{F}$-stopping time, then

$$H = {}^{o,\mathbb{G}}(h_\tau) = {}^{o,\mathbb{F}}(h_\tau) = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dD_u\right) = m^h.$$

Can we extend it to

$E(h_\tau \mid \mathcal{G}_t)$ **with** $h \in L^1(\widetilde{\Omega}, \mathcal{P}_{prog}(\mathbb{F}), P \otimes D)$**?**

How far can we go in extending this optional representation?

Theorem 4:

If $(\mathcal{M}(\mathbb{G}))^{\tau}$ is the set of $\mathbb{G}$-martingales stopped at τ, then

$$(\mathcal{M}(\mathbb{G}))^{\tau} = \mathcal{M}^{(1)}(\mathbb{G}) \oplus \mathcal{M}^{(2)}(\mathbb{G}) \oplus \mathcal{M}^{(3)}(\mathbb{G}). \qquad (6.4)$$

In other words, for any $\mathbb{G}$-martingale, $M^{\mathbb{G}}$, there exist $h \in L^1(P \otimes D, \mathcal{O}(\mathbb{F}))$ and $k \in L^1(P \otimes D, \mathcal{P}_{rog}(\mathbb{F}))$, such that $E(k_\tau | \mathcal{F}_\tau) = 0$ on $\{\tau < +\infty\}$, $k_\tau + h_\tau = M^{\mathbb{G}}_\tau$, and

$$\begin{aligned} \left(M^{\mathbb{G}}\right)^{\tau} &= M^{\mathbb{G}}_0 + (G_-)^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{M^h} - J^h_-\, (G_-)^{-1} I_{[\![0,\tau]\!]} \bullet \widehat{m} \\ &+ (h - J^h) \bullet N^{\mathbb{G}} + k \bullet D. \end{aligned}$$

Here M^h and J^h are defined in (6.1) and m is given by (2.2).

Jeulin's Theorem:

Suppose that τ is honest. Then,

$$\mathcal{M}(\mathbb{G}) = \text{generated by } \mathcal{M}^{(1bis)}(\mathbb{G}) \bigcup \mathcal{M}^{(2bis)}(\mathbb{G}) \bigcup \mathcal{M}^{(4)}(\mathbb{G}),$$

where

$$\mathcal{M}^{(1bis)}(\mathbb{G}) := \{ M^{\tau} - G_{-}^{-1} I_{[\![0,\tau]\!]} \centerdot \langle M, m \rangle^{\mathbb{F}} \mid M \in \mathcal{M}^2(\mathbb{F}) \}$$

$$\mathcal{M}^{(2bis)}(\mathbb{G}) := \{ H \centerdot \overline{N}^{\mathbb{G}} \mid H \text{ is } \mathbb{F}-\text{predictable and "integrable"} \}$$

Corollary

We have

$$(\mathcal{M}_{loc}(\mathbb{G}))^{\tau} = \mathcal{M}^{(1)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(2)}_{loc}(\mathbb{G}) \oplus \mathcal{M}^{(3)}_{loc}(\mathbb{G}).$$

Thanks to localization result of Aksamit/Choulli/Deng/Jeanblanc(2014)

Thank you for your attention